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On Waring's problem for squares

by

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Given an integral basis

$$A_k = \{a_1, a_2, \ldots, a_k\}, \quad 1 = a_1 < a_2 < \ldots < a_k.$$

For a positive integer h, we form all the combinations

$$\sum_{i=1}^{k} x_i a_i; \quad x_i \geqslant 0, \quad \sum_{i=1}^{k} x_i \leqslant h,$$

and ask for the smallest integer $N_h(A_k)$ which is not represented by such a combination. The number $n_h(A_k) = N_h(A_k)-1$ is called the *h-range of* A_k. For more details, see for instance Selmer [3].

A popular interpretation arises if we consider the integers a_i as *stamp* denominations, and h as the "size of the envelope".

A basis A_k is called *admissible* for a given h if there are no gaps in the representations below the largest basis element a_k. Thus A_k is admissible if and only if $h \geqslant h_0$, where

$$h_0 = h_0^{(k)} = \min\{h \in N \mid n_h(A_k) \geqslant a_k\}.$$

In our institute report [4], we tabulate extensive numerical information on the h-ranges $n_h(A_k)$ when A_k consists of the first k squares, cubes or triangular numbers. We give below our main theoretical result for squares, hence

$$A_k = \{1^2, 2^2, \ldots, k^2\}.$$

It then follows from Waring's theorem that $h_0^{(k)} = 4$ for all $k \geqslant 3$ (and trivially $h_0^{(2)} = 3$).

We shall determine $n_4(A_k)$ for all k. With $k \leqslant 100$, the values are listed in Table 1. We notice the striking fact that there are intervals for k, of increasing length, with constant $n_4(A_k)$.

For k sufficiently large, Table 1 indicates that the constancy intervals for

Table 1. The h-ranges $n_4(1^2, 2^2, \ldots, k^2)$ for $k \leqslant 100$

k	n_4	k	n_4	k	n_4	k	n_4
3	23	9	175	17	700	40–47	3583
4	38	10–11	223	18–19	703	48–55	6015
5	52	12	334	20–23	895	56–63	6143
6	82	13	375	24–27	1503	64–79	11263
7	95	14–15	383	28–31	1535	80–95	14335
8	154	16	686	32–39	2815	96–	24063

$n_4(A_k)$ are divided into four cases. Let

$$k = 2^s + t_1 \cdot 2^{s-1} + t_2 \cdot 2^{s-2} + \ldots \geqslant 4 \quad (s \geqslant 2)$$

be the binary representation of k. The cases are:

$$(1)\quad \begin{cases} 1.\ t_1 = 0, & t_2 = 0; \quad 2^s \leqslant k < 2^s + 2^{s-2} \\ 2.\ t_1 = 0, & t_2 = 1; \quad 2^s + 2^{s-2} \leqslant k < 2^s + 2^{s-1} \\ 3.\ t_1 = 1, & t_2 = 0; \quad 2^s + 2^{s-1} \leqslant k < 2^s + 2^{s-1} + 2^{s-2} \\ 4.\ t_1 = 1, & t_2 = 1; \quad 2^s + 2^{s-1} + 2^{s-2} \leqslant k < 2^{s+1}. \end{cases}$$

A closer study of (an extended) Table 1 led to the following

THEOREM. *Let*

$$k \geqslant 7, \quad k \neq 8, 12, 16, 17.$$

Then

$$(2)\quad n_4(1^2, 2^2, \ldots, k^2) = N_i \cdot 2^{2s-3} - 1,$$

where

$$N_1 = 22, \quad N_2 = 28, \quad N_3 = 47, \quad N_4 = 48$$

in cases 1–4 *respectively*.

Let x be a natural number, with a representation by four integer squares:

$$(3)\quad x = x_1^2 + x_2^2 + x_3^2 + x_4^2, \quad x_1 \geqslant x_2 \geqslant x_3 \geqslant x_4 \geqslant 0.$$

LEMMA. *Let* $s \geqslant 2$, *then*

$$x \equiv 0 \pmod{2^{2s-3}} \Rightarrow \text{all } x_j \equiv 0 \pmod{2^{s-2}}.$$

For if $2^\delta \| x_j$ (exactly divides) with $\delta \leqslant s-3$, then

$$(4)\quad \begin{aligned} x - x_j^2 &= t \cdot 2^{2s-3} - 2^{2\delta} \times \text{odd square} \\ &= 2^{2\delta}(\underbrace{t \cdot 2^{2s-3-2\delta}}_{\equiv 0 \pmod 8} - \underbrace{\text{odd square}}_{\equiv 1 \pmod 8}) \\ &= 4^\delta(8m+7). \end{aligned}$$

But by the famous theorem of Legendre and Gauss, a natural number has a representation by three squares if and only if it is not of the form (4).

Let k_i, $i = 1, 2, 3, 4$, denote the smallest k in each interval (1).

COROLLARY.

$$n_4(A_{k_i}) + 1 \equiv 0 \pmod{2^{2s-3}} \Rightarrow n_4(A_k) = n_4(A_{k_i}), \quad k_i < k < k_i + 2^{s-2}.$$

Since $2^{s-2} | k_i$, it follows from the lemma that no summand k^2 is possible in $n_4(A_{k_i}) + 1$. This explains the *constancy intervals* for $n_4(A_k)$, and shows that it suffices to prove the theorem for $k = k_i$, $i = 1, 2, 3, 4$.

For this purpose, we first show that $n_4(A_{k_i})$ is at least bounded by the expressions (2). It suffices to establish that $N_i \cdot 2^{2s-3}$ has no 4-representation by A_{k_i} in the cases 1–4. From the lemma, it follows that we only need to exclude representations where all summands x_j^2 have $2^{s-2} | x_j$.

In case 1, with $k_1 = 2^s$, we can only use $x_j = 2^{s-2}$, 2^{s-1}, $3 \cdot 2^{s-2}$ or 2^s. We must use 2^s at least once, since

$$4 \cdot (3 \cdot 2^{s-2})^2 = 18 \cdot 2^{2s-3} < N_1 \cdot 2^{2s-3} = 22 \cdot 2^{2s-3}.$$

But $22 \cdot 2^{2s-3} - (2^s)^2 = 4^{s-1} \cdot 7$, which by (4) has no 3-representation.

The cases 2–4 are treated similarly (with a few more possibilities to consider).

We must finally show that any natural number $x < N_i \cdot 2^{2s-3}$ really has a 4-representation by A_{k_i}. This is trivial for $x \leqslant k_i^2$, so we may assume $x > k_i^2$.

If $x = 4x'$, we can "double" the representation of x' from the same case with s reduced by 1. Using induction on s, we may thus assume that $4 \nmid x$. In what follows, this condition is very important.

For the representation (3), we consider the sum

$$\sigma_x = x_1 + x_2 + x_3 + x_4.$$

For given x, σ_x will generally increase if the "dispersion" of the summands decreases. The theoretical maximum of σ_x occurs when all the summands are equal, with $\sigma_x = 2\sqrt{x}$.

On the other hand, we clearly have $\sigma_x \equiv x \pmod 2$. Let

$$\bar{\sigma}_x = \max\{\sigma_x \in N | \ \sigma_x \leqslant 2\sqrt{x}, \ \sigma_x \equiv x \pmod 2\}.$$

It was shown by Cauchy (cf. Dickson [1], Vol. 2, p. 284) that there always exists a representation (3) with $\sigma_x = \bar{\sigma}_x$. (Cauchy's condition $x - \sigma_x^2/4 \neq 4^\delta(8m+7)$ for even x is automatically satisfied when $4 \nmid x$.) A proof is found in [2], Vol. 2, Ch. 6, § 1.

We now show that for sufficiently large k_i, any representation (3) with

$x_1 > k_i$ will have $\sigma_x < \bar{\sigma}_x$. By Cauchy's result, there must then exist a representation with $x_1 \leqslant k_i$, hence by A_{k_i}.

For this purpose, we write

$$x = (1+3\Delta_i^2)k_i^2 < N_i \cdot 2^{2s-3},$$

and so $\Delta_i < M_i < 1$, where

$$(5) \qquad M_1 = \tfrac{1}{2}\sqrt{\tfrac{7}{3}}, \quad M_2 = \tfrac{1}{5}\sqrt{\tfrac{31}{3}}, \quad M_3 = \tfrac{1}{3}\sqrt{\tfrac{29}{6}}, \quad M_4 = \tfrac{1}{7}\sqrt{\tfrac{47}{3}}.$$

Choose $x_1 > k_i$. By the "principle of dispersion", we then get a smaller σ_x than by choosing $x_1 = k_i$, $x_2 = x_3 = x_4 = \Delta_i k_i$, hence

$$\sigma_x < (1+3\Delta_i)k_i.$$

More concisely, this may be proved as follows: Write k for k_i and Δ for Δ_i, and put

$$x_1 = k+t_1 \quad (t_1 > 0); \qquad x_j = \Delta k + t_j, \quad j = 2, 3, 4.$$

We must show that $\sum_1^4 t_j < 0$. Now $x = (1+3\Delta^2)k^2 = \sum_1^4 x_j^2$ can be written as

$$2kt_1 + 2\Delta k(t_2+t_3+t_4) = -\sum_1^4 t_j^2 < 0,$$

hence from $t_1 > 0$ and $\Delta < 1$:

$$\Delta \cdot \sum_1^4 t_j < t_1 + \Delta(t_2+t_3+t_4) < 0.$$

On the other hand, we know that

$$\bar{\sigma}_x \leqslant 2\sqrt{x} = 2\sqrt{1+3\Delta_i^2}\,k_i.$$

We can thus find a $\bar{\sigma}_x > \sigma_x$, of appropriate parity, if there is "room" for two consecutive integers in the (real) interval

$$[(1+3\Delta_i)k_i,\ 2\sqrt{1+3\Delta_i^2}\,k_i].$$

This is always possible if the interval length is at least 2, hence

$$k_i \geqslant \frac{2}{2\sqrt{1+3\Delta_i^2}-(1+3\Delta_i)}.$$

The lower bound is an increasing function of Δ_i for $0 \leqslant \Delta_i < 1$. We thus get a k_i which is large enough for all the cases 1–4 if we replace Δ_i by the largest bound M_1 in (5):

$$k_1 \geqslant \frac{2}{2\sqrt{1+3M_1^2}-(1+3M_1)} \approx 79, \quad \text{hence} \quad k_i \geqslant 80.$$

Together with Table 1, this completes the proof of the theorem.

Our result clearly gives new information on Waring's problem for squares. It may safely be said that the theorem would not have been found without substantial numerical evidence.

Considered as a result in the theory of h-ranges, it is one (and probably the simplest) of the very few explicitly determined non-trivial h-ranges for arbitrarily large bases.

References

[1] L. E. Dickson, *History of the theory of numbers*, Stechert, New York 1934.

[2] A. M. Legendre, *Théorie des nombres*, 3 ed., Paris 1830. (German translation by H. Maser, 1886.)

[3] E. S. Selmer, *On the postage stamp problem with three stamp denominations*, Math. Scand. 47 (1980), pp. 29–71.

[4] — *Some h-bases for n related to Waring's problem*, Inst. report No. 28 (1983), Department of Pure Math., Univ. of Bergen, Norway, pp. 1–18.

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