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On S-integral solutions of the Catalan equation

by

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1. Introduction. In 1976 R. Tijdeman [15], employing a refined form of an inequality of A. Baker [1] on linear forms in logarithms, gave an effectively computable bound for the solutions of the Catalan equation. Later, A. J. van der Poorten proved the following p-adic generalization of Tijdeman's result.

Theorem A (A. J. van der Poorten [12]). *Let S be a finite set of distinct positive primes, $S = \{p_1, \ldots, p_s\}$. Then there is an effectively computable constant C_1 depending only on the set S, such that all rational integer solutions $x > 1$, $y > 1$, $u > 1$, $v > 1$, $\omega_1, \ldots, \omega_s$ with $(x, y) = 1$ and $uv > 4$ of the equation*

$$x^u - y^v = (p_1^{\omega_1} \ldots p_s^{\omega_s})^{\{u,v\}}$$

are bounded by C_1.

(We denote by (x, y) the g.c.d. of integers x, y and by $\{u, v\}$ the l.c.m. of integers u, v.)

Let K be an algebraic number field with ring of integers $\mathcal{O}_K$. Further, let $\overline{|\alpha|}$ denote the maximum absolute value of the conjugates of an algebraic number α. Recently, K. Győry, R. Tijdeman and the author have extended Tijdeman's result to the case of algebraic number fields.

Theorem B (B. Brindza, K. Győry, R. Tijdeman [3]). *There exists an effectively computable number C_2 which depends only on K such that all solutions of the equation*

$$x^p - y^q = 1 \quad \text{in} \quad x, y \in \mathcal{O}_K;\ p, q \in N \tag{1}$$

with x, y not roots of unity and $p > 1$, $q > 1$, $pq > 4$ satisfy

$$\max\{\overline{|x|}, \overline{|y|}, p, q\} < C_2.$$

For further results connected with the Catalan equation we refer to Shorey and Tijdeman [14], Ribenboim [13] and Tijdeman [15], [16].

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ $(t \geq 0)$ be distinct prime ideals in K, let $P = \max_i N\mathfrak{p}_i$ (with

$P = 1$ if $t = 0$) and let S denote the set of all additive valuations of K corresponding to $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. Further, let $\mathcal{O}_{K,S}$ denote the ring of S-integers of K. We recall that an element α of K is said to be *S-integral* if $v(\alpha) \geq 0$ for all valuations v of K not contained in S. The purpose of this paper is to prove the following result.

THEOREM. *There exists an effectively computable constant C which depends only on K, P and t such that all solutions of the equation* (1) *in* $x, y \in \mathcal{O}_{K,S}$, $p, q \in N$ *with* $p, q > 1$, $pq > 4$ *and* x, y *not roots of unity satisfy*

$$\max\{H(x), H(y), p, q\} < C. \text{ (}^1\text{)}$$

We note that the proof of this theorem is also based on the Gel'fond–Baker method and we shall use some arguments from Tijdeman's proof [15], and the proofs of Theorem A and Theorem B.

I would like to thank Professor K. Győry and Professor R. Tijdeman for their suggestions and valuable remarks.

2. Auxiliary results. Let $\alpha_1, \ldots, \alpha_k$ $(k > 1)$ be algebraic numbers in K with heights at most $A_1, \ldots, A_k$ respectively, and assume that $A_j \geq 4$, $1 \leq j \leq k$. Put

$$\Omega' = \prod_{i=1}^{k-1} \log A_i, \quad \Omega = \Omega' \log A_k, \quad n = [K:Q].$$

LEMMA 1 (A. Baker [1]). *There exist effectively computable constants* $C_3 > 0$ *and* $C_4 > 0$ *such that the inequalities*

$$0 < |\alpha_1^{b_1} \ldots \alpha_k^{b_k} - 1| < \exp\{-(C_3 kn)^{C_4 k} \Omega \log \Omega' \log B\}$$

have no solutions in rational $b_1, \ldots, b_k$ *with absolute values at most* B (≥ 2).

Denote by $\mathfrak{p}$ a prime ideal of K and suppose that $\mathfrak{p}$ divides the rational prime p.

LEMMA 2 (A. J. van der Poorten [12]). *For some effectively computable number* $C_p^* > 0$ *depending only on* p, k *and* n *the inequalities*

$$0 < |\alpha_1^{b_1} \ldots \alpha_k^{b_k} - 1|_{\mathfrak{p}} < \exp\{-C_p^* \Omega \log \Omega' (\log B)^2\}$$

have no solutions in rational integers $b_1, \ldots, b_k$ *with absolute values at most* B (≥ 4). (See "Added in proff", page 411.)

(Let $v_{\mathfrak{p}}$ denote the additive valuation of K corresponding to $\mathfrak{p}$ and $e_{\mathfrak{p}}$ is the exponent to which $\mathfrak{p}$ divides p, moreover, $f_{\mathfrak{p}}$ is given by $N\mathfrak{p} = p^{f_{\mathfrak{p}}}$. If α is any non-zero element of K then $|\alpha|_{\mathfrak{p}} = (N\mathfrak{p})^{-v_{\mathfrak{p}}(\alpha)/e_{\mathfrak{p}} f_{\mathfrak{p}}}$.

Let $\beta_1, \ldots, \beta_n, \pi_1, \ldots, \pi_s$ $(n \geq 2, s \geq 0)$ be algebraic integers in K with $\beta_i \neq \beta_j$ for $i \neq j$ and suppose that $0 \neq \pi_i$ is not a unit in K, $1 \leq i \leq s$.

(¹) By the *height* $H(\alpha)$ of an algebraic number α we mean the maximum of the absolute values of the relatively prime integer coefficients in its minimal defining polynomial over Z.

Put

$$S_1 = \{\pi_1^{k_1} \ldots \pi_s^{k_s} |\ 0 \leq k_i \in Z,\ i = 1, \ldots, s\},$$
$$f(X, Y) = \prod_{i=1}^{n} (X - \beta_i Y)^{r_i}.$$

Consider the equation

$$f(x, z) = \varepsilon\gamma y^m$$

where $z, \gamma \in S$, $x \in \mathcal{O}_K$, $m \in N$, ε is a unit and $0 \neq y \in \mathcal{O}_K$ is not a unit. Let τ be a positive number.

LEMMA 3 (Shorey and Tijdeman [14], Th. 10.3). *If*

$$\min\{\mathrm{ord}_{\mathfrak{p}} x, \mathrm{ord}_{\mathfrak{p}} z\} \leq \tau$$

for all prime ideals $\mathfrak{p}$ *then all solutions of the equation* $f(x, z) = \varepsilon\gamma y^m$ *in* $x, z, \varepsilon, \gamma, y, m$ *with above mentioned conditions satisfy* $m < C_5$, *where* C_5 *is an effectively computable constant depending on* K, S_1, τ *and the binary form* f.

The follwing lemma is an effective version of a well-known theorem of LeVeque [11].

LEMMA 4 (B. Brindza [2]). *Let*

$$f(x) = a \prod_{i=1}^{n} (X - \alpha_i)^{r_i} \in K[X]$$

be a polynomial with $a \neq 0$ *and* $\alpha_i \neq \alpha_j$ *for* $i \neq j$. *Further, let m be a positive integer and put* $t_i = m/(m, r_i)$ *for* $i = 1, \ldots, n$. *Suppose that* $t_1, \ldots, t_n$ *is not a permutation of the n-tuples* $\{t, 1, \ldots, 1\}$ *and* $\{2, 2, 1, \ldots, 1\}$. *Then all solutions* $x, y \in \mathcal{O}_{K,s}$ *of the equation*

$$f(x) = y^m$$

satisfy

$$\max\{H(x), H(y)\} < \exp\exp\{C_6 P^2 (s+1)^3\}$$

where C_6 *is an effectively computable constant depending only on* K, f *and* m.

LEMMA 5. *There are independent units* $\varepsilon_1, \ldots, \varepsilon_r$ *in* K (*r denotes the unit rank of* K) *and a root of unity* ε_0 *such that*

$$\max_i \overline{|\varepsilon_i|} < C_7, \quad \overline{|\varepsilon_0|} < C_8$$

and that every unit can be written as $\varepsilon = \varepsilon_0^{a_0} \varepsilon_1^{a_1} \ldots \varepsilon_r^{a_r}$ *with* $a_0, \ldots, a_r \in Z$ *where* C_7 *and* C_8 *are effectively computable numbers depending only on* K.

For a proof see [15, Corollaries A.4 and A.5] or [8, Lemma 3].

There are $n = [K:Q]$ isomorphisms $\sigma_1, \ldots, \sigma_n$ of K into the complex numbers; denote the images of an element α of K under these isomorphisms by

$$\sigma_i(\alpha) = \alpha^{(i)} \quad \text{for} \quad i = 1, \ldots, n.$$

LEMMA 6. *Let* $0 \neq \alpha \in K$ *with* $|N_{K/Q}(\alpha)| = M$. *Then there exists a* $\beta \in K$ *associated to* α *such that*

$$\left|\log(M^{-1/n}|\beta^{(i)}|)\right| \leqslant C_9 \quad \text{for} \quad i = 1, \ldots, n$$

where C_9 *is an effectively computable number which depends only on* K.

For a proof see [15, Lemma A.15] or [8, Lemma 3].

LEMMA 7. *There is an integral basis* $\omega_1, \ldots, \omega_n$ *of* K *such that*

$$\max_i \overline{|\omega_i|} < C_{10} |D_K|^{C_{11}}$$

where D_K *is the discriminant of* K *and* C_{10}, C_{11} *are effectively computable constants depending only on* n.

(Cf. K. M. Bartz, *On a theorem of Sokolovski*, Acta Arith. 34 (1978), pp. 113–126.)

The following lemma is a special case of a result of Győry (see Lemma 6 in [10]).

LEMMA 8. *Let* U_s *denote the group of S-units in* K. *If* x_1, x_2 *and* x_3 *are non-zero algebraic integers in* K *satisfying*

$$x_1 + x_2 + x_3 = 0 \quad \text{and} \quad x_1, x_2, x_3 \in U_s \cap \mathcal{O}_K$$

then for some $o \in U_s \cap \mathcal{O}_K$ *and* $\varrho_j \in \mathcal{O}_K$ *we have* $x_j = o\varrho_j$, $(j = 1, 2, 3)$ *and* $\max_j \overline{|\varrho_j|} < C_{12}$ *where* C_{12} *is an effectively computable constant depending only on* K *and* S.

LEMMA 9. *Let* α *be a non-zero algebraic integer of degree* n *which is not a root of unity. There exists an effectively computable positive number* C_{13} *depending only on* n *such that*

$$\overline{|\alpha|} > 1 + C_{13}.$$

Proof. This theorem is due to Schinzel and Zassenhaus who gave an explicit value for C_{13}. See also Cantor and Straus [4] and Dobrowolski [7].

3. Proof of the theorem. Let v be an arbitrary (additive) valuation of K. It is well known that if $v(\alpha) \neq v(\beta)$ for some $\alpha, \beta \in K$ then

$$v(\alpha+\beta) = \min\{v(\alpha), v(\beta)\}.$$

Suppose that $v(x^p) < 0$ or $v(y^q) < 0$. Then from equation (1) we have $v(x^p) = v(y^q)$. It means that the principal ideals $[x^p]$ and $[y^q]$ can be written in the following form

$$[x]^p = \mathfrak{X}/(\mathfrak{p}_1^{\alpha_1}\ldots\mathfrak{p}_s^{\alpha_s})^{(p,q)}, \quad [y]^q = \mathfrak{Y}/(\mathfrak{p}_1^{\alpha_1}\ldots\mathfrak{p}_s^{\alpha_s})^{(p,q)}$$

where $\mathfrak{X}$ and $\mathfrak{Y}$ are integral ideals, $\alpha_1, \ldots, \alpha_s$ are positive integers such that $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{p}_1^{\alpha_1}\ldots\mathfrak{p}_s^{\alpha_s}$ are relatively prime. We may assume that $0 < s$ $(\leqslant t)$. Supposing the contrary we get $x, y \in \mathcal{O}_K$ and we can apply Theorem B.

It is known that there are infinitely many prime ideals in every ideal class of K. Hence we can choose distinct prime ideals $\mathfrak{q}'_1, \ldots, \mathfrak{q}'_s$ such that $\mathfrak{p}_i\mathfrak{q}'_i = [f_i]$ $(i = 1, \ldots, s)$ for some $f_i \in \mathcal{O}_K$ and $\mathfrak{p}_i \| [f_i]$ but $\mathfrak{p}_j \nmid [f_i]$ for $j \neq i$. Let $\omega_1, \ldots, \omega_n$ be an integral basis of K such that

$$\max_i \overline{|\omega_i|} < C_{10}|D_K|^{C_{11}} =: c_1.$$

In the proof $c_1, c_2, \ldots$ denote effectively computable positive constants which depend only on K, P and t. Write $N\mathfrak{p}_i = n_i$ $(1 \leqslant i \leqslant s)$ and

$$f_i = x_{1i}\omega_1 + \ldots + x_{ni}\omega_n \quad (1 \leqslant i \leqslant s)$$

where $x_{ji} \in Z$ for $1 \leqslant j \leqslant n$, $1 \leqslant i \leqslant s$. Let x'_{ji} be defined by

$$x'_{ji} \equiv x_{ji} \,(\mathrm{mod}\, n_1^2 \ldots n_s^2), \quad 0 \leqslant x'_{ji} < n_1^2 \ldots n_s^2$$

for every pair (j, i) and

$$f'_i = x'_{1i}\omega_1 + \ldots + x'_{ni}\omega_n, \quad i = 1, \ldots, s.$$

Then $\prod_{l=1}^{s} \mathfrak{p}_l^2 | [f_i - f'_i]$ and $\mathfrak{p}_i \| [f'_i]$ but $\mathfrak{p}_j \nmid [f'_i]$ for $j \neq i$. We can write $[f'_i] = \mathfrak{p}_i\mathfrak{q}_i$ for some integral ideal $\mathfrak{q}_i$ such that $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ and $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$ are relatively prime ideals and

$$N\mathfrak{q}_i < |N(f'_i)| \leqslant [H(f'_i)]^n \leqslant (2\overline{|f'_i|})^{n^2} =: c_2.$$

Let $h = h_K$ denote the class number of K and $0 \leqslant m_i < h$ such that $m_i \equiv \alpha_i \,(\mathrm{mod}\, h)$, $i = 1, \ldots, s$. Then

$$\mathfrak{p}_1^{\alpha_1}\ldots\mathfrak{p}_s^{\alpha_s}\mathfrak{q}_1^{m_1}\ldots\mathfrak{q}_s^{m_s} = [z]$$

for some $z \in \mathcal{O}_K$ and by Lemma 6 we may assume that $\overline{|z|} < c_3|N(z)|$. Putting

$$x_1 = x \cdot z^{(p,q)p^{-1}} \quad \text{and} \quad y_1 = y \cdot z^{(p,q)q^{-1}}$$

we have

$$x_1^p - y_1^q = z^{(p,q)}, \quad x_1, y_1 \in \mathcal{O}_K \tag{2}$$

and

$$([x_1^p], [z]^{(p,q)}) = ([y_1^q], [z]^{(p,q)}) = (\mathfrak{q}_1^{m_1}\ldots\mathfrak{q}_s^{m_s})^{(p,q)}.$$

Put $\mathfrak{p}_i^h = [\pi_i]$, $i = 1, \ldots, s$ and $(\mathfrak{q}_1^{m_1}\ldots\mathfrak{q}_s^{m_s})^h = [\vartheta]$ with some fixed $\pi_1, \ldots, \pi_s$, ϑ and $q\alpha_i = h\alpha_i^{(1)} + \alpha_i^{(2)}$, $q = ah + b$ with $0 \leqslant \alpha_i^{(2)} < h$, $0 \leqslant b < h$ $(\alpha_i^{(1)}, \alpha_i^{(2)}, a, b \in Z)$. By Lemma 6 we can write

$$z^q\vartheta^{-a} = \pi_1^{\alpha_1}\ldots\pi_s^{\alpha_s}$$

where ε is a unit and

$$\overline{|\varphi|} < c_4|N(\varphi)| < c_4\prod_{i=1}^{s}(N\mathfrak{p}_i\mathfrak{q}_i^{m_i})^h.$$

At first we assume that $p = q$ (> 2). From (2) we obtain

$$x_1 y_1^q = y_1^q (y_1^q + z^q).$$

For brevity let us set $y_2 = y_1^q \vartheta^{-a}$, $y_3 = x_1^h y_1^h \vartheta^{-2}$ and $\gamma = \vartheta^{2b}$. T $y_2, y_3 \in \mathcal{O}_K$ and

(3) $$\gamma y_3^{\bar{q}} = y_2^h (y_2 + z^q \vartheta^{-a})^h.$$

Hence, by (3) we have

$$\gamma \varepsilon^{-2h} y_3^q = (y_2 \varepsilon^{-1})^h (y_2 \varepsilon^{-1} + \pi_1^{\alpha_1^{(1)}} \ldots \pi_s^{\alpha_s^{(1)}} \varphi)^h.$$

Moreover, $y_2 \varepsilon^{-1}$ and $\pi_1^{\alpha_1^{(1)}} \ldots \pi_s^{\alpha_s^{(1)}}$ are relatively prime integers in $\mathcal{O}_K$. I is not a unit then from Lemma 3 we get $q < c_5$. We suppose now $y_3 = (x_1^h \vartheta^{-1})(y_1^h \vartheta^{-1})$ is a unit. It means that $\varepsilon_1 = x_1^h \vartheta^{-1} \in \mathcal{O}_K$ anc $= y_1^h \vartheta^{-1} \in \mathcal{O}_K$ are also units. Therefore $\mathfrak{q}_1^{m_1} \ldots \mathfrak{q}_s^{m_s}$ and $\mathfrak{p}_1^{\alpha_1} \ldots \mathfrak{p}_s^{\alpha_s}$ are princ

$$\mathfrak{q}_1^{m_1} \ldots \mathfrak{q}_s^{m_s} = [Q], \quad \mathfrak{p}_1^{\alpha_1} \ldots \mathfrak{p}_s^{\alpha_s} = [R]$$

with some $Q, R \in \mathcal{O}_K$. From (2) we obtain

$$\varepsilon_3^q - \varepsilon_4^q = \varepsilon R^q$$

where $\varepsilon_3 = x_1 Q^{-1}$, $\varepsilon_4 = y_1 Q^{-1}$ and ε are units. Thus, by Lemma 8 we

$$\overline{|R^q \varepsilon \varepsilon_4^{-q}|} < c_6.$$

We infer

$$2^q \leqslant |N(R^q \varepsilon \varepsilon_4^{-q})| < (2c_6)^{n^2}$$

hence $q < c_7$.

It suffices to prove the theorem in case p and q are primes such $pq > 4$. Indeed, let p_1 and q_1 be the greatest prime factor of p and q. The implies

$$(x^{p/p_1})^{p_1} - (y^{q/q_1})^{q_1} = 1.$$

Hence, if $p_1 q_1 > 4$ then $\max\{p_1, q_1\} < C$ and

$$\max\{H(x^{p/p_1}), H(y^{q/q_1})\} < C.$$

There are $A, B \in \mathbf{Z}$ such that $\max\{|A|, |B|\} < C$ and Ax^{p/p_1}, By^{q/q_1} integers in K. But then $\mathfrak{p}_1^{p/p_1} \| [A]$ and $\mathfrak{p}_1^{q/q_1} \| [B]$. It means $\max\{p/p_1, q/q_1\} < C'$. In the remaining case we have $p_1 = q_1 = 2$ an can write

$$(x^{2^\alpha})^4 - (y^{2^\beta})^2 = 1 \quad \text{or} \quad (x^{2^\beta})^2 - (y^{2^\alpha})^4 = 1$$

for some non-negative integers α, β and by Lemma 4

$$\max\{H(x^{2^\alpha}), H(y^{2^\beta})\} < c_8 \quad \text{or} \quad \max\{H(x^{2^\beta}), H(y^{2^\alpha})\} < c_8.$$

Consequently, as above, $\max\{H(x), H(y), \alpha, \beta\} < c_9$. After this, we may assume without loss of generality that p and q are distinct primes and $pq > 4$.

Now suppose that p is fixed. Put $L = K(\eta)$ with $\eta = e^{2\pi i/p}$. From (2) we get

$$y_1^q = \prod_{i=1}^{p} (x_1 - \eta^i z^q).$$

Using the notation of the case $p = q$ we obtain

$$\vartheta^{bp} y_1^{qh} = \vartheta^{pq} \prod_{i=1}^{p} (x_1 \vartheta^{-a} - \eta^i z^q \vartheta^{-a})^h.$$

Putting $\gamma_1 = \vartheta^{bp}$, $y_2 = y_1^h \vartheta^{-p}$ and $x_2 = x_1 \vartheta^{-a}$ we have $y_2, x_2 \in \mathcal{O}_K$ and

$$\gamma_1 y_2^q = \prod_{i=1}^{p} (x_2 - \eta^i z^q \vartheta^{-a})^h$$

hence

$$\gamma_1 \varepsilon^{-ph} y_2^q = \prod_{i=1}^{p} (x_2 \varepsilon^{-1} - \eta^i \pi_1^{\alpha_1^{(1)}} \ldots \pi_s^{\alpha_s^{(1)}} \varphi)^h$$

where ε is a unit in K and $\overline{|\varphi|} < c_{10}$ again. Further, $x_2 \varepsilon^{-1}$ and $\pi_1^{\alpha_1^{(1)}} \ldots \pi_s^{\alpha_s^{(1)}}$ are relatively prime integers in $\mathcal{O}_K$ (and in $\mathcal{O}_L$ of course). If y_2 is a unit in $\mathcal{O}_L$, respectively $\mathcal{O}_K$ then y_1^h and ϑ^p are associated and $[y_1] = \mathfrak{q}_1^{m_1 p} \ldots \mathfrak{q}_s^{m_s p}$. Since y_1^q and z^{pq} have only fixed prime ideals, by Lemma 5 and Lemma 6 we can write

$$y_1^q = f_1 y_3^{3p} \quad \text{and} \quad z^{pq} = f_2 z_1^{3p}$$

where f_1, f_2, y_3 and z_1 are integers in K such that

$$\max_i \{\overline{|f_i|}\} < c_{11}.$$

Then $\xi_1 = y_3^p z_1^{-p}$ and $\xi_2 = x_1 z_1^{-3}$ are non-zero S'-integers, where S' is the set of all valuations of K corresponding to the prime ideal divisors of the product $\mathfrak{q}_1 \mathfrak{p}_1 \ldots \mathfrak{q}_s \mathfrak{p}_s$. From (2) we obtain

(4) $$\xi_2^p = f_1 \xi_1^3 + f_2.$$

We may apply Lemma 4 to (4) and we get $H(\xi_1) < c_{12}$. We deduce from definition of ξ_1 that

$$H(y_1^q / z^{pq}) < c_{13}.$$

Then there is an $a \in \mathbf{Z}$ such that $|a| < c_{14}$ and ay_1^q/z^{pq} is an integer in $\mathbf{K}$. Since $\mathfrak{p}_1 | [z]$ and $\mathfrak{p}_1 \nmid [y_1]$, we have $\mathfrak{p}_1^{pq} | [a]$ and hence $q < c_{15}$. If y_2 is not a unit in $\mathcal{O}_{\mathbf{L}}$ then, by Lemma 3, $q < c_{16}$.

It is easy to verify that if q is fixed then we can apply similar arguments. The equation (1) can be written

$$(-y)^q - (-x)^p = 1 \qquad (p,\ q > 2).$$

Thus we may assume that p and q are primes such that $p > q > c_{17}$, where c_{17} is large enough.

We have by (2)

$$[y_1]^q = [x_1 - z^q][\alpha(x_1 - z^q) + pz^{pq}]$$

for some $\alpha \in \mathcal{O}_{\mathbf{K}}$. We can write

$$[p] = P_1^{a_1} \ldots P_w^{a_w} \quad \text{and} \quad [q] = Q_1^{b_1} \ldots Q_t^{b_t}$$

where $P_1, \ldots, P_w, Q_1, \ldots, Q_t$ are distinct prime ideals in $\mathbf{K}$; $w, t, a_1, \ldots, a_w, b_1, \ldots, b_t$ are positive integers not exceeding n. If c_{17} is large enough then $(pq, z) = 1$ and if $\mathfrak{p}^k$ is a common divisor of $[x_1 - z^q]$ and $[\alpha(x_1 - z^q) + pz^{pq}]$ for some prime ideal $\mathfrak{p}$ and positive integer k, but $\mathfrak{p}_1 \nmid [z]$ then $\mathfrak{p}^k | [p]$ and $k \leqslant n$. Hence the ideal $[x_1 - z^q]$ can be written in the following form

$$[x_1 - z^q] = \mathfrak{q}_1^{n_1} \ldots \mathfrak{q}_s^{n_s} P_1^{k_1} \ldots P_w^{k_w} \mathfrak{X}_1^q$$

where $\mathfrak{X}_1$ is an integral ideal, $n_1, \ldots, n_s, k_1, \ldots, k_w$ are rational integers such that $|k_i| \leqslant n$ and $0 \leqslant n_j$. Further, we may assume that $n_j < q$ $(j = 1, \ldots, s)$. (The factor $\mathfrak{q}_i^{[n_j/q]}$ can be multiplied to $\mathfrak{X}_1$.) Setting $\mathfrak{q}_i^h = [\vartheta_i]$, $\mathfrak{X}_1^h = [X_1]$, $P_i^h = [u_i]$ for some $X_1, u_1, \ldots, u_w, \vartheta_1, \ldots, \vartheta_s \in \mathcal{O}_{\mathbf{K}}$ we obtain

$$(x_1 - z^q)^h = \varepsilon \vartheta_1^{n_1} \ldots \vartheta_s^{n_s} u_1^{k_1} \ldots u_w^{k_w} X_1^q$$

where ε is a unit in $\mathbf{K}$. By using Lemma 6 we can choose $\vartheta_1, \ldots, \vartheta_s, u_1, \ldots, u_w$ such that

$$\max\{\overline{|\vartheta_1|}, \ldots, \overline{|\vartheta_s|}\} < c_{18} \quad \text{and} \quad \max\{\overline{|u_1|}, \ldots, \overline{|u_w|}\} < c_{19}\, p.$$

By applying Lemma 5 we can write

$$(x_1 - z^q)^h = \varepsilon_0^{m_0} \ldots \varepsilon_r^{m_r} \vartheta_1^{n_1} \ldots \vartheta_s^{n_s} u_1^{k_1} \ldots u_w^{k_w} X^q \tag{5}$$

where $X \in \mathcal{O}_{\mathbf{K}}$, $\max_i |m_i| < q$ and $\varepsilon_0, \ldots, \varepsilon_r$ are units such that $\max\{\overline{|\varepsilon_0|}, \ldots, \overline{|\varepsilon_r|}\}$ is bounded.

By similar arguments we have

$$(y_1 + z^p)^h = \varepsilon_0^{i_0} \ldots \varepsilon_r^{i_r} \vartheta_1^{j_1} \ldots \vartheta_s^{j_s} v_1^{l_1} \ldots v_t^{l_t} Y^p \tag{6}$$

where $Y \in \mathcal{O}_{\mathbf{K}}$, $|l_i| \leqslant n$, $\max\{|i_0|, \ldots, |i_r|, |j_1|, \ldots, |j_s|\} < p$ and $\max_i \overline{|v_i|} < c_{20}\, q$.

Firstly we aim to show that

$$q < \chi_1 (\log p)^{\chi_2} \tag{7}$$

for some effective constants χ_1 and χ_2. We suppose that

$$\Lambda = (x_1 - z^q)^{hp} (y_1 + z^p)^{-hq} - 1 \neq 0$$

(later, we shall deal with case $\Lambda = 0$). By construction of the element z we have $\mathfrak{p}_i^{\alpha_i} | [z]$, $\mathfrak{p}_i^{\alpha_i q} | [z]^p$, $z^{pq} | x_1^{hp} - y_1^{hq}$ and $\mathfrak{p}_i \nmid [y_1 + z^p]$. It means that (using the notation introduced between Lemma 2 and Lemma 3)

$$|\Lambda|_{\mathfrak{p}_i} \leqslant (N\mathfrak{p}_i)^{-\alpha_i q / e_{\mathfrak{p}_i} f_{\mathfrak{p}_i}} \leqslant (N\mathfrak{p}_i)^{-\alpha_i q n^{-1}}. \tag{8}$$

Now, we shall apply Lemma 2 to Λ with $k = r + s + w + t + 2$, $B = 2p^2$, $\alpha_k = XY^{-1}$,

$$\{\alpha_1, \ldots, \alpha_{k-1}\} = \{\varepsilon_0, \ldots, \varepsilon_r, \vartheta_1, \ldots, \vartheta_s, u_1, \ldots, u_w, v_1, \ldots, v_t\}.$$

Since

$$\max\{H(\varepsilon_0), \ldots, H(\varepsilon_r), H(\vartheta_1), \ldots, H(\vartheta_s)\} < c_{21},$$

$$\max\{H(u_1), \ldots, H(u_w), H(v_1), \ldots, H(v_t)\} < p^{c_{22}}$$

and

$$H(XY^{-1}) \leqslant (\overline{|X|} + \overline{|Y|})^n \leqslant M^{2n},$$

where $M = \max\{\overline{|X|}, \overline{|Y|}, 2\}$, we have

$$|\Lambda|_{\mathfrak{p}_i} > \exp\{-c_{23} (\log p)^{c_{24}} \log M\}. \tag{9}$$

Comparing (8) with (9) we obtain

$$(N\mathfrak{p}_i)^{\alpha_i q} < M^{c_{25} (\log p)^{c_{24}}} \qquad (c_{24} > 1). \tag{10}$$

As we want to show that $q < \chi_1 (\log p)^{\chi_2}$ we may assume that $M > c_{26}$ where c_{26} is to be determined later. Taking the product of the inequalities (10) for all $\mathfrak{p}_i$ $(i = 1, \ldots, s)$ we get

$$N^q(\mathfrak{p}_1^{\alpha_1} \ldots \mathfrak{p}_s^{\alpha_s}) < M^{c_{27} (\log p)^{c_{24}}}$$

and

$$\overline{|z|}^q < c_{28}^q M^{c_{27} (\log p)^{c_{24}}}. \tag{11}$$

Supposing $q > 12 c_{27} h (\log p)^{c_{24}}$ and $c_{26} > c_{28}^{12h}$ we have

$$\overline{|z|} < M^{1/6h}. \tag{12}$$

Write

$$U = \varepsilon_0^{m_0} \ldots \varepsilon_r^{m_r} \vartheta_1^{n_1} \ldots \vartheta_s^{n_s} u_1^{k_1} \ldots u_w^{k_w}$$

and

$$V = \varepsilon_0^{i_0} \ldots \varepsilon_r^{i_r} \vartheta_1^{j_1} \ldots \vartheta_s^{j_s} v_1^{l_1} \ldots v_t^{l_t}.$$

Then $\overline{|U|} < c_{29}^q$, $\overline{|V|} < c_{30}^p$ and

$$U^{(j)} = |N(U)| \prod_{i\neq j} |U^{(i)}|^{-1} > c_{31}^{-q}, \quad j=1,\dots,n,$$

and similarly

$$V^{(j)} > c_{32}^{-p}, \quad j=1,\dots,n.$$

Using (5), (6) and (12) we obtain at least one of the following inequalities

$$\overline{|x_1|} > \overline{|X|}^{q/h} c_{31}^{-q} - \overline{|z|}^q > M^{q/h} c_{31}^{-q} - M^{q/6h}$$

or

$$\overline{|y_1|} > \overline{|Y|}^{p/h} c_{32}^{-p} - \overline{|z|}^p < M^{p/h} c_{32}^{-p} - M^{p/6h}$$

hold.

Supposing $M > \max\{c_{31}^{2h}, c_{32}^{2h}\}$ and $q > 6h$ we get

$$\overline{|x_1|} > M^{q/3h}$$

or

$$\overline{|y_1|} > M^{p/3h}.$$

If $\overline{|x_1|} > M^{q/3h}$ $(> \overline{|z|}^{2q})$ then

$$|y_1^{(j)}|^q \geq |x_1^{(j)}|^p - |z^{(j)}|^{pq}, \quad j=1,\dots,n$$

and

$$\overline{|y_1|}^q \geq \overline{|x_1|}^p - \overline{|z|}^{pq} > M^{pq/3h} - M^{pq/6h} > M^{pq/4h} > \overline{|z|}^{3pq/2}.$$

In the other case, when $\overline{|y_1|} > M^{p/3h}$, we have by a similar argument

$$\overline{|x_1|}^p > \overline{|z|}^{3pq/2}.$$

So, we may assume that

(13) $$\min\{\overline{|x_1|}^p, \overline{|y_1|}^q\} > M^{pq/4h} > \overline{|z|}^{3pq/2}.$$

Choose j such that $\overline{|x_1|} = |x_1^{(j)}|$. Then we have

$$\left|\left(\frac{x_1^{(j)} - (z^{(j)})^q}{x_1^{(j)}}\right)^{hp} - 1\right| \leq \binom{hp}{1}\frac{\overline{|z|}^q}{\overline{|x_1|}} + \dots + \binom{hp}{hp}\frac{\overline{|z|}^{hpq}}{\overline{|x_1|}^{hp}}.$$

Since we intend to show that $q < \chi_1 (\log p)^{\chi_2}$ we may assume that $(hp)^2 < M^{q/12h}$.

Then by (13)

$$\overline{|x_1|}/\overline{|z|}^q > M^{q/4h}/M^{q/6h}(hp)^2,$$

hence

$$\binom{hp}{i}\frac{\overline{|z|}^{qi}}{\overline{|x_1|}^i} \geq \binom{hp}{i+1}\frac{\overline{|z|}^{q(i+1)}}{\overline{|x_1|}^{i+1}}, \quad i=1,\dots,hp-1.$$

Therefore from (11) we obtain

(14) $$\left|\left(\frac{x_1^{(j)} - (z^{(j)})^q}{x_1^{(j)}}\right)^{hp} - 1\right| < (hp)^2 \frac{\overline{|z|}^q}{\overline{|x_1|}} < \frac{M^{q/12h} c_{28}^q M^{c_{27}(\log p)^{c_{24}}}}{M^{q/4h}} < \frac{M^{c_{27}(\log p)^{c_{24}}}}{M^{c_{33}q}}.$$

Now, we shall give an upper bound for

$$|(y_1^{(j)})^{hq}(y_1^{(j)} + (z^{(j)})^p)^{-hq} - 1|.$$

By taking c_{17} $(< q)$ large enough we obtain

$$\left|\frac{y_1^{(j)}}{(z^{(j)})^p} + 1\right| > \left|\frac{y_1^{(j)}}{(z^{(j)})^p}\right| - 1 = \left|\frac{(x_1^{(j)})^p}{(z^{(j)})^{pq}} - 1\right|^{1/q} - 1 > \left(\frac{\overline{|x_1|}}{2\overline{|z|}^q}\right)^{p/q} - 1 > \frac{\overline{|x_1|}}{4\overline{|z|}^q} > \frac{M^{q/4h}}{4M^{q/6h}} > (hq)^2.$$

Then

$$\binom{hq}{i}\frac{(4\overline{|z|}^q)^i}{\overline{|x_1|}^i} > \binom{hq}{i+1}\frac{(4\overline{|z|}^q)^{i+1}}{\overline{|x_1|}^{i+1}}, \quad i=1,\dots,hq-1$$

and therefore

(15) $$\left|\left(\frac{y_1^{(j)}}{y_1^{(j)} + (z^{(j)})^p}\right)^{hq} - 1\right| < (2hq)^2 \frac{\overline{|z|}^q}{\overline{|x_1|}} < \frac{M^{c_{27}(\log p)^{c_{24}}}}{M^{c_{34}q}}.$$

Since

$$|(x_1^{(j)})^p (y_1^{(j)})^{-q}| = |1 + (z^{(j)})^{pq}(y_1^{(j)})^{-q}| < 1 + \frac{\overline{|z|}^{pq}}{\overline{|x_1|}^p - \overline{|z|}^{pq}} < c_{35},$$

we have, by (11),

(16) $$\left|\frac{(x_1^{(j)})^{hp}}{(y_1^{(j)})^{hq}} - 1\right| < c_{36}\frac{c_{28}^{pq} M^{c_{27}(\log p)^{c_{24}}}}{M^{c_{37}pq}} < \frac{M^{c_{27}p(\log p)^{c_{24}}}}{M^{c_{38}pq}} < \frac{M^{c_{27}(\log p)^{c_{24}}}}{M^{c_{38}q}}$$

if M is large enough and $c_{27}(\log q)^{c_{24}} < c_{38}q$.

For any complex numbers z_1, z_2, z_3

$$z_1 z_2 z_3 - 1 = \prod_{i=1}^{3}(z_i - 1) + \sum_{1\leq i<j\leq 3}(z_i-1)(z_j-1) + \sum_{i=1}^{3}(z_i-1).$$

Using (14), (15) and (16) we obtain

$$(17)\qquad \left|(x_1^{(j)}-(z^{(j)})^q)^{hp}(y_1^{(j)}+(z^{(j)})^p)^{-hq}-1\right| < \frac{M^{c_{39}(\log p)^{c_{40}}}}{M^{c_{41}q}}.$$

On the other hand, from Lemma 1 we have

$$(18)\qquad |\Lambda^{(j)}| > \exp\left(-c_{42}(\log p)^{c_{43}}\log M\right).$$

Comparing (17) with (18) we see that

$$M^{-c_{42}(\log p)^{c_{43}}} < M^{c_{39}(\log p)^{c_{40}}-c_{41}q}$$

and this yields $q < \chi_1(\log p)^{\chi_2}$.

We assumed that $\Lambda \neq 0$; supposing the contrary we have

$$(x_1-z^q)^{hp} = (y_1+z^p)^{hq}.$$

If $\mathfrak{p}|[x_1-z^q]$ for some prime ideal $\mathfrak{p}$ then, by (2),

$$\mathfrak{p}|[x_1^p-z^{pq}],\quad \mathfrak{p}|[y_1]\quad \text{and}\quad \mathfrak{p}|[y_1+z^p].$$

It means that $\mathfrak{p}|\mathfrak{q}_1\dots\mathfrak{q}_s$. Put

$$\mathfrak{q}_1\dots\mathfrak{q}_s = \mathfrak{R}_1^{\delta_1}\dots\mathfrak{R}_{s_1}^{\delta_{s_1}},$$

where $\mathfrak{R}_1, \dots, \mathfrak{R}_{s_1}$ are distinct prime ideals. Then

$$[x_1-z^q] = (\mathfrak{R}_1^{\tau_1}\dots\mathfrak{R}_{s_1}^{\tau_{s_1}})^q,$$
$$[y_1+z^p] = (\mathfrak{R}_1^{\omega_1}\dots\mathfrak{R}_{s_1}^{\omega_{s_1}})^p$$

where $\tau_1, \dots, \tau_{s_1}, \omega_1, \dots, \omega_{s_1}$ are non-negative integers. We may assume that $(pq, h) = 1$. Therefore $\mathfrak{R}_1^{\tau_1}\dots\mathfrak{R}_{s_1}^{\tau_{s_1}}$ and $\mathfrak{R}_1^{\omega_1}\dots\mathfrak{R}_{s_1}^{\omega_{s_1}}$ are principal ideals and we have as before (cf. (5))

$$(5')\qquad x_1-z^q = \varepsilon_0^{\mu_1}\dots\varepsilon_r^{\mu_r}X_1^q,$$

$$(6')\qquad y_1+z^p = \varepsilon_0^{\nu_1}\dots\varepsilon_r^{\nu_r}Y_1^p$$

for some $X_1, Y_1 \in \mathcal{O}_K$ and $\max_i|\mu_i| < q$, $\max_i|\nu_i| < p$. Now, we show that

$$\Lambda_1 = (x_1-z^q)^p(y_1+z^p)^{-q}-1 \neq 0.$$

Indeed, if

$$(x_1-z^q)^p = (y_1+z^p)^q$$

then

$$(19)\qquad \binom{p}{1}x_1^{p-1}z^q = x_1^p-y_1^q+\beta z^{q+1}+\gamma z^p$$

for some $\beta, \gamma \in \mathcal{O}_K$. From (19) we get

$$(20)\qquad z|x_1^{p-1}p.$$

By taking c_{17} $(<p)$ large enough we have

$$\mathfrak{p}_1 \nmid [p],\quad \mathfrak{p}_1 \nmid [x_1]\quad \text{and}\quad \mathfrak{p}_1^{\alpha_1 q}\|[px_1^{p-1}]$$

which contradicts (20).

Now, we can repeat the arguments with Λ_1 instead of Λ, starting from (5′) and (6′) in place of (5) and (6), respectively, and we again arrive at (7), but with other constants.

In second part of the proof we shall show that

$$(21)\qquad p < \chi_3(\log p)^{\chi_4}$$

where χ_3 and χ_4 are effectively computable constants.

Write

$$(22)\qquad \Lambda_2 = x_1^{ph}(y_1+z^p)^{-qh}-1.$$

We postpone discussion of the case $\Lambda_2 = 0$. Since $\mathfrak{p}_i^{\alpha_i}\|[z]$ and $z^p|x_1^{ph}-(y_1+z^p)^{qh}$ but $\mathfrak{p}_i \nmid [y_1+z^p]$, we obtain

$$(23)\qquad |\Lambda_2|_{\mathfrak{p}_i} \leqslant (N\mathfrak{p}_i)^{-\alpha_i p/e_{\mathfrak{p}_i}f_{\mathfrak{p}_i}} \leqslant (N\mathfrak{p}_i)^{-\alpha_i p/n}.$$

On the other hand Λ_2 can be written in the form (cf. (6))

$$\Lambda_2 = \varepsilon_0^{-i_0 q}\dots\varepsilon_r^{-i_r q}\vartheta_1^{-j_1 q}\dots\vartheta_s^{-j_s q}\upsilon_1^{-l_1 q}\dots\upsilon_t^{-l_t q}(x_1^h Y^{-q})^p-1$$

and we may apply Lemma 2 to give a lower bound for $|\Lambda_2|_{\mathfrak{p}_i}$. Putting

$$M_1 = \max\{\overline{|X|}, \overline{|Y|}, \overline{|z|}\}$$

we have by (5)

$$\overline{|x_1|} < c_{44}^q M_1^q p^{c_{45}},$$

hence

$$H(x_1^h Y^{-q}) \leqslant (\overline{|x_1|}^h+\overline{|Y|}^q)^n < c_{46}^q M_1^{c_{47}q} p^{c_{48}}.$$

Therefore, by (7),

$$(24)\qquad \log H(x_1^h Y^{-q}) < c_{49}(\log p)^{c_{50}}\log M_1.$$

Consequently, by Lemma 2 we obtain

$$(25)\qquad |\Lambda_2|_{\mathfrak{p}_i} > \exp\left(-c_{51}(\log p)^{c_{52}}\log M_1\right).$$

Comparing (23) with (25) we have

$$(26)\qquad 2^p \leqslant (N\mathfrak{p}_i)^{\alpha_i p} < M_1^{c_{53}(\log p)^{c_{52}}}.$$

As we aim to show (21) we may assume that $M_1 > c_{53}$ where c_{53} is to be determined later. Taking the product of inequalities (26) we have

$$\overline{|z|}^p \leqslant (c_3 N(z))^p < c_{54}^p M_1^{c_{55}(\log p)^{c_{52}}}.$$

By assuming $p/(12h) > c_{55}(\log p)^{c_{52}}$ and $c_{54}^{12h} < c_{53}$ we obtain

$$\overline{|z|} < c_{54} M_1^{1/12h} < M_1^{1/6h}$$

and by definition of M_1 we have

$$M_1 = M \quad (= \max\{\overline{|X|}, \overline{|Y|}\})$$

and

$$\overline{|z|} < M^{1/6h} \quad \text{(cf. (12))}.$$

Hence, we may assume that

$$\min\{\overline{|x_1|}^p, \overline{|y_1|}^q\} > M^{pq/4h} > \overline{|z|}^{3pq/2} \quad \text{(cf. (13))}.$$

By $|x_1^{(j)}| = \overline{|x_1|}$ we get

$$|x_1^{(j)}|^p \leqslant |y_1^{(j)}|^q + \overline{|z|}^{pq}, \quad \overline{|y_1|}^q \leqslant \overline{|x_1|}^p + \overline{|z|}^{pq}.$$

Consequently,

$$\frac{|y_1^{(j)}|^q}{\overline{|y_1|}^q} \geqslant \frac{\overline{|x_1|}^p - \overline{|z|}^{pq}}{\overline{|x_1|}^p + \overline{|z|}^{pq}} > 1 - \frac{2}{\overline{|z|}^{pq/2}+1}.$$

By Lemma 9 we have $\overline{|z|} > 1 + c_{56}$ (z is not a root of unity because of $\mathfrak{p}_1 | [z]$). Therefore by taking c_{17} $(< p)$ large enough we obtain $|y_1^{(j)}| > \frac{1}{2}\overline{|y_1|}$ and

$$(27) \qquad |(y_1^{(j)})^{hq}(y_1^{(j)} + (z^{(j)})^p)^{-hq} - 1| \leqslant \sum_{i=1}^{hq} \binom{hq}{i} \frac{\overline{|z|}^{pi}}{|y_1^{(j)} + (z^{(j)})^p|^i},$$

$$\sum_{i=1}^{hq} \binom{hq}{i} \frac{\overline{|z|}^{pi}}{(\frac{1}{2}\overline{|y_1|} - \overline{|z|}^p)^i} < \sum_{i=1}^{hq} \binom{hq}{i} \frac{\overline{|z|}^{pi}}{(\frac{1}{2}\overline{|y_1|} - \overline{|y_1|}^{2/3})^i}$$

$$< \sum_{i=1}^{hq} \binom{hq}{i} (\overline{|z|}^p \overline{|y_1|}^{-2/3})^i < 2^{hq} \frac{\overline{|z|}^p}{\overline{|y_1|}^{2/3}}$$

$$< \frac{2^{hq} c_{54}^p M^{c_{55}(\log p)^{c_{52}}}}{M^{p/6h}}$$

$$< \frac{M^{c_{55}(\log p)^{c_{52}}}}{M^{c_{57} p}}.$$

Further, since $|(x_1^{(j)})^p (y_1^{(j)})^{-q}| < 2$, we have

$$(28) \qquad |(x_1^{(j)})^{ph} (y_1^{(j)})^{-qh} - 1| < c_{58} |(x_1^{(j)})^p - (y_1^{(j)})^q| \, |y_1^{(j)}|^{-q} < 2^q c_{58} \overline{|z|}^{pq} \overline{|y_1|}^{-q}$$

$$< c_{58} 2^q c_{54}^{pq} M^{c_{55} q (\log p)^{c_{52}}} M^{-pq/4h}$$

$$< M^{c_{59}(\log p)^{c_{60}} - c_{61} p}.$$

For any complex numbers z_1, z_2

$$z_1 z_2 - 1 = (z_1 - 1)(z_2 - 1) + (z_1 - 1) + (z_2 - 1).$$

From (27) and (28) we have

$$(29) \qquad |\Lambda_2^{(j)}| < M^{c_{62}(\log p)^{c_{63}} - c_{64} p}$$

Moreover, by Lemma 1, we obtain

$$(30) \qquad |\Lambda_2^{(j)}| > \exp(-c_{65}(\log p)^{c_{66}} \log M).$$

Comparing (29) with (30) we have $p < \chi_3 (\log p)^{\chi_4}$ or $q < p < c_{67}$. Finally, by virtue of Lemma 4, $\max\{H(x), H(y)\} < c_{68}$.

To complete our argument we consider the case $\Lambda_2 = 0$. Then

$$x_1^{ph} = (y_1 + z^p)^{hq}$$

and

$$-\binom{hq}{1} y_1^{hq-1} z^p = y_1^{hq} - x_1^{hp} + E z^{2p} = F z^{2p}$$

for some $E, F \in \mathcal{O}_K$. Since $\mathfrak{p}_1 | [z]$ and $\mathfrak{p}_1 \nmid [y_1]$, we get $\mathfrak{p}_1^p | [hq]$ and

$$2^p \leqslant (N\mathfrak{p}_1)^p \leqslant |N(hq)| < c_{69} p^{c_{70}}$$

which implies $p < c_{71}$.

Added in proof. Recent work of Yu Kunrui (*Linear forms in logarithms in the p-adic case*, to appear in the *Proceedings of the Durham conference on Transcendental Number Theory, 1986*) has thrown doubt on the validity of van der Poorten's proof of Lemma 2. The matters in question, however, could only affect the constant C_p and thus do not affect the results of this paper. In particular, observe that in the present work we may assume that we have the strong independence condition required according to Yu, in order that the inequalities of J. H. Loxton and A. J. van der Poorten (*Multiplicative independence in number fields*, Acta Arith. 42 (1983), pp. 291–302) allow us to appropriately transform the expressions to which we apply Lemma 2.

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