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On unit solutions of the equation $xyz = x+y+z$ in the ring of integers of a quadratic field

by

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1. Introduction. This work was inspired by a study of the equation $xyz = x+y+z = 1$ which is known to have no solutions in the rational number field Q (see [1], [2] and [3]). In [4] this equation is studied over finite fields, and a precise count is given therein of the number of solutions in the finite fields. It is natural to ask the more general question: What are the solutions of $xyz = x+y+z = u$ where u is a unit in the ring of integers of a number field? Equivalently; what are the solutions of $xyz = x+y+z$ where x, y, z are units in the ring of integers of a number field? It is the purpose of this paper to completely solve this problem in the quadratic number field case.

2. Results. In what follows U_K denotes the units of the ring of integers of $K = Q(\sqrt{d})$, where d is a square-free rational integer.

Theorem. *There exist solutions to:*

$$(*) \qquad u_1 u_2 u_3 = u_1 + u_2 + u_3$$

where $u_i \in U_K$ for $i = 1, 2, 3$ if and only if $d = -1, 2$ or 5.

A complete classification of the solutions for each d is given in Table 4 following the proof of the theorem.

Proof. First we consider the case $d < 0$. If $d \neq -1$ or -3 then $U_K = \{\pm 1\}$ and the equation $(*)$ is clearly not solvable. If $d = -3$ then we claim there are no solutions. Let w denote a primitive 6th root of unity. Then $u_i = w^{l_i}$ where $0 \leqslant l_i \leqslant 5$. If any two of the l_i's are equal, say $l_1 = l_2$ without loss of generality, then $w^{2l_1+l_3} = 2w^{l_1} + w^{l_3}$ implies $w^{l_1+l_3} = 2 + w^{l_3-l_1}$ whence $w^{l_1+l_3} - w^{l_3-l_1} = 2$, and so $w^{l_3}(w^{l_1} - w^{-l_1}) = 2$. However for $0 \leqslant l_1 \leqslant 5$ we get $w^{l_1} - w^{-l_1} = 0$ or $\pm\sqrt{-3}$, which yields a contradiction in

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any case. Therefore all of the l_i's are distinct. If any l_i is 3, say l_1 without loss of generality, then $-1+w^{l_2}+w^{l_3} = -w^{l_2+l_3}$, whence

$$w^{l_3} = (1-w^{l_2})/(1+w^{l_2}).$$

Therefore $(1-w^{l_2})^3 = \pm(1+w^{l_2})^3$ and it is straightforward to check that this leads to a contradiction. By a similar argument no l_i can be 0. Only four cases remain for the l_i. They are dismissed in the following chart where $0 < l_1 < l_2 < l_3 \leqslant 5$.

Table 1

l_1	l_2	l_3	$w^{l_1+l_2+l_3}$	$w^{l_1}+w^{l_2}+w^{l_3}$
1	2	4	w	w^2
1	2	5	w^2	w
1	4	5	w^4	w^5
2	4	5	w^5	w^4

The remaining case for $d<0$ is $d=-1$. Here $U_K = \{\pm 1, \pm i\}$ where $i^2=-1$. Let $i^{l_1+l_2+l_3} = i^{l_1}+i^{l_2}+i^{l_3}$. Using similar arguments to the above it can be shown that any two of the l_j's are equal if and only if all the l_j's are odd and this case yields solutions of (*) which are permutations of $\pm(i, i, -i)$. The remaining cases where the l_j's are distinct yields solutions of (*) which are permutations of $\pm(1, i, -i)$.

Now we may restrict our attention to $d>0$. Let $E = (a_1+b_1\sqrt{d})$ be the fundamental unit of K, and set $E^l = (a_1+b_1\sqrt{d})^l = a_l + b_l\sqrt{d}$ for any integer l (with the convention that $a_0 = 1$ and $b_0 = 0$). Since $U_K = \{\pm E^l\colon\ l \in Z\}$ then we may assume without loss of generality that $u_1 u_2 u_3 = E^{l_1+l_2+l_3}$ (since we may multiply by -1 otherwise). Therefore only two possibilities occur, namely either:

$$E^{l_1+l_2+l_3} = E^{l_1}+E^{l_2}+E^{l_3}$$

or

$$E^{l_1+l_2+l_3} = E^{l_1}-E^{l_2}-E^{l_3}$$

(up to order). For convenience sake set $\delta = \pm 1$ and set

$$u_4 = u_1 u_2 u_3 = E^{l_1+l_2+l_3} = E^{l_1}+\delta E^{l_2}+\delta E^{l_3} = u_1+\delta u_2'+\delta u_3'$$

where $\delta u_i' = u_i$ for $i = 2, 3$. Hence:

(1.1) $$a_{l_1+l_2+l_3} = a_{l_1} a_{l_2} a_{l_3} + a_{l_3} b_{l_1} b_{l_2} d + a_{l_1} b_{l_2} b_{l_3} d + a_{l_2} b_{l_1} b_{l_3} d = a_{l_1}+\delta a_{l_2}+\delta a_{l_3}$$

and

(1.2) $$b_{l_1+l_2+l_3} = a_{l_1} a_{l_3} b_{l_2} + a_{l_2} a_{l_3} b_{l_1} + b_{l_1} b_{l_2} b_{l_3} d + a_{l_1} a_{l_2} b_{l_3} = b_{l_1}+\delta b_{l_2}+\delta b_{l_3}.$$

Multiplying (1.1) by a_{l_1} and subtracting (1.2) times $b_{l_1} d$ yields:

(1.3) $$N(u_1)[a_{l_2} a_{l_3} + b_{l_2} b_{l_3} d - 1] = \delta[a_{l_1} a_{l_2} + a_{l_1} a_{l_3} - b_{l_1} b_{l_2} d - b_{l_1} b_{l_3} d]$$

where $N(\cdot)$ denotes the norm from K to Q. Also:

$$N(u_4) = (a_{l_1}+\delta a_{l_2}+\delta a_{l_3})^2 - (b_{l_1}+\delta b_{l_2}+\delta b_{l_3})^2 d$$

whence:

(1.4) $$[N(u_4)-N(u_1)-\delta N(u_2)-\delta N(u_3)]/2 = \delta(a_{l_1} a_{l_2} + a_{l_1} a_{l_3} - b_{l_1} b_{l_2} d - b_{l_1} b_{l_3} d) + a_{l_2} a_{l_3} - b_{l_2} b_{l_3} d.$$

Combining (1.3) and (1.4) yields:

(1.5) $$[N(u_1)-\delta N(u_2)-\delta N(u_3)+N(u_4)]/2 = N(u_1)[a_{l_2} a_{l_3} + b_{l_2} b_{l_3} d] + a_{l_2} a_{l_3} - b_{l_2} b_{l_3} d.$$

Now it remains to analyze (1.5) in terms of the ordered 4-tuples $(N(u_1), N(u_2), N(u_3), \delta)$ of ± 1's. The following chart contains the values (exactly half) of these 4-tuples which lead to either $a_{l_2} a_{l_3} = 0$ or $b_{l_2} b_{l_3} d = 1$, both of which cannot hold. Hence these values yield no solutions.

Table 2

$(N(u_1), N(u_2), N(u_3), \delta)$	result
$(1, 1, 1, 1)$	$a_{l_2} a_{l_3} = 0$
$(1, 1, -1, 1)$	$a_{l_2} a_{l_3} = 0$
$(1, 1, -1, -1)$	$a_{l_2} a_{l_3} = 0$
$(1, -1, 1, 1)$	$a_{l_2} a_{l_3} = 0$
$(1, -1, 1, -1)$	$a_{l_2} a_{l_3} = 0$
$(1, -1, -1, -1)$	$a_{l_2} a_{l_3} = 0$
$(-1, 1, 1, 1)$	$b_{l_2} b_{l_3} d = 1$
$(-1, -1, -1, -1)$	$b_{l_2} b_{l_3} d = 1$

The next table yields the remaining half of the values of the four-tuples which do yield solutions. They imply either that $a_{l_2} a_{l_3} = 1$ or $b_{l_2} b_{l_3} = 0$. In either case we get the same set of solutions, the details of which will be discussed after that table.

Table 3

$(N(u_1), N(u_2), N(u_3), \delta)$	result
$(1, 1, 1, -1)$	$a_{l_2} a_{l_3} = 1$
$(1, -1, -1, 1)$	$a_{l_2} a_{l_3} = 1$
$(-1, 1, 1, -1)$	$b_{l_2} b_{l_3} = 0$
$(-1, -1, 1, 1)$	$b_{l_2} b_{l_3} = 0$
$(-1, -1, 1, -1)$	$b_{l_2} b_{l_3} = 0$
$(-1, 1, -1, 1)$	$b_{l_2} b_{l_3} = 0$
$(-1, 1, -1, -1)$	$b_{l_2} b_{l_3} = 0$
$(-1, -1, -1, 1)$	$b_{l_2} b_{l_3} = 0$

First we consider $b_{l_2} b_{l_3} = 0$. We may assume without loss of generality that $b_{l_3} = 0$. Therefore $a_{l_3} = 1$ and we are left with

$$u_1 = \delta u_2' + \delta = u_1 u_2'.$$

Therefore

$$u_2' = (u_1 + \delta)/(u_1 - \delta) = (a_{l_1} + b_{l_1}\sqrt{d} + \delta)/(a_{l_1} + b_{l_1}\sqrt{d} - \delta).$$

Multiplying numerator and denominator by $(a_{l_1} - b_{l_1}\sqrt{d} - \delta)$ we get:

$$u_2' = (N(u_1) - 2\delta b_{l_1}\sqrt{d} - 1)/(N(u_1) - 2\delta a_{l_1} + 1).$$

However, from Table 3 we see that $N(u_1) = -1$ so:

$$u_2 + \delta u_2' = (1 + \delta b_{l_1}\sqrt{d})/a_{l_1}. \tag{1.6}$$

Since $2a_{l_1} \in Z$ then $a_{l_1} \in \{\pm 1, \pm 2, \pm 1/2\}$. We now analyze (1.6) for the various values of a_{l_1}.

If $a_{l_1} = \pm 1$ then $d = 2$ and $b_{l_1} = \pm 1$. If $a_{l_1} = 1$ and $\delta = 1$ then $u_1 = 1 \pm \sqrt{2}$, $\delta u_2' = 1 \pm \sqrt{2}$ and $\delta u_3' = 1$. If $a_{l_1} = -1$ and $\delta = 1$ then $u_1 = -1 \pm \sqrt{2}$, $\delta u_2' = -1 \mp \sqrt{2}$ and $\delta u_3' = 1$. If $a_{l_1} = 1$ and $\delta = -1$ then $u_1 = 1 \pm \sqrt{2}$, $\delta u_2' = 1 \mp \sqrt{2}$ and $\delta u_3' = -1$. If $a_{l_1} = -1$ and $\delta = -1$ then $u_1 = -1 \pm \sqrt{2}$, $\delta u_2' = -1 \pm \sqrt{2}$, and $\delta u_3' = -1$. Hence all solutions for the case $a_{l_1} = \pm 1$ are permutations of $\pm(1+\sqrt{2}, 1+\sqrt{2}, 1)$, $\pm(1-\sqrt{2}, 1-\sqrt{2}, 1)$ and $\pm(1+\sqrt{2}, 1-\sqrt{2}, -1)$.

If $a_{l_1} = \pm 1/2$ then $b_{l_1} = \pm 2$ and $d = 5$. If $a_{l_1} = 1/2$ and $\delta = 1$ then $u_1 = (1 \pm \sqrt{5})/2$, $\delta u_2' = 2 \pm \sqrt{5}$ and $\delta u_3' = 1$. If $a_{l_1} = -1/2$ and $\delta = 1$ then $u_1 = (-1 \pm \sqrt{5})/2$, $\delta u_2' = -2 \mp \sqrt{5}$ and $\delta u_3' = 1$. If $a_{l_1} = 1/2$ and $\delta = -1$ then $u_1 = (1 \pm \sqrt{5})/2$, $\delta u_2' = 2 \mp \sqrt{5}$, and $\delta u_3' = -1$. If $a_{l_1} = -1/2$ and $\delta = -1$ then $u_1 = (-1 \pm \sqrt{5})/2$, $\delta u_2' = -2 \pm \sqrt{5}$ and $\delta u_3' = -1$. Hence all solutions for the case $a_{l_1} = \pm 1/2$ are permutations of $\pm((1+\sqrt{5})/2, 2+\sqrt{5}, 1)$, $\pm((1-\sqrt{5})/2, 2-\sqrt{5}, 1)$, $\pm((1+\sqrt{5})/2, 2-\sqrt{5}, -1)$ and $\pm((1-\sqrt{5})/2, 2+\sqrt{5}, -1)$.

If $a_{l_1} = \pm 2$ then the roles of u_1 and u_2 are reversed in the previous case and no new solutions are found.

Finally one may check that an analysis of $a_{l_2} a_{l_3} = 1$ yields exactly the same solutions as in the above cases. This completes the proof of the theorem, and the results are summarized in the following table. ∎

Table 4. Classification of all solutions to $u_1 u_2 u_3 = u_1 + u_2 + u_3$ for $u_i \in U_k$ for any quadratic field $K = Q(\sqrt{d})$

d	Solutions are permutations of:
-1	$\pm(i, i, -i)$ and $\pm(1, i, -i)$
2	$\pm(1+\sqrt{2}, 1+\sqrt{2}, 1)$; $\pm(1-\sqrt{2}, 1-\sqrt{2}, 1)$ and $\pm(1+\sqrt{2}, 1-\sqrt{2}, -1)$
5	$\pm((1+\sqrt{5})/2, 2+\sqrt{5}, 1)$; $\pm((1-\sqrt{5})/2, 2-\sqrt{5}, 1)$;
	$\pm((1+\sqrt{5})/2, 2-\sqrt{5}, -1)$ and $\pm((1-\sqrt{5})/2, 2+\sqrt{5}, -1)$
	No solutions exist for $d \neq -1, 2$ or 5.

It remains open to classify all solutions of $(*)$ for the ring of integers of an arbitrary number field K. For example, in view of the Kronecker–Weber Theorem, to answer the question for abelian extensions of Q it would be of value to know the solutions of $(*)$ in $Z[\xi]$ where ξ is a primitive root of unity. In this paper we have solved the case where ξ is a primitive third or fourth root of unity since these are the only roots of unity which generate quadratic fields.

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