On unit solutions of the equation \(xyz = x + y + z\) in the ring of integers of a quadratic field

by

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1. Introduction. This work was inspired by a study of the equation \(xyz = x + y + z = 1\) which is known to have no solutions in the rational number field \(\mathbb{Q}\) (see [1], [2] and [3]). In [4] this equation is studied over finite fields, and a precise count is given therein of the number of solutions in the finite fields. It is natural to ask the more general question: What are the solutions of \(xyz = x + y + z = w\) where \(w\) is a unit in the ring of integers of a number field? Equivalently, what are the solutions of \(xyz = x + y + z\) where \(x, y, z\) are units in the ring of integers of a number field? It is the purpose of this paper to completely solve this problem in the quadratic number field case.

2. Results. In what follows \(U_K\) denotes the units of the ring of integers of \(K = Q(\sqrt{d})\), where \(d\) is a square-free rational integer.

Theorem. There exist solutions to:

\[
\begin{align*}
\{w_1, w_2, w_3\} &= \{\pm 1\} \\
\end{align*}
\]

where \(w_i \in U_K\) for \(i = 1, 2, 3\) if and only if \(d = -1, 2\) or 5.

A complete classification of the solutions for each \(d\) is given in Table 4 following the proof of the theorem.

Proof. First we consider the case \(d < 0\). If \(d \neq -1\) or \(-3\) then \(U_K = \{\pm 1\}\) and the equation (*) is clearly not solvable. If \(d = -3\) then we claim there are no solutions. Let \(w\) denote a primitive 6th root of unity. Then \(w^i = w^j\) where \(0 \leq i \leq 5\). If any two of the \(i\)'s are equal, say \(i = j\) without loss of generality, then \(w^{2i+6} = w^{2j+6}\) implies \(w^{i+1} = w^{j+1}\), whence \(w^{i+1} = w^{j+1}\) and so \(w^{i+1}(w^1 - w^{-1}) = 2\). However for \(0 \leq i \leq 5\) we get \(w^{i+1} = w^{-1}\) or \(\pm \sqrt{-3}\), which yields a contradiction in

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any case. Therefore all of the $l_i$’s are distinct. If any $l_i$ is 3, say $l_1$ without loss of generality, then $-1 + w^3 + w^3 = -w^{1+2+3}$ whence

$$w^3 = (1 - w^2)/(1 + w^2).$$

Therefore $(1 - w^2)^2 = 1 - (1 + w^2)^2$ and it is straightforward to check that this leads to a contradiction. By a similar argument no $l_i$ can be 0. Only four cases remain for the $l_i$. They are dismissed in the following chart where $0 < l_1 < l_2 < l_3 \leq 5$.

<table>
<thead>
<tr>
<th>$l_1$</th>
<th>$l_2$</th>
<th>$l_3$</th>
<th>$w^{l_1 + l_2 + l_3}$</th>
<th>$w^{l_1 + l_2 + l_3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>4</td>
<td>$w$</td>
<td>$w^2$</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>5</td>
<td>$w^2$</td>
<td>$w$</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>5</td>
<td>$w^3$</td>
<td>$w^3$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>5</td>
<td>$w^5$</td>
<td>$w^6$</td>
</tr>
</tbody>
</table>

The remaining case for $d < 0$ is $d = -1$. Here $U_K = \{ \pm 1, \pm i \}$ where $i^2 = -1$. Let $i^{1+2+3} = i^1 + i^2 + i^3$. Using similar arguments to the above it can be shown that any two of the $l_i$’s are equal if and only if all the $l_i$’s are odd and this case yields solutions of (1) which are permutations of $(i, i, -i)$. The remaining cases where the $l_i$’s are distinct yields solutions of (1) which are permutations of $(1, i, -i)$.

Now we may restrict our attention to $d > 0$. Let $E = (a_1 + b_1 \sqrt{d})$ be the fundamental unit of $K$, and set $E^l = (a_1 + b_1 \sqrt{d})^l = a_l + b_l \sqrt{d}$ for any integer $l$ (with the convention that $a_0 = 1$ and $b_0 = 0$). Since $U_K = \{ \pm E^l : l \in \mathbb{Z} \}$ then we may assume without loss of generality that $u_1 u_2 u_3 = E^{l_1 + l_2 + l_3}$ (since we may multiply by $-1$ otherwise). Therefore only two possibilities occur, namely either:

$$E^{l_1 + l_2 + l_3} = E^{l_1} + E^{l_2} + E^{l_3}$$

or

$$E^{l_1 + l_2 + l_3} = E^{l_1} - E^{l_2} - E^{l_3}$$

(up to order). For convenience sake set $\delta = \pm 1$ and set

$$u_0 = u_1 u_2 u_3 = E^{l_1 + l_2 + l_3} = E^{l_1} + \delta E^{l_2} + \delta E^{l_3} = u_1 + \delta u_2 + \delta u_3$$

where $\delta u_i = u_i$ for $i = 2, 3$. Hence:

$$a_1 + b_1 + c_1 = a_1 a_2 a_3 + a_3 b_1 b_3 d + a_1 b_1 b_3 d + a_1 b_1 b_3 d$$

and

$$b_1 + b_2 + b_3 = a_1 a_2 a_3 + a_3 b_1 b_3 d + a_1 b_1 b_3 d + a_1 b_1 b_3 d.$$
Table 3

<table>
<thead>
<tr>
<th>((N(u_1), N(u_2), N(u_3), \delta))</th>
<th>result</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 1, 1, -1))</td>
<td>(u_2 u_3 = 1)</td>
</tr>
<tr>
<td>((1, -1, -1, 1))</td>
<td>(u_2 u_3 = 1)</td>
</tr>
<tr>
<td>((-1, 1, -1, 1))</td>
<td>(b_2 b_3 = 0)</td>
</tr>
<tr>
<td>((-1, -1, 1, 1))</td>
<td>(b_2 b_3 = 0)</td>
</tr>
<tr>
<td>((-1, -1, -1, -1))</td>
<td>(b_2 b_3 = 0)</td>
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<tr>
<td>((-1, -1, -1, 1))</td>
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<tr>
<td>((-1, 1, -1, 1))</td>
<td>(b_2 b_3 = 0)</td>
</tr>
<tr>
<td>((-1, 1, -1, -1))</td>
<td>(b_2 b_3 = 0)</td>
</tr>
</tbody>
</table>

First we consider \(b_2 b_3 = 0\). We may assume without loss of generality that \(b_3 = 0\). Therefore \(a_3 = 1\) and we are left with

\(u_1 = \delta u_2^2 + \delta = u_1 u_2^2\).

Therefore

\(u_2^2 = (u_1 + \delta)(u_1 - \delta) = (a_1 + b_1 \sqrt{d + \delta})(a_1 + b_1 \sqrt{d - \delta}).\)

Multiplying numerator and denominator by \((a_1 - b_1 \sqrt{d - \delta})\) we get:

\(u_2^2 = (N(u_1) - 2 \delta b_1 \sqrt{d - \delta})/N(u_2 - 2 \delta a_1 + 1).\)

However, from Table 3 we see that \(N(u_1) = -1\) so:

\[(1.6) \quad u_2^2 + \delta u_2 = (1 + \delta b_1 \sqrt{d - \delta})/a_1.\]

Since \(2a_1 \in \mathbb{Z}\) then \(a_1 \in \{\pm 1, \pm 2, \pm 1/2\}\). We now analyze (1.6) for the various values of \(a_1\).

If \(a_1 = \pm 1\) then \(d = 2\) and \(b_1 = \pm 1\). If \(a_1 = 1\) and \(\delta = 1\) then

\(u_1 = 1 + \sqrt{2}, \quad \delta u_2 = 1 + \sqrt{2} \quad \text{and} \quad \delta u_3 = 1.\)

If \(a_1 = -1\) and \(\delta = 1\) then

\(u_1 = -1 + \sqrt{2}, \quad \delta u_2 = -1 + \sqrt{2} \quad \text{and} \quad \delta u_3 = 1.\)

If \(a_1 = 1\) and \(\delta = -1\) then \(u_1 = 1 - \sqrt{2}, \quad \delta u_2 = 1 - \sqrt{2} \quad \text{and} \quad \delta u_3 = -1.\)

If \(a_1 = -1\) and \(\delta = -1\) then

\(u_1 = -1 - \sqrt{2}, \quad \delta u_2 = -1 - \sqrt{2} \quad \text{and} \quad \delta u_3 = -1.\)

All solutions for the case \(a_1 = \pm 1\) are permutations of \(\pm (1 + \sqrt{2}, 1 + \sqrt{2}, 1), \quad \pm (1 - \sqrt{2}, 1 - \sqrt{2}, 1), \quad \pm (1 + \sqrt{2}, 1 + \sqrt{2}, 1), \quad \pm (1 - \sqrt{2}, 1 - \sqrt{2}, 1).\)

If \(a_1 = \pm 1/2\) then \(b_1 = \pm 1/2\) and \(d = 5\). If \(a_1 = 1/2\) and \(\delta = 1\) then

\(u_1 = (1 + \sqrt{5})/2, \quad \delta u_2 = 2 + \sqrt{5} \quad \text{and} \quad \delta u_3 = 1.\)

If \(a_1 = -1/2\) and \(\delta = 1\) then

\(u_1 = -(1 + \sqrt{5})/2, \quad \delta u_2 = -2 + \sqrt{5} \quad \text{and} \quad \delta u_3 = 1.\)

If \(a_1 = 1/2\) and \(\delta = -1\) then

\(u_1 = (1 + \sqrt{5})/2, \quad \delta u_2 = 2 - \sqrt{5} \quad \text{and} \quad \delta u_3 = -1.\)

If \(a_1 = -1/2\) and \(\delta = -1\) then

\(u_1 = -(1 + \sqrt{5})/2, \quad \delta u_2 = -2 - \sqrt{5} \quad \text{and} \quad \delta u_3 = -1.\)

Hence all solutions for the case \(a_1 = \pm 1/2\) are permutations of \(\pm (1 + \sqrt{5}/2, 2 + \sqrt{5}, 1), \quad \pm (1 - \sqrt{5}/2, 2 - \sqrt{5}, 1), \quad \pm (1 + \sqrt{5}/2, 2 + \sqrt{5}, -1) \quad \text{and} \quad \pm (1 - \sqrt{5}/2, 2 - \sqrt{5}, -1).\)

If \(a_1 = \pm 2\) then the roles of \(u_1\) and \(u_2\) are reversed in the previous case and no new solutions are found.

Finally one may check that an analysis of \(a_1 a_2 a_3 = 1\) yields exactly the same solutions as in the above cases. This completes the proof of the theorem, and the results are summarized in the following table:

Table 4. Classification of all solutions to \(a_1 a_2 a_3 = 1 + u_1 + u_2 + u_3\) for \(u_i \in \mathbb{O}_2\) for any quadratic field \(K = \mathbb{Q}(\sqrt{d})\)

<table>
<thead>
<tr>
<th>(d)</th>
<th>Solutions are permutations of:</th>
</tr>
</thead>
<tbody>
<tr>
<td>(-1)</td>
<td>(\pm (i, i, -i)) and (\pm (1, i, -1))</td>
</tr>
<tr>
<td>(2)</td>
<td>(\pm (i + \sqrt{2}, 1 + \sqrt{2}, 1); \pm (i + \sqrt{2}, 1 - \sqrt{2}, 1))</td>
</tr>
<tr>
<td>(5)</td>
<td>(\pm (2 + \sqrt{5}, 2 + \sqrt{5}, 1); \pm (2 + \sqrt{5}, 2 - \sqrt{5}, 1))</td>
</tr>
<tr>
<td></td>
<td>(\pm (1 + \sqrt{5}/2, 2 + \sqrt{5}, -1))</td>
</tr>
<tr>
<td></td>
<td>(\pm (1 + \sqrt{5}/2, 2 + \sqrt{5}, 1))</td>
</tr>
</tbody>
</table>

No solutions exist for \(d \neq -1, 2, 5\).

It remains open to classify all solutions of (1.6) for the ring of integers of an arbitrary number field \(K\). For example, in view of the Kronecker–Weber Theorem, to answer the question for abelian extensions of \(Q\) it would be of value to know the solutions of (1.6) in \(\mathbb{Z}[\xi]\) where \(\xi\) is a primitive root of unity. In this paper we have solved the case where \(\xi\) is a primitive third or fourth root of unity since these are the only roots of unity which generate quadratic fields.

References


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