nal equation which can be derived from the functional equation for $f(z)$, as follows.

By definition,

$$f(z) = p(z) f(z^g), \quad \text{where} \quad p(z) = p_0 + p_1 z + \ldots + p_d z^d,$$

hence

$$f(e^{-Z}) = (p_0 + p_1 e^{-Z} + p_2 e^{-2Z} + \ldots + p_d e^{-dZ}) f(e^{-dZ}).$$

Therefore

$$\Gamma(s)\varphi(s|a) = \int_0^\infty e^{-aZ} f(e^{-gZ})(p_0 + p_1 e^{-Z} + \ldots + p_d e^{-dZ}) Z^{s-1}\, dZ$$

$$= \sum_{h=0}^{d} p_h \int_0^\infty e^{-aZ} f(e^{-gZ}) e^{-hZ} Z^{s-1}\, dZ.$$

Here replace gZ by the new variable ζ. Then this formula becomes

$$\Gamma(s)\varphi(s|a) = \sum_{h=0}^{d} p_h \int_0^\infty e^{-\frac{a+h}{g}\zeta} f(e^{-\zeta})(\zeta/g)^{s-1}\, d\zeta/g$$

$$= \sum_{h=0}^{d} p_h g^{-s} \Gamma(s) \varphi\left(s \,\middle|\, \frac{a+h}{g}\right).$$

Hence $\varphi(s|a)$ satisfies the functional equation

$$\varphi(s|a) = \sum_{h=0}^{d} p_h g^{-s} \varphi\left(s \,\middle|\, \frac{a+h}{g}\right). \tag{3}$$

On differentiating the integral for $\Gamma(s)\varphi(s|a)$ partially with respect to a, we obtain the further identity

$$\frac{\partial}{\partial a}\varphi(s|a) = -\varphi(s+1|a). \tag{4}$$

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On some estimates involving the number of prime divisors of an integer

by

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Dedicated to Professor Paul Erdős on the occasion of his 75th birthday

1. Introduction and statement of results. Let as usual $\Omega(n)$ and $\omega(n)$ denote the number of all prime factors of $n \geqslant 1$ and the number of distinct prime factors of n, respectively. Further let $P(n)$ denote the largest prime factor of $n \geqslant 2$, and let $P(1) = 1$. The functions $\Omega(n)$, $\omega(n)$ and $P(n)$ determine to a large extent the distribution of prime divisors of n. In many problems involving $P(n)$ one often encounters the function

$$\psi(x, y) = \sum_{n \leqslant x, P(n) \leqslant y} 1, \tag{1.1}$$

which represents the number of positive integers $\leqslant x$ all of whose prime factors are $\leqslant y$. An extensive literature on $\psi(x, y)$ exists, and recently (see [7], [8]) important developments in this field have been made. The new results on $\psi(x, y)$ are likely to find many applications, and in [11] they were used to obtain information about local densities of a certain class of arithmetical functions over integers with small prime factors. Several results concerning the local behaviour of $\psi(x, y)$ were derived in [11], and some of these will be needed in the proof of

THEOREM 1. *Let $y \leqslant x$, $\log y/\log\log x \to \infty$ as $x \to \infty$, and let p denote prime numbers. Then we have uniformly*

$$\sum_{n \leqslant x, P(n) \leqslant y} (\Omega(n) - \omega(n)) = \psi(x, y)\left(\sum_p \frac{1}{p^2 - p} + O\left(\frac{\log\log x}{\log y}\right)\right). \tag{1.2}$$

Asymptotic estimates of sums involving $\Omega(n)$, $\omega(n)$ and reciprocals of $P(n)$ elucidate the distribution of prime factors of n, and they were studied in [5], [6], and [10]. In particular, it was proved in [6] that

$$\sum_{n \leqslant x} \frac{\Omega(n) - \omega(n)}{P(n)} = \left\{c + O\left(\frac{(\log\log x)^{3/2}}{\log^{1/2} x}\right)\right\} \sum_{n \leqslant x} \frac{1}{P(n)} \tag{1.3}$$

holds for a suitable constant $c > 0$, and that, as $x \to \infty$,

$$\sum_{n\leqslant x}\frac{\omega(n)}{P(n)} \sim \sum_{n\leqslant x}\frac{\Omega(n)}{P(n)} \sim \left(\frac{2\log x}{\log\log x}\right)^{1/2}\sum_{n\leqslant x}\frac{1}{P(n)}. \tag{1.4}$$

In these formulas certain sums are compared with the sum of reciprocals of $P(n)$, which was estimated in [3], [6], [9] and [10]. A precise estimate for this sum is obtained in [6], where it is shown that

$$\sum_{n\leqslant x}\frac{1}{P(n)} = \left\{1+O\left(\left(\frac{\log\log x}{\log x}\right)^{1/2}\right)\right\}x\delta(x). \tag{1.5}$$

The function $\delta(x)$ is explicitly given, although it is fairly complicated. Its first approximation by elementary functions (see [10]) is

$$\delta(x) = \exp\left\{-(2\log x\log_2 x)^{1/2}\left(1+\frac{\log_3 x-2-\log 2}{2\log_2 x} -\left(\tfrac{1}{8}+o(1)\right)\frac{\log_3^2 x}{\log_2^2 x}\right)\right\} \quad (x\to\infty), \tag{1.6}$$

where here and in the sequel $\log_2 x = \log\log x$ and $\log_3 x = \log\log\log x$. One of the aims of this paper is to provide sharpenings of (1.3) and (1.4). Our approach is different and simpler than the one employed in [6]. The results are contained in

THEOREM 2.

$$\sum_{n\leqslant x}\frac{\Omega(n)-\omega(n)}{P(n)} = \left\{\sum_p\frac{1}{p^2-p}+O\left(\left(\frac{\log\log x}{\log x}\right)^{1/2}\right)\right\}\sum_{n\leqslant x}\frac{1}{P(n)}. \tag{1.7}$$

THEOREM 3.

$$\sum_{n\leqslant x}\frac{\omega(n)}{P(n)} = \left\{\left(\frac{2\log x}{\log\log x}\right)^{1/2}\left(1+O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right\}\sum_{n\leqslant x}\frac{1}{P(n)}. \tag{1.8}$$

The last formula remains true if $\omega(n)$ is replaced by $\Omega(n)$, which follows trivially from (1.7). By more elaborate arguments the error term in (1.8) could be replaced by a more precise expression. Our methods of proof are capable of dealing with several other sums which are similar to those on the left-hand side of (1.7) and (1.8). Thus it may be shown that (1.7) and (1.8) remain true if $P(n)$ is replaced by $P^r(n)$ for any fixed $r>0$, but for simplicity only the most interesting case $r=1$ is considered. Also it seems interesting to compare the sum of reciprocals of $P(n)$ over squarefree integers with the sum of reciprocals of $P(n)$. The result is

THEOREM 4.

$$\sum_{n\leqslant x}\frac{\mu^2(n)}{P(n)} = \left\{6\pi^{-2}+O\left(\left(\frac{\log\log x}{\log x}\right)^{1/2}\right)\right\}\sum_{n\leqslant x}\frac{1}{P(n)}. \tag{1.9}$$

This is exactly the type of formula one expects heuristically, since the density of squarefree numbers is well known to be $6\pi^{-2}$, and $\mu^2(n)$ is the characteristic function of squarefree numbers. It will transpire from the proof that the major contribution to the sums in (1.5), (1.7)–(1.9) comes from those n for which

$$P(n) = \exp\left\{\left(1/\sqrt{2}+o(1)\right)(\log x\log\log x)^{1/2}\right\} \quad (x\to\infty). \tag{1.10}$$

Since it is known (see [7]) that the error term in the asymptotic formula (2.3) for $\psi(x, y)$ is best possible, it is reasonable to conjecture that the error terms in (1.5), (1.7) and (1.9) are also best possible.

2. Auxiliary estimates. In this section we shall formulate several results which will be needed in the sequel.

LEMMA 1. *For any additive function $f(n)$ and $2\leqslant y\leqslant x$ we have*

$$\sum_{n\leqslant x, P(n)\leqslant y} f(n) = \sum_{p^\alpha\leqslant x, p\leqslant y}\left(f(p^\alpha)-f(p^{\alpha-1})\right)\psi(xp^{-\alpha}, y). \tag{2.1}$$

This is a straightforward result, stated as Lemma 6 in [4]. Namely, because of additivity the left-hand side of (2.1) equals

$$\begin{aligned}\sum_{n\leqslant x, P(n)\leqslant y}\ \sum_{p^\alpha\| n} f(p^\alpha) &= \sum_{\substack{p^\alpha m\leqslant x, (p,m)=1\\ p\leqslant y, P(m)\leqslant y}} f(p^\alpha)\\ &= \sum_{p^\alpha\leqslant x, p\leqslant y}\left(f(p^\alpha)-f(p^{\alpha-1})\right)\sum_{m\leqslant x/p^\alpha, P(m)\leqslant y} 1\\ &= \sum_{p^\alpha\leqslant x, p\leqslant y}\left(f(p^\alpha)-f(p^{\alpha-1})\right)\psi(xp^{-\alpha}, y).\end{aligned}$$

LEMMA 2. *For any fixed $\varepsilon>0$ and*

$$\exp\left((\log\log x)^{5/3+\varepsilon}\right)\leqslant y\leqslant x \tag{2.2}$$

we have uniformly

$$\psi(x, y) = \sum_{n\leqslant x, P(n)\leqslant y} 1 = x\varrho(u)\left(1+O\left(\frac{\log(u+1)}{\log y}\right)\right), \quad u=\frac{\log x}{\log y}, \tag{2.3}$$

where $\varrho(v)$ is the solution of the differential delay equation $v\varrho'(v)+\varrho(v-1)=0$ with the initial condition $\varrho(v)=1$ for $0\leqslant v\leqslant 1$. An approximation to $\varrho(v)$ in terms of elementary functions is

$$\varrho(v) = \exp\left\{-v\left(\log v+\log\log v-1+\frac{\log\log v-1}{\log v} +O\left(\left(\frac{\log\log v}{\log v}\right)^2\right)\right)\right\}. \tag{2.4}$$

Lemma 2 contains standard results on $\psi(x, y)$. The proof of (2.3) for the wide range (2.2) is given by A. Hildebrand [7], who extended considerably the earlier known range, due to N. G. de Bruijn. The asymptotic formula (2.4) was established much earlier by de Bruijn [2].

LEMMA 3. *For* $1 \leqslant d \leqslant y$, $\log^{1+\varepsilon} x \leqslant y \leqslant x$, $0 < \varepsilon < 1$ *fixed, we have uniformly*

$$\psi(x/d, y) = \psi(x, y) d^{-\beta}\left\{1+O\left(\frac{\log(d\log y)}{\log y}\right)\right\},$$

where

$$\beta = \beta(x, y) = 1-\frac{\xi(\log x/\log y)}{\log y}.$$

Here $\xi = \xi(v)$ *for* $v > 1$ *is the solution of the equation* $e^{\xi} = 1+\xi v$, *so that asymptotically*

$$\xi(v) = \log v+\log\log v+O(1).$$

LEMMA 4. *For* $1 \leqslant d \leqslant x$, $\log^{1+\varepsilon} x \leqslant y \leqslant x$, $0 < \varepsilon < 1$ *fixed, we have uniformly*

$$\psi(x/d, y) \ll \psi(x, y) d^{-\beta+c/\log y}$$

for some absolute $c > 0$, *where* $\beta = \beta(x, y)$ *is as in Lemma* 3.

LEMMA 5. *For* y *satisfying* (2.2) *we have uniformly*

$$\sum_{n\leqslant x, P(n)\leqslant y} \mu^2(n) = \left(6\pi^{-2}+O\left(\frac{\log\log y}{\log y}\right)\right)\psi(x, y).$$

Lemmas 3–5 are proved in [11] by using results on $\psi(x, y)$ which were obtained in [8]. Actually a more precise result than Lemma 5 is proved in [11], but for our purposes Lemma 5 is more than sufficient.

LEMMA 6. *For*

$$y_0 = \exp\left\{(\tfrac{1}{2}\log x\log\log x)^{1/2}\left(1+O\left(\frac{\log\log\log x}{\log\log x}\right)\right)\right\}$$

we have

$$\sum_{n\leqslant x, P(n)\leqslant y_0} \omega(n) = \psi(x, y_0)\left(\frac{2\log x}{\log\log x}\right)^{1/2}\left(1+O\left(\frac{\log\log\log x}{\log\log x}\right)\right).$$

Proof. Taking $f(n) = \omega(n)$ in (2.1) we obtain

$$\sum_{n\leqslant x, P(n)\leqslant y} \omega(n) = \sum_{p\leqslant y} \psi(x/p, y). \tag{2.5}$$

If y satisfies (2.2), then

$$\sum_{p\leqslant y} \psi(x/p, y) = x\sum_{p\leqslant y}\frac{1}{p}\varrho\left(\frac{\log x-\log p}{\log x}\right)\left(1+O\left(\frac{\log(u+1)}{\log y}\right)\right) \tag{2.6}$$
$$= \psi(x, y)\eta(x, y)\left(1+O\left(\frac{\log(u+1)}{\log y}\right)\right),$$

where

$$\eta(x, y) = \sum_{p\leqslant y}\frac{1}{p}\varrho\left(\frac{\log x-\log p}{\log y}\right)\Big/\varrho\left(\frac{\log x}{\log y}\right).$$

Here we used the notation of K. Alladi [1], who investigated the sums in (2.5). At the time of his writing the asymptotic formula (2.3) was not known to hold in the range (2.2), but his arguments clearly remain valid if y satisfies (2.2). Alladi proved that

$$\eta(x, y) = \operatorname{li}(u\xi(u))-\log\log u+\log\log y+O(1)+O(\xi(u)/\log y),$$

where $u = \log x/\log y$, and $\operatorname{li} x = \int_2^x dt/\log t$ is the logarithmic integral. We have

$$\operatorname{li}(u\xi(u)) = \frac{u\xi(u)}{\log(u\xi(u))}+O\left(\frac{u\xi(u)}{\log^2(u\xi(u))}\right).$$

But $e^{\xi} = 1+u\xi$, whence

$$\log(u\xi(u)) = \log(e^{\xi}-1) = \xi+\log(1-e^{-\xi}) = \xi(u)+O(e^{-\xi(u)}),$$
$$u\xi(u)/\log(u\xi(u)) = u+O(1).$$

Taking $y = y_0$ we find that

$$\log y_0 = (\tfrac{1}{2}\log x\log_2 x)^{1/2}\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right),$$
$$u = \left(\frac{2\log x}{\log_2 x}\right)^{1/2}\left(1+O\left(\frac{\log_3 x}{\log_2 x}\right)\right),$$
$$\xi(u) = \log u+\log\log u+O(1) = \tfrac{1}{2}\log\log x+O(\log\log\log x).$$

Therefore

$$\eta(x, y_0) = \left(\frac{2\log x}{\log\log x}\right)^{1/2}\left(1+O\left(\frac{\log\log\log x}{\log\log x}\right)\right),$$

and the assertion of the lemma follows from (2.5) and (2.6).

The lemma can be also obtained from Theorem 3 of [8], which gives

$$\sum_{p\leqslant y}\psi(x/p, y) = \psi(x, y)\sum_{p\leqslant y} p^{-\alpha}(1+O(1/u)+O(y^{-1}\log y))$$

uniformly for $2 \leqslant y \leqslant x^{1/2}$, where $u = \log x/\log y$, and $\alpha = \alpha(x, y)$ is precisely defined and evaluated in [8]. A standard application of the prime number theorem gives uniformly

$$\sum_{p \leqslant y} p^{-\alpha} = u + O(u/\log u), \quad \log^{1+\varepsilon} x \leqslant y \leqslant \exp\left(\frac{\log x}{(\log\log x)^2}\right),$$

and the lemma follows as before.

3. Proof of Theorem 1. We remark first that in the special case $y = x$ (1.2) reduces to

$$\sum_{n \leqslant x} (\Omega(n) - \omega(n)) = x\left(\sum_p \frac{1}{p^2 - p} + O\left(\frac{\log\log x}{\log x}\right)\right).$$

This is a well-known formula (see p. 30 of [3]) which shows that $\Omega(n) - \omega(n)$ has a mean value. The true order of the O-term above is known to be $O(1/\log x)$, which shows that the error term in (1.2) cannot be in general much improved.

We begin the proof by setting $f(n) = \Omega(n) - \omega(n)$ in (2.1) and noting that $\Omega(p^a) - \omega(p^a) = a - 1$ for $a \geqslant 1$ and any prime p. It follows that

$$\begin{aligned} (3.1) \quad \sum_{n \leqslant x, P(n) \leqslant y} (\Omega(n) - \omega(n)) &= \sum_{p^a \leqslant x, p \leqslant y, a \geqslant 2} \psi(xp^{-a}, y) \\ &= \sum_{p^a \leqslant y, a \geqslant 2} \psi(xp^{-a}, y) + \sum_{y < p^a \leqslant x, p \leqslant y, a \geqslant 2} \psi(xp^{-a}, y) \\ &= S' + S'', \end{aligned}$$

say. Using Lemma 3 we obtain

$$\begin{aligned} (3.2) \quad S' &= \psi(x, y) \sum_{p^a \leqslant y, a \geqslant 2} p^{-a\beta}\left(1 + O\left(\frac{a \log p + \log\log y}{\log y}\right)\right) \\ &= \psi(x, y) \sum_{p^a \leqslant y, a \geqslant 2} p^{-a\beta} + O\left(\psi(x, y)\frac{\log\log y}{\log y}\right). \end{aligned}$$

Further we have

$$\begin{aligned} (3.3) \quad \sum_{p^a \leqslant y, a \geqslant 2} p^{-a\beta} &= \sum_{p^a \leqslant y, a \geqslant 2} p^{-a}\left(1 + O\left(\frac{a\xi(u) \log p}{\log y}\right)\right) \\ &= \sum_{p, a \geqslant 2} p^{-a} + O\left(\frac{\log\log x}{\log y}\right) \\ &= \sum_p \frac{1}{p^2 - p} + O\left(\frac{\log\log x}{\log y}\right). \end{aligned}$$

To estimate S'' we use Lemma 4. Thus we obtain

$$\begin{aligned} (3.4) \quad S'' &\ll \psi(x, y) \sum_{y < p^a \leqslant x, p \leqslant y, a \geqslant 2} p^{-a(\beta - c/\log y)} \\ &= \psi(x, y) \sum_{y < p^a \leqslant x, p \leqslant y, a \geqslant 2} p^{-a(1 + o(1))} \ll \psi(x, y)/\log x, \end{aligned}$$

since $\lim_{x \to \infty} \beta(x, y) = 1$ if $\log y/\log\log x \to \infty$. Theorem 1 follows then from (3.1)–(3.4).

It seems interesting to note that there is another approach to Theorem 1, which can be generalized to sums of the type

$$S(x, y; F) = \sum_{n \leqslant x, P(n) \leqslant y} F(\Omega(n) - \omega(n))$$

for any function F such that the series $\sum_{k \geqslant 0} k2^{-k} F(k)$ converges. This may be obtained, even in a more general setting, from Theorem 1 of [11], in whose notation we have

$$S(x, y; F) = \sum_{k=0}^{x} F(k) \sum_{\substack{n \leqslant x, P(n) \leqslant y \\ \Omega(n) - \omega(n) = k}} 1 = \sum_{k=0}^{x} F(k)\psi_k(x, y; \Omega - \omega).$$

Theorem 1 of [11] gives a sharp approximation of $\psi_k(x, y; \Omega - \omega)$ in the range $\log^{2+\varepsilon} \leqslant y \leqslant x$, $0 < \varepsilon < 1$ fixed, which eventually leads to a suitable expression for $S(x, y; F)$. In the case $F(n) = n$ and $\log y/\log\log x \to \infty$ as $x \to \infty$ this expression will reduce after some calculation to (1.2). If $m > 0$ is fixed and $\log y/\log\log x \to \infty$, then we obtain with some constant $C_m > 0$ that

$$\sum_{n \leqslant x, P(n) \leqslant y} (\Omega(n) - \omega(n))^m = \psi(x, y)\left(C_m + O_m\left(\frac{\log\log x}{\log y}\right)\right).$$

4. Proof of Theorem 2. Let $L = \exp((\log x \log_2 x)^{1/2})$. Using Lemma 2, (1.5) and (1.6) we easily obtain

$$\begin{aligned} (4.1) \quad \sum_{n \leqslant x} \frac{1}{P(n)} &= 1 + \sum_{pm \leqslant x, P(m) \leqslant p} \frac{1}{p} = 1 + \sum_{p \leqslant x} \frac{1}{p}\psi\left(\frac{x}{p}, p\right) \\ &= (1 + O(\log^{-A} x)) \sum_{L^{1/10} \leqslant p \leqslant L^{10}} \frac{1}{p}\psi\left(\frac{x}{p}, p\right) \end{aligned}$$

for any fixed $A > 0$. The crucial step in the proof is to show that the range for p in the last sum can be further restricted. Namely, let

$$\begin{aligned} (4.2) \quad L_1 &= \exp\left\{(\tfrac{1}{2}\log x \log_2 x)^{1/2}\left(1 - 2\frac{\log_3 x}{\log_2 x}\right)\right\}, \\ L_2 &= \exp\left\{(\tfrac{1}{2}\log x \log_2 x)^{1/2}\left(1 + 2\frac{\log_3 x}{\log_2 x}\right)\right\}. \end{aligned}$$

Then we shall show that

$$\sum_{n\leqslant x}\frac{1}{P(n)}=\left(1+O(\log^{-A}x)\right)\sum_{L_1\leqslant p\leqslant L_2}\frac{1}{p}\psi\left(\frac{x}{p},\,p\right) \tag{4.3}$$

for any fixed $A>0$. To accomplish this we shall use Lemma 2 to obtain

$$S_1:=\sum_{L^{1/10}\leqslant p\leqslant L_1}\frac{1}{p}\psi\left(\frac{x}{p},\,p\right)\ll\exp\left(C\log^{1/2}x(\log_2 x)^{-3/2}\log_3 x\right)$$
$$\times\max_{L^{1/10}\leqslant t\leqslant L_1} t^{-1}\exp(-v\log v-v\log\log v+v),$$

where $v=\log(x/t)/\log t$, and $C>0$ denotes generic absolute constants. Setting further $w=\log t$ it is seen that

$$v\log v+v\log\log v-v$$
$$=\frac{\log x}{w}(\log\log x-\log w+\log(\log\log x-\log w)-1)+O(\log\log x).$$

Hence

$$S_1\ll\exp\left(C\log^{1/2}x(\log_2 x)^{-3/2}\log_3 x\right)\max_{(\log L)/10\leqslant w\leqslant\log L_1}\exp(-F(w)),$$

where we have set

$$F(w)=w+\frac{\log x}{w}(\log\log x-\log w+\log(\log\log x-\log w)-1).$$

But we have

$$F'(w)=1-\frac{\log x}{w^2}\left(\log\log x-\log w+\log(\log\log x-\log w)+\frac{1}{\log\log x-\log w}\right).$$

Therefore $F'(w)=0$ for

$$w^2=\log x\left(\log\log x-\log w+\log(\log\log x-\log w)+O(1)\right),$$

whence

$$\log w=\tfrac{1}{2}\log_2 x+\tfrac{1}{2}\log_3 x-\tfrac{1}{2}\log 2+O(\log_3 x/\log_2 x),$$

and then

$$w=w_0=\left\{\tfrac{1}{2}\log x\left(\log_2 x+\log_3 x+O(1)\right)\right\}^{1/2}.$$

Since $F''(w_0)>0$, the function $F(w)$ attains its minimal value for $w=w_0$ in the interval $(\log L)/10\leqslant w\leqslant 10\log L$. For our choice of L_1 and L_2 we have

$$\log L_1<w_0\quad\text{and}\quad\log L_2>w_0.$$

Moreover, for any fixed real B and

$$L(B)=\exp\left\{\left(\tfrac{1}{2}\log x\log_2 x\right)^{1/2}\left(1-B\frac{\log_3 x}{\log_2 x}\right)\right\}$$

we compute that, as $x\to\infty$,

$$F(-\log L(B))=\exp\left\{-(2\log x\log_2 x)^{1/2}\right.$$
$$\left.\times\left(1+\frac{\log_3 x-2-\log 2}{2\log_2 x}+\frac{B^2+B}{2}(1+o(1))\frac{\log_3^2 x}{\log_2^2 x}\right)\right\}.$$

But in L_1 and L_2 we have $B=2$ and $B=-2$, respectively. Hence in view of (1.5) and (1.6) we obtain

$$S_1\ll\exp\left(C\log^{1/2}x(\log_2 x)^{-3/2}\log_3 x\right)F(-\log L_1)$$
$$\ll\sum_{n\leqslant x}\frac{1}{P(n)\log^A x}$$

for any fixed $A>0$. Analogously we obtain

$$S_2:=\sum_{L_2<p\leqslant L^{10}}\frac{1}{p}\psi(x/p,\,p)$$
$$\ll\exp\left(C\log^{1/2}x(\log_2 x)^{-3/2}\log_3 x\right)F(-\log L_2)$$
$$\ll\sum_{n\leqslant x}\frac{1}{P(n)\log^A x}.$$

Formula (4.3) follows then from (4.1) and the estimates for S_1 and S_2.

We proceed now to estimate the sum on the left-hand side of (1.7). The contribution of integers n for which $P^2(n)|n$ and $L_1\leqslant P(n)\leqslant L_2$ is

$$\ll\log x\sum_{\substack{n\leqslant x,\,P^2(n)|n\\ L_1\leqslant P(n)\leqslant L_2}}\frac{1}{P(n)}=\log x\sum_{L_1\leqslant p\leqslant L_2}\frac{1}{p}\psi\left(\frac{x}{p^2},\,p\right) \tag{4.4}$$
$$=x\exp\left(-(2+o(1))(\log x\log_2 x)^{1/2}\right)\ll\sum_{n\leqslant x}\frac{1}{P(n)\log^A x}$$

for any fixed $A>0$. This follows by using Lemma 2, and a more precise formula for sums of $1/P(n)$ when $P^2(n)|n$ has been given in [10]. Using the additivity of $\Omega(n)-\omega(n)$ we obtain

$$\sum_{n\leqslant x}\frac{\Omega(n)-\omega(n)}{P(n)} \tag{4.5}$$
$$=\sum_{L_1\leqslant p\leqslant L_2}\frac{1}{p}\sum_{m\leqslant x/p,\,P(m)<p}(\Omega(m)-\omega(m))+O\left(\sum_{n\leqslant x}\frac{1}{P(n)\log x}\right)$$
$$=\sum+O\left(\sum_{n\leqslant x}\frac{1}{P(n)\log x}\right),$$

where we have set

$$\sum = \sum_{L_1 \leqslant p \leqslant L_2} \frac{1}{p} \sum_{m \leqslant x/p, P(m) \leqslant p} (\Omega(m) - \omega(m)). \tag{4.6}$$

The inner sum in (4.6) may be estimated by Theorem 1 with $y = p$ and x replaced by x/p. The condition $\log y/\log\log(x/p) \to \infty$ is satisfied because $L_1 \leqslant p \leqslant L_2$, and it follows that

$$\begin{aligned} \sum &= \sum_{L_1 \leqslant p \leqslant L_2} \frac{1}{p} \psi\left(\frac{x}{p}, p\right)\left(c + O\left(\frac{\log\log x}{\log p}\right)\right) \\ &= \left\{c + O\left(\left(\frac{\log\log x}{\log x}\right)^{1/2}\right)\right\} \sum_{L_1 \leqslant p \leqslant L_2} \frac{1}{p} \psi\left(\frac{x}{p}, p\right) \\ &= \left\{c + O\left(\left(\frac{\log\log x}{\log x}\right)^{1/2}\right)\right\} \sum_{n \leqslant x} \frac{1}{P(n)}, \end{aligned} \tag{4.7}$$

where we used (4.3) and $c = \sum_p 1/(p^2 - p)$ (it is not difficult to show that the value for c obtained in [6] is the same). Theorem 2 follows from (4.5)–(4.7).

5. Proof of Theorems 3 and 4. Both proofs will use (4.3). To prove (1.8) we remark first that, reasoning as in (4.4), we may suppose that $P^2(n)$ does not divide n. Because of the additivity of $\omega(n)$ and (4.3) we see that the main contribution to the sum on the left-hand side of (1.8) will be

$$\begin{aligned} &\sum_{L_1 \leqslant p \leqslant L_2} \frac{1}{p} \sum_{m \leqslant x/p, P(m) \leqslant p} \omega(m) \\ &= \sum_{L_1 \leqslant p \leqslant L_2} \frac{1}{p} \psi\left(\frac{x}{p}, p\right)\left(\frac{2\log(x/p)}{\log\log(x/p)}\right)^{1/2}\left(1 + O\left(\frac{\log\log\log(x/p)}{\log\log(x/p)}\right)\right) \\ &= \left(\frac{2\log x}{\log\log x}\right)^{1/2}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right) \sum_{L_1 \leqslant p \leqslant L_2} \frac{1}{p} \psi\left(\frac{x}{p}, p\right) \\ &= \left(\frac{2\log x}{\log\log x}\right)^{1/2}\left(1 + O\left(\frac{\log\log\log x}{\log\log x}\right)\right) \sum_{n \leqslant x} \frac{1}{P(n)}. \end{aligned}$$

Here we used Lemma 6, since for $L_1 \leqslant p \leqslant L_2$ we have

$$p = \exp\left\{\left(\tfrac{1}{2}\log\frac{x}{p}\log\log\frac{x}{p}\right)^{1/2}\left(1 + O\left(\frac{\log\log\log(x/p)}{\log\log(x/p)}\right)\right)\right\}.$$

Finally we remark that the sum on the left-hand side of (1.9) is estimated analogously. The main contribution is by Lemma 5 equal to

$$\begin{aligned} &\sum_{L_1 \leqslant p \leqslant L_2} \frac{1}{p} \sum_{m \leqslant x/p, P(m) \leqslant p} \mu^2(m) \\ &= \sum_{L_1 \leqslant p \leqslant L_2} \frac{1}{p} \psi\left(\frac{x}{p}, p\right)\left(6\pi^{-2} + O\left(\left(\frac{\log\log\log(x/p)}{\log\log(x/p)}\right)^{1/2}\right)\right) \\ &= \left\{6\pi^{-2} + O\left(\left(\frac{\log\log\log x}{\log\log x}\right)^{1/2}\right)\right\} \sum_{L_1 \leqslant p \leqslant L_2} \frac{1}{p} \psi\left(\frac{x}{p}, p\right) \\ &= \left\{6\pi^{-2} + O\left(\left(\frac{\log\log\log x}{\log\log x}\right)^{1/2}\right)\right\} \sum_{n \leqslant x} \frac{1}{P(n)}, \end{aligned}$$

where in the last step (4.3) was used.

6. Concluding remarks. It was mentioned in Section 1 that our methods are capable of dealing with certain other sums which involve $\Omega(n)$, $\omega(n)$ and the reciprocals of $P(n)$. This is to be understood in the sense that the sums in question are compared with the sum of reciprocals of $P(n)$, for which a precise expression (furnished by (1.5) and (1.6)) is known. In view of (4.3) the main contribution comes from $P(n)$ lying in a relatively short interval. As shown by Lemma 5, the average order of $\omega(n)$ in that case is $\sim (2\log x/\log_2 x)^{1/2}$. However, the normal order of $\omega(n)$ is then also the same, which was shown by K. Alladi [1]. Alladi proves that, uniformly for $\exp(\log^{2/3} x) \leqslant y \leqslant x$,

$$\sum_{n \leqslant x, P(n) \leqslant y} (\omega(n) - \eta(x, y))^2 \sim \psi(x, y)\left(\eta(x, y) - \frac{u\xi(u)}{\xi(u) - 1}\right) \tag{6.1}$$

as $x \to \infty$, where $u = \log x/\log y$, and η, ξ are as in Section 2. Since (2.3) is now known to hold for (2.2), it is easily seen that (6.1) holds also for (2.2), in particular if $P(n)$ satisfies (1.10), which is the relevant range for our problems involving the reciprocals of $P(n)$. Using (6.1) and the foregoing methods a number of further results may be proved without difficulty. Simple examples are:

$$\sum_{2 \leqslant n \leqslant x} \frac{1}{\omega(n) P(n)} = (1 + o(1))\left(\frac{\log\log x}{2\log x}\right)^{1/2} \sum_{n \leqslant x} \frac{1}{P(n)} \quad (x \to \infty), \tag{6.2}$$

$$\sum_{2 \leqslant n \leqslant x} \frac{\Omega(n)}{\omega(n) P(n)} = (1 + O(\log^{\varepsilon - 1/2} x)) \sum_{n \leqslant x} \frac{1}{P(n)} \quad (0 < \varepsilon < 1/2), \tag{6.3}$$

$$\sum_{2 \leqslant n \leqslant x} \frac{\omega(n)}{\Omega(n) P(n)} = (1 + O(\log^{\varepsilon - 1/2} x)) \sum_{n \leqslant x} \frac{1}{P(n)}. \tag{6.4}$$

For (6.3) the easiest way seems to write

$$\frac{\Omega(n)}{\omega(n)} = \frac{\Omega(n) - \omega(n)}{\omega(n)} + 1.$$

To obtain (6.4) note that trivially $\omega(n)/\Omega(n) \leqslant 1$, which yields an upper bound for the sum in question. To derive a lower bound, note that by the

Cauchy–Schwarz inequality

$$\sum_{n\leqslant x}\frac{1}{P(n)}-1\leqslant\Big(\sum_{2\leqslant n\leqslant x}\frac{\omega(n)}{\Omega(n)\,P(n)}\Big)^{1/2}\Big(\sum_{2\leqslant n\leqslant x}\frac{\Omega(n)}{\omega(n)\,P(n)}\Big)^{1/2}.$$

Thus using (6.3) we obtain (6.4).

It is possible to generalize both Theorem 2 and Theorem 4 by considering sums of the type $\sum_{n\leqslant x} f(n)/P(n)$ and comparing them with the sum of reciprocals of $P(n)$ when $f(n)$ is an s-function. This class of functions was defined in [11] as the class of nonnegative, integer-valued arithmetical functions $f(n)$ such that $f(n)=f(s(n))$ for all $n\geqslant 1$. Here by $s(n)$ we mean the largest squarefull divisor of n (an integer m is squarefull if $p|m$ implies $p^2|m$). Clearly both $\Omega(n)-\omega(n)$ and $\mu^2(n)$ are examples of s-functions. By using Lemma 4 of [11] and the method of proof of Theorem 2 it is possible to obtain that

$$\sum_{n\leqslant x} f(n)/P(n)=\big(C_f+o(1)\big)\sum_{n\leqslant x}\frac{1}{P(n)}$$

as $x\to\infty$, where $C_f\geqslant 0$ is an absolute constant depending on f, provided that the average order of f is not too large.

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