Discrepancy estimates for the value-distribution of the Riemann zeta-function I

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1. Introduction. The purpose of this paper is to refine a theorem of Bohr–Jessen related with the value-distribution of the Riemann zeta-function.

Let $s = \sigma + it$ be a complex variable, and $\zeta(s)$ the Riemann zeta-function. Bohr–Jessen [1] discussed the value-distribution of $\zeta(s)$ on the fixed line $\sigma = \sigma_0$ ($> 1$). Let $R$ be any closed rectangle in the complex $z$-plane with the edges parallel to the axes, and $L(T, R)$ the (Jordan) measure of the set $\{t \in [0, T] \mid \log \zeta(\sigma_0 + it) \in R\}$. Then, Bohr–Jessen proved that there exists the limit

\begin{equation}
W(R) = \lim_{T \to \infty} \frac{L(T, R)}{T},
\end{equation}

which depends only on $\sigma_0$ and $R$. We can consider $W(R)$ as a “probability” of how many values of $\log \zeta(s)$ on the line $\sigma = \sigma_0$ belong to the rectangle $R$.

We cannot obtain any quantitative version of (1.1) by Bohr–Jessen’s method only. Hence, we will introduce the method of discrepancies, and will prove the following sharpening of (1.1):

**Theorem.** For any $\varepsilon > 0$,

\begin{equation}
L(T, R) = W(R) T + O \left( \left( \min(R) + \varepsilon \right) (\log \log T)^{-\sigma(\sigma_0 - 1) + \varepsilon} \right),
\end{equation}

where $m(R)$ is the measure of the rectangle $R$, and the $O$-constant depends only on $\sigma_0$ and $\varepsilon$.

In [1], Bohr–Jessen used the celebrated Kronecker–Weyl theorem on the uniform distribution of sequences. We put

\[ f_N(t) = - \sum_{n=1}^{N} \log \left(1 - p_n^{-\sigma_0} \exp(-it \cdot \log p_n)\right), \]

where $p_n$ is the $n$th prime number. Of course, $f_N(t)$ tends to the function $\log \zeta(\sigma_0 + it)$ as $N$ tends to infinity. Then, by using the Kronecker–Weyl
theorem, Bohr–Jessen proved that there exists the limit
\[ W_{N}(R) = \lim_{T \to \infty} L_{N}(T, R)/T, \]
where \( L_{N}(T, R) \) is the measure of the set \( \{ t \in [0, T] \mid f_{n}(t) \in R \} \). Hence, our
refinement will require essentially the information on discrepancies—the information on how rapidly \( L_{N}(T, R)/T \) converges to the value \( W_{N}(R) \). The
use of the multi-dimensional version of the Erdős–Turán inequality, which was
proved by J. F. Koksma [3] and P. Szűsz [9]. (See Kuipers–Niederreiter [4], Chap. 2. A further generalization is proved by Niederreiter–Philipp [7].) Combining this inequality with a result on transcendental number theory (Waldschmidt [10]), we can deduce the estimate of
\[ E(T, N, R) = L_{N}(T, R)/T - W_{N}(R). \]
We note that a geometric lemma is indispensable in order to apply the
Koksma–Szűsz inequality to our case. Such a lemma was already shown by
T. Miyazaki and the author [5].
Furthermore, we shall prove an upper-bound estimate of \( |W_{N}(R) - W(R)| \). To prove this estimate, we must refine Bohr–Jessen's theory on the
sums of convex curves [2], and the details of our argument are rather
complicated. These estimates of \( E(T, N, R) \) and \( |W_{N}(R) - W(R)| \) lead to the
result of our theorem.
Throughout this paper, the rectangles we consider are closed and have
the edges parallel to the axes. For any subset \( X \) of (real or complex)
Euclidean space, we denote the boundary of \( X \) by \( \partial X \). By \( m(X) \) we mean the
Jordan measure of \( X \). The symbol \( \text{dist}(, ) \) signifies the usual Euclidean
metric. The \( O \)-constants depend only on \( \sigma_{0} \) in the following sections. Also, \( C \)
denotes a positive constant, which depends only on \( \sigma_{0} \) and is not necessarily
the same in each occurrence.
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valuable comments.

2. Outline of the proof. Let \( R \) be the given rectangle, \( \delta_{N} \) a small positive
number, and \( A_{1} + iB_{1} (1 \leq p, q \leq 2, i = \sqrt{-1}, A_{1} < A_{2}, B_{1} < B_{2}) \) the four
vertices of \( R \):
\[ R = \{ z \mid A_{1} \leq \text{Re} \ z \leq A_{2}, B_{1} \leq \text{Im} \ z \leq B_{2} \}. \]
We consider two rectangles \( R_{i} \) and \( R_{j} \) defined by
\[ R_{i} = \{ z \mid A_{1} + \delta_{N} \leq \text{Re} \ z \leq A_{2} - \delta_{N}, B_{1} + \delta_{N} \leq \text{Im} \ z \leq B_{2} - \delta_{N} \}. \]
and
\[ R_{y} = \{ z \mid A_{1} - \delta_{N} \leq \text{Re} \ z \leq A_{2} + \delta_{N}, B_{1} - \delta_{N} \leq \text{Im} \ z \leq B_{2} + \delta_{N} \}. \]
If \( \delta_{N} \) satisfies
\[ |\log \zeta(\sigma_{0} + it) - f_{n}(t)| < \delta_{N} \]
for any real \( t \), then we see that, if \( \log \zeta(\sigma_{0} + it) \in R \) then \( f_{n}(t) \in R_{y} \), and, if
\( f_{n}(t) \in R_{y} \) then \( \log \zeta(\sigma_{0} + it) \in R \). Hence we obtain
\[ L_{N}(T, R_{i}) \leq L(T, R) \leq L_{N}(T, R_{j}). \]
Now we specify the value of \( \delta_{N} \). We have
\[ |\log \zeta(\sigma_{0} + it) - f_{n}(t)| \]
\[ = \left| \sum_{n=1}^{\infty} \log(1 - p_{n}^{-\sigma_{0}} \exp(-it \log p_{n})) \right| \]
\[ \leq \sum_{n=1}^{\infty} p_{n}^{-\sigma_{0}} \leq \sum_{n=1}^{\infty} (n \log n)^{-\sigma_{0}} \]
\[ \leq (\log N)^{-\sigma_{0}} \gamma x^{-\sigma_{0}} dx \leq N^{1-\sigma_{0}} (\log N)^{-\sigma_{0}}. \]
So we can choose \( \delta_{N} = CN^{1-\sigma_{0}} (\log N)^{-\sigma_{0}} \) for some constant \( C \). Under this choice of \( \delta_{N} \), the inequalities (2.1) and (2.2) hold.
We already mentioned in Section 1 that we shall obtain an estimate for
\( E(T, N, R) \), by using the Koksma–Szűsz inequality. In fact, we obtain the
following estimate:

PROPOSITION 1. For any large positive integers \( m \) and \( r \), we have
\[ E(T, N, R) \leq N^{2}(3r)^{N}(m^{-1} + D_{r}) + r^{-N-1} N^{3/2+2\sigma_{0}} + T^{-1} \]
\[ = A_{1} + A_{2} + T^{-1}, \] (say)
where
\[ D_{r} = T^{-1}(3 + 2 \cdot \log m)^{N} \exp(C(mN \cdot \log N)^{3} (\log (mN)^{2})). \]

Since the right-hand side of (2.4) is independent of \( R \), we can apply this
inequality to \( R_{i} \) and \( R_{j} \), and get
\[ L_{N}(T, R_{i})/T - W_{N}(R_{i}) \leq A_{1} + A_{2} + T^{-1}, \]
(2.5)
\[ L_{N}(T, R_{j})/T - W_{N}(R_{j}) \leq A_{1} + A_{2} + T^{-1}. \]
(2.6)

Next we treat the evaluation of \( |W_{N}(R_{i}) - W_{N}(R)| \) and \( |W_{N}(R_{j}) - W_{N}(R)| \). We
first introduce some notations. Let \( Q_{N} \) be the real \( N \)-dimensional unit
cube: \( Q_{N} = \{ (\theta_{1}, \ldots, \theta_{N}) \mid 0 \leq \theta_{n} < 1 \} \). We define the mapping \( S_{N} \) from \( Q_{N} \) to
the complex $z$-plane by
\[ S_N: (\theta_1, \ldots, \theta_N) \mapsto -\sum_{n=1}^{N} \log(1 - p_n^{-\sigma_0} \exp(2\pi i \theta_n)). \]

and we put
\[ \Omega_N(R) = \{(\theta_1, \ldots, \theta_N) \in Q_N | S_N(\theta_1, \ldots, \theta_N) \leq R\}. \]

Applying Kronecker–Weyl's theorem, we can show that $L_N(T; R)/T$ tends to $m(\Omega_N(R))$ as $T$ tends to infinity. This, so far, is an outline of Bohr–Jessen's proof of (1.3). In particular, we see $W_N(R) = m(\Omega_N(R))$. Hence, our present problem is reduced to a geometric one. We will compare the volume of $\Omega_N(R)$, $\Omega_N(R_a)$, and $\Omega_N(R)$, and will show the following result:

**Proposition 2.** If we choose $\delta_N = CN^{-1-\varepsilon_0} (\log N)^{-\varepsilon_0}$, then
\[ W_N(R) - W(R) \leq N^{1-\varepsilon_0/2} (\log N)^{-\varepsilon_0/2} \quad (\text{say}), \]

and the same estimate holds for $W_N(R_a) - W(R)$.

Combining (2.2), (2.5) and (2.6) with the above proposition, we have
\[ L(T, R)/T - W(N) \leq A_1 + A_2 + A_3 + T^{-1}. \]

Now we proceed to the estimation of $|W_N(R) - W(R)|$. Bohr–Jessen [2] showed the following results:

1. For any sufficiently large $N$, there is a function $F_N(z)$, continuous in the whole plane, for which
\[ W_N(R) = \iiint_R F_N(x) \, dx \, dy \quad (x = z + iy), \]
holds for any rectangle $R$.

2. $F_N(z)$ converges uniformly to a continuous function $F(z)$ as $N \to \infty$, and

3. $W(R) = \iiint_R F(x) \, dx \, dy$.

In [2], Bohr–Jessen constructed concretely the functions $F_N(z)$. In this paper, we will investigate their construction in detail, and will obtain the following

**Proposition 3.**
\[ F_N(z) - F(z) \leq N^{1-\varepsilon_0/7} (\log N)^{-(\varepsilon_0/7)} \quad (\text{say}) \]
holds uniformly in the whole plane.

A. Hence we have
\[ |W_N(R) - W(R)| \leq \iiint_R |F_N(z) - F(z)| \, dx \, dy \leq m(R) A_4, \]

This inequality and (2.7) show that
\[ L(T, R)/T - W(N) \leq A_1 + A_2 + A_3 + m(R) A_4 + T^{-1}. \]

Lastly we choose the values of the parameters $m$, $r$, and $N$, and obtain the result of the theorem. We first note that it is obvious that $A_1 \leq A_4$. We can decide the value of $m$ by the equation $m^{-1} = D_T$, then decide $r$ by requiring $A_1 = A_2$, and finally decide $N$ by requiring $A_3 = A_4$. This is the author's original method, but it requires rather complicated calculations. Here we show the following simple choice of the parameters, which is due to Mr. S. Egami (by a private communication to the author, April 20, 1984).

Let $a = \frac{1}{4} + 3\sigma_0$, $m = \lceil (\log T)^{1/2} \rceil$ (where $[x]$ is the integer part of $x$), $N = \lceil (6a \cdot \log m \log T)^{-1} \rceil$ and $r = [N^2]$. Then we have
\[ D_T \leq T^{-1} \exp(C \cdot (\log m + C \cdot (\log m)^2)) \leq T^{-1} \exp(m) \leq T^{-12}, \]
\[ N^2 (31)^N \leq \exp(2a N \cdot \log N) \leq \exp(\frac{1}{2} \cdot \log m) \leq (\log T)^{1/10}. \]

So,
\[ A_1 \leq (\log T)^{1/8} (\log T)^{-1/4} + T^{-1/2} \leq (\log T)^{-1/8}. \]

Next we have
\[ A_2 \leq N^{-(\sigma_0 + 1/2)} \leq (\log \log T)^{-(\sigma_0 + 1/2)}, \]
\[ A_3 \leq (\log T)^{-(\sigma_0 + 1/2)} \leq (\log T)^{-\varepsilon_0/7}, \]
\[ A_4 \leq (\log \log T)^{3/7} (\log \log T)^{-(\varepsilon_0 + 1/7)}, \]

and these estimates lead to the conclusion. In fact, the above estimate yields a slightly stronger result stated as follows:
\[ L(T, R) = W(R) + O(m(R) T \exp \left( -\frac{1}{2} (\sigma_0 - 1/7) \log \log \log T + \frac{6}{7} \log \log \log T \frac{1}{2} \right) + T(\log \log T)^{-\varepsilon_0/7} (\log \log T)^{-1/2} \].

We remark that if we follow the original proof of the author, we can write down explicitly the value of the constant $C$, appearing in the right-hand side of (2.10). An elementary (but complicated) calculation shows that
\[ C \leq (\sigma_0 - 1/7) \log (90\sigma_0 + 57/14) + 6(\log \log \log T \log \log T \log \log T). \]

Now our remaining task is to prove the Propositions 1, 2, and 3.

Before starting the proof of these propositions, we must sketch Bohr–Jessen's theory developed in [2]. Then we will start with the proof of Proposition 3, which will be completed in Sections 5 and 6. Next, Sections 7 and 8 will be devoted to the proof of Proposition 1. And in the last section we shall prove Proposition 2.
3. Sketch of Bohr–Jessen's theory on the sums of convex curves. In this and the next section we sketch Bohr–Jessen's theory [2] on the sums of convex curves, and quote some of the lemmas proved in [2], which we shall use later.

Let 
\[ \Gamma_n = \{ z_n = z_n(\theta_n) = -\log (1 - g_n^{-\theta_0} \exp(2\pi i \theta_n)) \mid 0 \leq \theta_n < 1 \} \]

for some positive integer \( g_n \). Then, \( \Gamma_n \) describes a closed convex Jordan curve in the \( z \)-plane. We can consider \( \theta_n \) as a function of \( z_n \) defined on \( \Gamma_n \). If we put \( f_n(z_n) = |d\theta_n(z_n)| / dz_n \), then Bohr–Jessen proved that
\[ f_n(z_n) = |1 - q_n^{-\theta_0} \exp(2\pi i \theta_n) / 2\pi q_n^{-\theta_0} | \]

We remark that
\[ d\theta_n = f_n(z_n) \cdot dz_n \]
on the curve \( \Gamma_n \).

To study the geometric properties of \( \Gamma_n \), we introduce some general notions on closed convex curves. At first we define the "inner radius" and the "outer radius" of a closed convex Jordan curve \( \Gamma \) as follows. For any point \( z \) on \( \Gamma \), we denote by \( L_z \) the tangent line of \( \Gamma \) at \( z \). The curve \( \Gamma \) separates the \( z \)-plane into three parts: A bounded open set \( I(\Gamma) \), another open set \( Y(\Gamma) \), and \( \Gamma \) itself. Let
\[ C_1 = \{ C : \text{circle} \ z \in C, \text{the tangent line to} \ C \text{ at} \ z \} \]
\[ C_2 = \{ C : \text{circle} \ z \in C, \text{the tangent line to} \ C \text{ at} \ z \} \]

By \( g(C) \) we mean the radius of the circle \( C \). Now we define the inner radius \( r_1(\Gamma) \) and the outer radius \( r_2(\Gamma) \) by
\[ r_1(\Gamma) = \inf \sup \ g(C), \quad r_2(\Gamma) = \sup \inf \ g(C). \]

Then Bohr–Jessen showed that
\[ r_1(\Gamma_n) = q_n^{-\theta_0}, \quad r_2(\Gamma_n) = q_n^{-\theta_0} / (1 - q_n^{-2\theta_0})^{1/2}. \]

Next we define the "parallel curves" of the convex curve \( \Gamma \). Let \( z \in \Gamma \) and \( n(z) \) be the unit normal vector of \( \Gamma \) at \( z \), oriented to the outside of \( \Gamma \). We put \( z(\delta) = z + \delta \cdot n \) for any real \( \delta \). Then, we define the parallel curve of \( \Gamma \) with the distance \( \delta \) by \( \Gamma(\delta) = \{ z(\delta) \mid z \in \Gamma \} \). Then, \( \Gamma(\delta) \) is again a closed convex curve, and it can be shown that
\[ r_1(\Gamma(\delta)) = r_1(\Gamma) + \delta, \quad r_2(\Gamma(\delta)) = r_2(\Gamma) + \delta. \]

Now we discuss the "sum" \( \Sigma \) of two convex curves \( \Gamma \) and \( \Gamma' \):
\[ \Sigma = \{ z + z' \mid z \in \Gamma, z' \in \Gamma' \}. \]
We assume that
\[ r_1(\Gamma) > r_2(\Gamma) > r_1(\Gamma'). \]

Then we have that \( \partial \Sigma \) consists of two convex curves \( \Gamma_1 \) and \( \Gamma_2 \), satisfying \( \Gamma_1 \cap \Gamma_2 = \emptyset \), and \( \Sigma = \Gamma_1 \cup \Gamma_2 \). For any point \( z \), we denote the set \( \{ z \mid w \in \Gamma \} \) by \( z - \Gamma \). The convex curves \( \Gamma_1 \) and \( \Gamma_2 \) have the following properties:
\[ I(\Gamma_1) \cup Y(\Gamma_2) = \{ z \mid \Gamma \cap (z - \Gamma) = \emptyset \}, \]
\[ I(\Gamma_1) \cap Y(\Gamma_2) = \{ z \mid \Gamma \cap (z - \Gamma) \text{ consists of one point} \}, \]

and
\[ Y(\Gamma_1) \cap I(\Gamma_2) = \{ z \mid \Gamma \cap (z - \Gamma) \text{ consists of two points} \}. \]

And also we have
\[ \text{Lemma 2 (Bohr–Jessen [2], § 19).} \]
\[ r_1(\Gamma) - r_2(\Gamma) \leq r_1(\Gamma) - r_2(\Gamma'), \]
\[ r_1(\Gamma) + r_2(\Gamma) \leq r_1(\Gamma) + r_2(\Gamma'), \]
\[ r_1(\Gamma) - r_2(\Gamma) \leq r_1(\Gamma) - r_2(\Gamma'). \]

This Lemma 2 is quite useful in the sections below. Now we return to the convex curves \( \Gamma_n \), introduced at the beginning of this section. We put
\[ \Sigma_n = \{ z_1 + z_2 + \ldots + z_k \mid z_j \in \Gamma_j \ (1 \leq j \leq N) \}. \]

Then, for any \( N \), one of the following two cases happens:

Case I. The boundary \( \partial \Sigma_n \) consists of one convex curve \( \Gamma_n \), and
\[ \Sigma_n = \Gamma_{\Gamma_n} \cup \Gamma_{\Gamma_n}. \]
Case II. The boundary $\partial \Sigma_N$ consists of two convex curves $\Gamma_{y,N}$ and $\Gamma_{y,N}$ and 

$$\Sigma_N = \Gamma_{y,N} \cup (\Gamma_{y,N} \cap Y(\Gamma_{y,N})) \cup \Gamma_{y,N}.$$ 

Now we decide the values of $q_n$. We can choose the primes $p_{m(1)}$, $p_{m(2)}$, $p_{m(3)}$, and $p_{m(4)}$ for which the inequality

$$r_{k}(\Gamma_{y,N}) \geq 2r_{k}(\Gamma_{y,N})$$

holds for $k = 1, 2, 3$, where $q_n = p_{m(k)}$ ($1 \leq k \leq 4$). We note that the choices of the values of $m(k)$ depend only on $\sigma_0$. (See (3.3).) Further we define $q_n = p_{m(2)}$ for $M \leq n \leq m(2) + 2$, $q_n = p_{m(3)}$ for $M + 3 \leq n \leq m(3) + 1$, $q_n = p_{m(4)}$ for $M + 2 < n \leq m(4)$, and $q_n = p_{m(1)}$ for $n \geq m(4) + 1$ (that is, $N_0$, say). Then, for any $N \geq N_0$, $\Sigma_N$ is the range of the values of the mapping $S_N$; $\Sigma_N = S_N(Q_N)$. Under the condition (3.6), Case II holds for $\Sigma_2$, $\Sigma_3$, and $\Sigma_4$.

Our aim is to construct the continuous function $F_N$ which satisfies the relation (2.8). We put

$$S_N: (\theta_1, \ldots, \theta_N) \mapsto -\sum_{n=1}^{N} \log (1-q_n^{-a} \exp(2\pi i \theta_n))$$

and

$$Q_N(R) = \{(\theta_1, \ldots, \theta_N) \in Q_N | S_N(\theta_1, \ldots, \theta_N) \in R\}.$$

Then it is obvious that $m(Q_N(R)) = W_N(R)$ for $N \geq N_0$. Bohr-Jessen constructed concrete the continuous function $F_N(z)$, which satisfies

$$m(Q_N(R)) = \frac{1}{\pi} \int_{R} F_N(z) \, dx \, dy$$

for $N = 4$. Then we define inductively

$$F_N(z) = \frac{1}{\pi} \int_{0}^{1} F_{N-1}(z-z_{N}(\theta_N)) \, d\theta_N$$

for $N \geq 5$. We can easily show that each of these $F_N$'s again satisfies (3.7), and therefore, satisfies (2.8) in the next section, we sketch Bohr-Jessen's method to construct the function $F_N$.

4. Bohr-Jessen's construction of the functions $F_2$, $F_3$, and $F_4$. We remark first of all that our $F_2$, $F_3$, $F_4$ are written as $F_1$, $F_2$, $F_3$ in Bohr-Jessen's paper.

Let $\Gamma_1 = \Gamma_{y,1}$, $\Gamma_2 = \Gamma_{y,2}$. For any $z \in Y(\Gamma_1) \cap I(\Gamma_2)$, the set $\Gamma_1 \cap (z+\Gamma_2)$ consists of two points, $z_1$ and $z_1'$. Then, $z_2 = z - z_2$ and $z_2' = z - z_2'$ are on $\Gamma_2$. We denote the cross angles of $\Gamma_1$ and $\Gamma_2$ at $z_1$ and $z_1'$ by $\theta$ and $\theta'$ respectively. (We can assume that $0 < \theta < \pi/2, 0 < \theta' < \pi/2$. We always keep this assumption in the following.) We define the function $F_2(z)$ as follows:

$$F_2(z) = \begin{cases} 0 & \text{if } z \in I(\Gamma_1) \cup Y(\Gamma_2), \\ \infty & \text{if } z \in \Gamma_1 \cap \Gamma_2, \\ (\sin \theta)^{-1} f_1(z_1) f_1(z_1') + (\sin \theta')^{-1} f_2(z_2) f_2(z_2') & \text{if } z \in Y(\Gamma_1) \cap I(\Gamma_2). \end{cases}$$

Then we have

**Lemma 3** (Bohr-Jessen [2], § 31). For any $z \notin \Gamma_1 \cup \Gamma_2$,

$$F_2(z) \leq d^{-1/2}$$

holds, where $d = \min (\text{dist}(z, \Gamma_1), \text{dist}(z, \Gamma_2))$.

We "cut off" the values of $F_2(z)$ if $z$ is close to the curve $\Gamma_1$ or $\Gamma_2$, and define the function $F_2^2(z)$. For small positive $d$, we define

$$F_2^2(z) = \begin{cases} F_2(z) & \text{if } \text{dist}(z, \Gamma_1) > d \text{ and dist}(z, \Gamma_2) > d, \\ 0 & \text{otherwise.} \end{cases}$$

And further, we define

$$F_2^2(z) = \frac{1}{\pi} \int_{0}^{1} F_2^2(z-z_2(\theta_2)) \, d\theta_2.$$

Then, $F_2(z)$ tends to the function

$$F_3(z) = \frac{1}{\pi} \int_{0}^{1} F_2(z-z_2(\theta_2)) \, d\theta_2$$

as $d$ tends to 0. To estimate the speed of this convergence, we first investigate the shape of the set $S_3$. The boundary $\partial S_3$ consists of two convex curves $\Gamma_{y} = \Gamma_{y,3}$ and $\Gamma_{y} = \Gamma_{y,3}$, and $S_3$ contains other two convex curves $\Gamma_{y}$ and $\Gamma_{y}$. ($\Gamma_{y} \subset I(\Gamma_{y})$, $\Gamma_{y} \subset I(\Gamma_{y})$, $\Gamma_{y} \subset I(\Gamma_{y})$). These curves have the following properties:

1. If $z \in I(\Gamma_{y}) \cup Y(\Gamma_{y})$, then $S_3 \cap (z-\Gamma_{y}) = \emptyset$.
2. If $z \in \Gamma_{y}$, then $S_3 \cap (z-\Gamma_{y})$ consists of one point $z'$, and $z' \in \Gamma_{y}$.
3. If $z \in I(\Gamma_{y}) \cap Y(\Gamma_{y})$, then $\Gamma_{y} \cap \Gamma_{y}$ consists of two points.
4. If $z \in I(\Gamma_{y}) \cap \Gamma_{y}$, then $\Gamma_{y} \cap (z-\Gamma_{y})$ consists of one point $z'$, and similarly, if $z \in \Gamma_{y}$.
5. If $z \in \Gamma_{y}$, then $\Gamma_{y} \cap (z-\Gamma_{y})$ consists of one point $z'$. In both cases, $S_3 \cap (z-\Gamma_{y}) = \emptyset$.
6. If $z \in I(\Gamma_{y}) \cap I(\Gamma_{y})$, then $S_3 \cap (z-\Gamma_{y})$ consists of one point $z'$. In both cases, $S_3 \cap (z-\Gamma_{y}) = \emptyset$.

Let $\alpha$ be a small positive number with the condition $\alpha > 2d$, and put

$$A = \frac{|z| \text{ dist}(z, \Gamma_{y})}{\alpha}, \text{ dist}(z, \Gamma_{y}) \geq \alpha, \text{ dist}(z, \Gamma_{y}) \geq \alpha, \text{ dist}(z, \Gamma_{y}) \geq \alpha.$$
Then, for sufficiently small \( a \), the set \( A \) consists of five closed subsets

\[
A_{ii} = A \cap I(\Gamma_i), \quad A_{iy} = A \cap Y(\Gamma_i) \cap I(\Gamma_{iy}),
\]

\[
A^* = A \cap Y(\Gamma_{iy}) \cap I(\Gamma_{iy}), \quad A_{yy} = A \cap Y(\Gamma_{yy}).
\]

Let \( z \in A_{yy} \). Then, \( \Gamma_i \cap (z - \Gamma_z) \) consists of two points. Furthermore, since \( z \in A_i \), \( \Gamma_i(\delta) \cap (z - \Gamma_z) \) consists of two points \( z' \) and \( z'' \), for any positive \( \delta \leq d \).

We denote the cross angles of \( \Gamma_i(\delta) \) and \( z - \Gamma_z \) at \( z' \) and \( z'' \) by \( \theta(\delta) \) and \( \theta''(\delta) \), respectively, and put

\[
\vartheta_i = \min \left\{ \inf_{\delta \in \delta_i} \sin \theta(\delta), \quad \inf_{\delta \in \delta_i} \sin \theta''(\delta) \right\}.
\]

Also we define \( \vartheta_i \) in a similar way. Then we have

**Lemma 4** (Bohr–Jessen [2], § 33). If \( z \in A_{yy} \), then

\[
B_3(z) - B_3^*(z) \leq d^{1/2}/\vartheta_i.
\]

If \( z \in A_{ii} \), then

\[
B_3(z) - B_3^*(z) \leq d^{1/2}/\vartheta_i.
\]

If \( z \in A_{ii} \cup A^* \cup A_{iy} \), then

\[
B_3(z) = B_3^*(z).
\]

Now we arrive at the stage to define the function \( F_4(z) \). We define

\[
F_4(z) = \int_0^1 B_3(z - z_4(\vartheta_4))d\vartheta_4.
\]

and

\[
F_4(z) = \int_0^1 B_3(z - z_4(\vartheta_4))d\vartheta_4.
\]

By using the definition of \( F_4(z) \), we can easily prove (3.7) for \( N = 2 \).

However, \( F_2 \) is not a continuous function, so we integrate \( F_2 \) twice to secure the continuity. In fact, we can show that the continuous function \( F_4(z) \) converges to \( F_4(z) \) uniformly in the whole plane as \( d \) tends to 0. Hence \( F_4 \) is continuous, and it can be proved that this \( F_4 \) satisfies (3.7) for \( N = 4 \).

Therefore, this \( F_4(z) \) is our desired function.

We quote a lemma of Bohr–Jessen related with the difference between \( F_4(z) \) and \( F_4(z) \). Let

\[
\Omega_{n+1}(a) = \{ z | z \in \Sigma_3, \text{dist}(z, \Gamma_n) < a \},
\]

\[
\Omega_{n+1}(a) = \{ z | \text{dist}(z, \Gamma_n) < a \}, \quad \Omega_{n+1}(a) = \{ z | \text{dist}(z, \Gamma_n) < a \},
\]

\[
\Omega_{n+1}(a) = \{ z | \text{dist}(z, \Gamma_n) < a \},
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\[
\Omega_{n+1}(a) = \{ z | \text{dist}(z, \Gamma_n) < a \}.
\]

Thus, the uniform convergence of \( F_4(z) \). The function \( F = \lim F_N \) satisfies the relation (2.9).
5. Outline of the proof of Proposition 3. Now we shall start on the proof of Proposition 3. For any fixed \( z \), by \( w = w(z) \) we mean any point satisfying the condition \( \text{dist}(w, z) \leq \delta(N) \). From Lemma 6 it follows that

\[
|F_N(z) - F(z)| \leq \sup_{z, w} |F_{N_0}(z) - F_{N_0}(w)|.
\]

By (3.8) we see that

\[
|F_N(z) - F_N(w)| \leq \frac{1}{\delta} |F_{N-1}(z - z_N(0)) - F_{N-1}(w - z_N(0))|\, d\theta_n.
\]

Hence,

\[
\sup_{z, w} |F_N(z) - F_N(w)| \leq \sup_{z, w} |F_{N-1}(z) - F_{N-1}(w)|.
\]

By using this inequality repeatedly, we have from (5.1) that, for any small positive \( d \),

\[
F_N(z) - F(z) \leq \sup_{z, w} |F_N(z) - F_N(w)| \leq \sup_{z, w} |F_N(z) - F_N(w)| + \sup_{z, w} |F_N(w) - F_N(w)| = X_1(z) + X_2 + X_3(w),
\]

The estimation of \( X_2 \) is the most essential part of the proof of Proposition 3, and we devote the next section to the discussion of this point. The result obtained there states as follows:

\[
X_2 \leq d^{-3/2} \delta(N) + d^{-1/2} \delta(N)^{1/3}.
\]

On the other hand, we shall prove in this section that

\[
X_1(z), X_2(w) \leq d^{1/4} \log(1/d).
\]

Hence, if we choose \( d = \delta(N)^{a_7} \), then by (5.3), (5.4) and (5.5) we have

\[
F_N(z) - F(z) \leq \delta(N)^{1/3} \log(1/\delta(N)).
\]

Now we estimate the value of \( \delta(N) \). By (4.3) and (3.3) we have

\[
\delta(N) \leq \sum_{n=n+1}^{N_0+1} r_j(\Gamma_n) \leq \sum_{n=n+1}^{N_0+1} \rho_\sigma^{s_0} \leq N^{1-s_0} \log(N)^{-s_0}
\]

for \( N \geq N_0 \). (See (2.3).) Substituting this result in (5.6), we arrive at the result of Proposition 3.

We shall estimate \( X_1(z) \) in the rest of this section, by using several lemmas prepared by Bohr and Jessen. The same argument, of course, holds for \( X_1(w) \).

Our starting point is the inequality (4.1). This inequality, with (4.2), reduces our problem to the estimation of \( U_\delta^\pi(z) \). And the integrand of the term \( U_\delta^\pi(z) \) was already estimated in Lemma 4. To use the result of Lemma 4, we need a lower bound of \( y_\beta \) and \( y_\beta' \). We can get such a bound by virtue of Lemma 2; we apply Lemma 2 in the case of \( \Gamma = \Gamma_1(\delta) \) and \( \Gamma = \Gamma_3 \), where \( \delta \) is the parameter appearing in the definition of \( y_\beta \). We remark that \( \delta(\Gamma_1 + \Gamma_3) \) consists of two convex curves \( \Gamma_1(\delta) \) and \( \Gamma_3(\delta) \). The boundary of the set \( \Gamma_1(\delta) + \Gamma_3(\delta) \) also consists of two convex curves \( \Gamma_1^\beta \) and \( \Gamma_3^\beta \). (\( \Gamma_1^\beta \subset I(\Gamma_1^\beta) \).) Then we have

\[
\text{Lemma 7.} \quad \Gamma_1^\beta = \Gamma_1(\delta), \quad \Gamma_3^\beta = \Gamma_3(\delta).
\]

Proof. We only prove the former; the proof of the latter is similar. Let \( z \in \Gamma_1 \). Then \( \Gamma_1 \cap (z - \Gamma_3) \) consists of one point \( z' \). So the curves \( \Gamma_1 \) and \( z - \Gamma_3 \) are tangent to each other at \( z' \), and we denote by \( L \) the common tangent line of \( \Gamma_1 \) and \( z - \Gamma_3 \) at \( z' \). Then \( L \) is parallel to the tangent line \( L \) of \( \Gamma_1 \) at \( z \). (See Bohr-Jessen [2], § 17.) Hence, for the point \( z(\delta) = z + \delta \cdot n(\delta) \in \Gamma_1(\delta) \), the curves \( \Gamma_1(\delta) \) and \( z(\delta) - \Gamma_3 \) are tangent to each other at \( z'(\delta) = z' + \delta \cdot n'(\delta) \) (where \( n \) and \( n' \) are the unit normal vector of \( \Gamma_1 \) and \( \Gamma_3 \), respectively, see Fig. 1). This implies that \( \Gamma_1(\delta) \subset \Gamma_1^\beta \). Both \( \Gamma_1(\delta) \) and \( \Gamma_1^\beta \) are closed convex curves, so we have \( \Gamma_1(\delta) = \Gamma_1^\beta \). This proves the lemma.

Fig. 1

Now we deduce the lower bound of \( y_\beta \). If \( z \in A_1(\delta) \), then by Lemma 7, we have \( \text{dist}(z, \Gamma_1^\beta) \geq \frac{1}{2} \alpha - \delta = \frac{1}{2} z \), and similarly \( \text{dist}(z, \Gamma_3^\beta) \geq \frac{1}{2} z \). Applying Lemma 2 in this case, we have

\[
y_\beta > \left( \frac{1}{2} \alpha (r_1(\Gamma_1(\delta)) - r_2(\Gamma_3(\delta))) / 2 r_1(\Gamma_1(\delta)) r_2(\Gamma_3(\delta)) \right)^{1/2}.
\]

By (3.4) and Lemma 1, we have

\[
r_1(\Gamma_1(\delta)) - r_2(\Gamma_2(\delta)) + \delta \leq r_1(\Gamma_2(\delta)) \leq r_1(\Gamma_3(\delta)) - r_1(\Gamma_2(\delta)) + \delta.
\]
Since we can assume \( r_1(\Gamma_2) - r_2(\Gamma_2) - r_3(\Gamma_3) = C = C(\sigma_0) > 0 \), by (5.7) we can say that
\[ g_q \gg a^{1/2}. \]
And similarly, we can show the estimate \( g_q \gg a^{1/2} \). Hence, by using Lemma 4, we have
\[ U_q^*(z) \ll a^{-1/2} d^{1/2} \int_{r_{\Sigma_0}^b} |dz_4| \ll a^{-1/2} d^{1/2} \int_{r_{\Sigma_0}^b}. \]
From this result and (4.2), we obtain
\[ F_4(z) - F_2(z) \ll a^{-1/2} d^{1/2} + a^{1/2} \log(1/a). \]
Now we choose \( a = d^{1/2} \). We remark that this choice satisfies the condition \( a \gg 2d \) for sufficiently small \( d \). Under this choice of \( a \), (5.8) leads to the estimate \( F_4(z) - F_2(z) \ll d^{1/2} \log(1/d) \). This completes the proof of (5.5).

6. Estimation of \( X_2 \). Our aim in this section is to estimate \( |F_4(z) - F_2(w)| \) under the condition \( \text{dist}(z, w) \leq \delta(N) \). We first note that by using (5.2) again, we have
\[ F_4(z) = F_2(z) \quad \text{if} \quad z \in \Sigma_2^*; \]
\[ F_4(z) = 0 \quad \text{otherwise.} \]
We first assume that both \( z \) and \( w \) belong to \( \Sigma_2^* \). Then we have
\[ |F_4(z) - F_2(w)| = |F_4(z) - F_2(w)| \ll (\sin \theta')^{-1} f_1(z_1) f_2(z_2) - (\sin \phi')^{-1} f_1(w_1) f_2(w_2) + \]
\[ + (\sin \theta')^{-1} f_1(w_1) f_2(z_2) - (\sin \phi')^{-1} f_1(w_1) f_2(w_2) \]
\[ = X_2^1 + X_2^2, \quad \text{say}. \]
Here, \( z_1, z_2, z_2, z_1, \theta' \) and \( \phi' \) are the same as those which appeared in the definition of \( F_2(z) \), and we define \( w_1, w_2, w_2, w_1, \phi' \) and \( \phi' \) in a similar way. Furthermore, for \( X_2^2 \) we have
\[ X_2^2 \ll (\sin \theta')^{-1} f_1(z_2) |f_1(z_2) - f_1(w_1)| + \]
\[ + (\sin \theta')^{-1} f_1(w_1) |f_2(z_2) - f_2(w_2)| + \]
\[ + (\sin \theta' \sin \phi')^{-1} f_1(w_1) f_2(w_2) |\sin \theta' - \sin \phi'| \]
\[ = Y_1 + Y_2 + Y_3, \quad \text{say}. \]
To estimate the terms \( Y_1, Y_2 \) and \( Y_3 \), we first prove the following lemma:

**Lemma 8.** If \( \text{dist}(z, w) \leq \delta(N) \), then
\[ \text{dist}(z_1, w_1) \ll d^{-1/2} \delta(N), \quad \text{dist}(z_2, w_2) \ll d^{-1/2} \delta(N). \]

**Proof.** Let \( z_3 = z_1 + (w - z) \in w - \Gamma_2 \). We define the four circles \( C_1, C_1', C_2, C_2' \) as follows (see Fig. 2):

![Fig. 2](image)

\( C_1 \): tangent to \( \Gamma_1 \) at \( z_1' \), \( C_1 \subset I(\Gamma_1) \cup \Gamma_1 \), and
\[ \varphi(C_1) = q_1^{-\sigma_0}. \]
\( C_1' \): tangent to \( \Gamma_1 \) at \( z_1' \), \( \Gamma_1 \subset I(C_1') \cup C_1' \), and
\[ \varphi(C_1) = q_1^{-\sigma_0} / (1 - q_1^{-2\sigma_0})^{1/2}. \]
\( C_2 \): tangent to \( w - \Gamma_2 \) at \( z_2 \), \( C_2 \subset I(w - \Gamma_2) \cup (w - \Gamma_2) \), and
\[ \varphi(C_2) = q_2^{-\sigma_0}. \]
\( C_2' \): tangent to \( w - \Gamma_2 \) at \( z_2 \), \( w - \Gamma_2 \subset I(C_2') \cup C_2' \), and
\[ \varphi(C_2) = q_2^{-\sigma_0} / (1 - q_2^{-2\sigma_0})^{1/2}. \]

We denote then by \( z_4, z_5, z_6 \) and \( z_7 \), the intersection points of \( C_1 \) and \( C_2 \), \( C_1' \) and \( C_2' \), and \( C_1' \) and \( C_2' \) respectively. Then, by an elementary calculation, we can show that
\[ \text{dist}(z_1', z_j) \ll (\sin \theta')^{-1} \delta(N) \quad (j = 4, 5, 6, 7). \]
We apply Lemma 2 to \( \Gamma = \Gamma_1 \) and \( \Gamma^* = \Gamma_{2*} \), and obtain that
\[
\sin \theta' \ll d^{1/2},
\]
so it follows that \( \text{dist}(z_1', z_2') \ll d^{-1/2} \delta(N) \). Hence we have that
\[
\text{dist}(z_1', w_1') \ll d^{-1/2} \delta(N).
\]
And also, since \( w_2' - z_1' = ((w - z_1') - (z - z_2')) + (z_1' - w_1') \), it follows that
\[
\text{dist}(z_2', w_2') \ll d^{-1/2} \delta(N).
\]
This implies the lemma.

Let \( z_1' = z_1(\theta) \) and \( w_1' = z_1(\varphi) \). Then, by (3.1), we have
\[
|f_1(z_1') - f_1(w_1')| = (2\pi a_{\theta}^{-e_0} - 1) \exp(L_\gamma) - \exp(L_\eta),
\]
where
\[
L_\gamma = \log |1 - q_{1*}^{-e_0} \exp(2\pi i \theta)| \quad \text{and} \quad L_\eta = \log |1 - q_{1*}^{-e_0} \exp(2\pi i \varphi)|.
\]
There exists a real number \( \xi \) between \( L_\gamma \) and \( L_\eta \) for which
\[
|\exp(L_\gamma) - \exp(L_\eta)| \ll \exp(\xi) |L_\gamma - L_\eta| \ll |L_\gamma - L_\eta|
\]
holds. So, by Lemma 8, it follows that
\[
|f_1(z_1') - f_1(w_1')| \ll |L_\gamma - L_\eta| \ll |z_1' - w_1'| \ll d^{-1/2} \delta(N).
\]
Hence, by using (6.4), we have that \( Y_1 \ll d^{-1} \delta(N) \). Since the same argument holds for \( Y_2 \), we now obtain the following estimate:
\[
Y_1 + Y_2 \ll d^{-1} \delta(N).
\]

Next we estimate the term \( Y_3 \). Using (6.4) again, we see
\[
Y_3 \ll d^{-1} |\sin \theta' - \sin \varphi'| \ll d^{-1} \theta' - \varphi'|
\]
so our problem is reduced to proving the following lemma:

**Lemma 9.** Let \( z = z_1(\theta) \) and \( w = z_1(\varphi) \) be points on \( \Gamma_1 \), let \( K_z \) and \( K_w \) be the tangent lines of \( \Gamma_1 \) at \( z \) and \( w \), respectively, and let \( \Theta \) be the cross angle of \( K_z \) and \( K_w \). If \( \text{dist}(z, w) \ll \delta \) for some small positive \( \delta \), then it follows that
\[
|\theta - \varphi| \ll \delta \quad \text{and} \quad \Theta \ll \delta.
\]

Also, similar results hold for any pair of points \( z \) and \( w \) on \( \Gamma_2 \).

By Lemma 8 we have \( \text{dist}(z_1', w_1') \ll d^{-1/2} \delta(N) \), and also,
\[
\text{dist}(z_2', w_2') \ll \text{dist}(z_1', w_1') + \text{dist}(z_1', z_2') \ll d^{-1/2} \delta(N).
\]
Hence we obtain \( |\theta' - \varphi'| \ll d^{-1/2} \delta(N) \) by the above lemma. This result and (6.6) lead to the following estimate:
\[
Y_3 \ll d^{-3/2} \delta(N).
\]

The proof of Lemma 9 is quite elementary, so we omit the details. We only note the fact that we can assume \( q_1 = q_1(\sigma_0) \) is sufficiently large.

By (6.3), (6.5) and (6.7) we have
\[
X_2' \ll d^{-3/2} \delta(N).
\]
The same estimate holds for \( X_2' \), so by (6.2) we obtain
\[
|F_2'(z) - F_2'(w)| \ll d^{-3/2} \delta(N)
\]
if both \( z \) and \( w \) belong to \( \Sigma_2^* \).

Let
\[
\Sigma' = \{ z \in \Sigma \mid \text{dist}(z, \Gamma_1) = d \},
\]
\[
\Sigma_\gamma' = \{ z \in \Sigma_\gamma \mid \text{dist}(z, \Gamma_\gamma) = d \},
\]
and
\[
K = \{ x \in z - \Sigma \mid \text{min} \{ \text{dist}(x, \Gamma_\gamma'), \text{dist}(x, \Gamma_\gamma) \} \ll \delta(N) \}.
\]
(See Fig. 3)

![Fig. 3](https://example.com/fig3.png)

Let \( z_3 \in \Gamma_3 \) and \( z_4 \in \Gamma_4 \). If \( z_3 - z_4 \in \Sigma_2^* \setminus K \), then \( z_3 - z_4 \in \Sigma_2^* \), so by (6.8) we have
\[
|F_2'(z - z_3 - z_4) - F_2'(w - z_3 - z_4)| \ll d^{-3/2} \delta(N).
\]
If \( z_3 - z_4 \notin \Sigma_2^* \cup K \), then \( w - z_3 - z_4 \notin \Sigma_2^* \), so
\[
F_2'(z - z_3 - z_4) - F_2'(w - z_3 - z_4) = 0.
\]
Hence, by (6.1) and Lemma 3, it follows that
\[ |F^k_w(z) - F^k_w(w)| \leq \int_{z-z_3}^{z-z_4} d^2 + \int_{z-z_5}^{z-z_6} d^2 \]
\[ \leq d^{1/2} \int_{z-z_3}^{z-z_4} |dz_3| + d^{3/2} \delta(N). \]

For any point \( u = x + iy \in K \), the set \( (z - T_n) \cap (u + T_n) \) consists of two points. We denote them by \( z - z_3' \) and \( z - z_3'' \). If \( u \in \partial K \), then \( z_3' = z_3'' \).

Let \( \theta(u) \) and \( \theta''(u) \) be the cross angles of \( z - T_3 \) and \( -T_4 \) at \( z_3' \) and \( z_3'' \), respectively. Then we have
\[ \int_{z-z_3}^{z-z_4} |dz_3| \leq \int_{0}^{2\pi} (\sin \theta''(u))^{-1} dx \, dy + \int_{0}^{2\pi} (\sin \theta''(u))^{-1} dx \, dy \]
\[ = I' + I'', \quad \text{say}. \]

We remark that we can choose the values of \( z_3' \) and \( z_3'' \) for which the functions \( \theta(u) \) and \( \theta''(u) \) are continuous in \( K \).

Let
\[ \Sigma = \{ u \in z - \Sigma \mid \delta(u, \partial(z - \Sigma)) \leq 2\delta(N) \}. \]

For any \( u \in \Sigma \), we put \( r = \delta(u, \partial(z - \Sigma)) \). Then, by Lemma 2, we have \( \sin \theta(u) \geq 1/2 \). Hence it follows that
\[ \int_{r = 2\delta(N)}^{0} (\sin \theta(u))^{-1} dx \, dy \leq \int_{0}^{\delta(N)} I' \leq \delta(N)^{1/2}. \]

On the other hand, since \( m(K) \leq \delta(N) \), we have
\[ \int_{K \setminus \partial K} (\sin \theta(u))^{-1} dx \, dy \leq \delta(N)^{-1/2} \int_{K \setminus \partial K} dx \, dy \leq \delta(N)^{1/2}, \]

by using Lemma 2 again. Hence we have \( I' \leq \delta(N)^{1/2} \), and the same estimate holds for \( I'' \). Substituting these estimates in (6.9), we obtain the inequality (5.4), so that the proof of Proposition 3 is now completed.

**7. Some reductions for the proof of Proposition 1.** Our next aim is the estimation of the term
\[ E(T, N, R) = L_N(T, R) - W_N(R). \]

We first quote the Koksma-Szüsz inequality, which plays the essential role in our proof of Proposition 1. Let \( x_1, \ldots, x_N \) be any (given) points of the real \( N \)-dimensional Euclidean space \( \mathbb{R}^N \). A sub-interval \( J \) of \( Q_N \) is a direct product of one-dimensional sub-intervals \( [x_{i_1}, \beta_{i_1}], \ldots, [x_{i_N}, \beta_{i_N}] \) of \([0, 1)\):
\[ J = ([0, 1) \cap \mathbb{Z}) \times \cdots \times ([0, 1) \cap \mathbb{Z}) \]

We denote the set of all sub-intervals of \( Q_N \) by \( \mathcal{A} \). And we define the \( N \)-

dimensional discrepancy of the sequence \( x_1, \ldots, x_N \) by
\[ D(x_1, \ldots, x_N) = \sup_{J \in \mathcal{A}} (J) \frac{1}{k} - m(J), \]
where \( (J) \) is the number of elements of the set \( \{ j \mid 1 \leq j \leq k, x_j \in J \} \). Then, the following inequality holds:

**Lemma 10 (Koksma [3], Szüsz [9]).** For any \( h = (h_1, \ldots, h_N) \in \mathbb{Z}^N \) (\( \mathbb{Z} \) is the ring of rational integers), we define
\[ ||h|| = \max_{1 \leq j \leq N} |h_j| \quad \text{and} \quad r(h) = \prod_{j=1}^{N} (\max(|h_j|, 1)). \]

Then, for any positive integer \( m \), we have
\[ D(x_1, \ldots, x_N) \leq 2N^2 3^{N+1} (m^{-1} + \sum_{0 < |h_j| \leq m} r(h)^{-1} k^{-1} \sum_{n=1}^{\infty} \exp(2\pi i \langle h, x_n \rangle)), \]

where \( \langle \cdot, \cdot \rangle \) is the standard inner product of \( \mathbb{R}^N \).

This lemma is a multi-dimensional generalization of the famous inequality of Erdoes and Turan. (See Kuipers-Niederreiter [4].)

Now we start to estimate \( E(T, N, R) \). Let \( C_R(z) \) be the characteristic function of the rectangle \( R \):
\[ C_R(z) = \begin{cases} 1 & \text{if } z \in R, \\ 0 & \text{if } z \not\in R. \end{cases} \]

Then we have
\[ L_N(T, R) = \int_{0}^{T} C_R(f_R(t)) \, dt = \int_{0}^{T} C_R(f_R(n+t)) \, dt + \int_{\mathbb{T}} C_R(f_R(t)) \, dt. \]

Hence
\[ E(T, N, R) = [T]^{-1} \sum_{n=0}^{[T]} C_R(f_R(n+t)) \, dt - W_N(R) + \int_{\mathbb{T}} C_R(f_R(t)) \, dt - (L_N(T, R)[T]) \cdot ([T] - T)/T \]
\[ = \int_{\mathbb{T}} C_R(f_R(t)) \, dt - W_N(R) \, dt + O(T^{-1}). \]

It is easy to see that, if we put
\[ x_n(t) = \left( \left\lfloor -((n+t)/2\pi) \log p_1 \right\rfloor, \ldots, \left\lfloor -((n+t)/2\pi) \log p_N \right\rfloor \right) \]
(where \( \lfloor x \rfloor = x - [x] \)), then \( f_R(n+t) \in R \) if and only if \( x_n(t) \in Q_N(R) \). For any subset \( S \subset Q_N \), we denote the number of elements of the set \( \{ n \mid 0 \leq n \leq [T] \} \)


Lemma 11 ([5]). For \( j = 1, 2, \) we have
\[
|m(R_j) - m(O_N(R))| \leq r^{-\eta(N+1)} N^{(3/2)+2\varepsilon_0}.
\]

On the other hand, in the next section we shall prove the following estimate of \( D_T(t) \):
\[
D_T(t) \leq N^2 3^{N+1} (m^{-1} + D_T),
\]
where \( D_T \) is the term defined in the statement of Proposition 1.

Substituting these results in (7.2), we have
\[
\left[ [T]^{-1} A_j(O_N(R)) - m(O_N(R)) \right] \leq N^2 3^N (m^{-1} + D_T) + r^{-N(N+1)} N^{3/2} + 2\varepsilon_0.
\]

Since the right-hand side of the above is independent of \( t \), by combining this result with (7.1), the assertion of Proposition 1 follows.

Remark. Professor H. Niederreiter kindly informed the author of the existence of his paper [3]. (See also [6], p. 982.) We can apply the result in [3], by virtue of the result proved in [5]. If we use the theorem of Niederreiter–Wills, the result of Proposition 1 can be improved slightly. This improvement, however, has no effect on our final result.

8. Estimation of discrepancies. The purpose of this section is to prove (7.3). By Lemma 10 we have
\[
D_T(t) \leq 2N^2 3^{N+1} (m^{-1} + \sum_{|h| \leq m} r(h)^{-1} \left[ [T]^{-1} \sum_{n=0}^{[T]-1} \exp(2\pi i \langle h, x_n(t) \rangle) \right]).
\]

We first evaluate the exponential sum
\[
S = \sum_{n=0}^{[T]-1} \exp(2\pi i \langle h, x_n(t) \rangle).
\]

We have
\[
[|S| = \left| \exp(-iA) \sum_{n=0}^{[T]-1} \exp(-iAn) \right| \leq 2|1 - \exp(-iA)|,
\]
where \( A = h_1 \log p_1 + \ldots + h_N \log p_N = \log(p_1^{h_1} \cdots p_N^{h_N}) \).

We shall obtain a lower bound of \([1 - e^{-|A|}] \). We first remark that we can assume \( A > 0 \); if \( A < 0 \), then we can use the relation \(|1 - e^{-|A|}| = |1 - e^{-|A|}| \), and the case \( A = 0 \) is impossible because of the uniqueness of the decomposition of integers into prime divisors.

Let \( k \geq 0 \) be the nearest integer to \( A/2\pi \), and put \( \eta = (A/2\pi) - k \).

Namely,
\[
p_1^{h_1} \cdots p_N^{h_N} = e^A = e^{2\pi k + \eta}.
\]
We first assume \( k \geq 1 \). In this case we have

\[
|1 - e^{-1/k}| = |1 - e^{-2\pi i/k}| > |h| = |\pi - A/2k| > |\pi - A/2k|.
\]

On the other hand, there exists a number \( \xi \) between \( \pi \) and \( A/2k \), for which

\[
e^\xi - e^{\xi/2k} = (\pi - A/2k)e^\xi
\]

holds. We claim that \( \xi < (3/2)\pi \). In fact, if \( k > 2 \), then \( A/2\pi > 3/2 \), so we have \( A/2k = \pi A/(A - 2\pi) < \pi A/(A - \pi) \leq 3\pi^2/(3\pi - \pi) = (3/2)\pi \), and this implies \( \xi < (3/2)\pi \). And if \( k = 1 \), then our claim follows from the inequality \( A/2\pi \leq 3/2 \).

Hence, from (8.4), we have \( |e^\xi - e^{\xi/2k}| < |\pi - A/2k| \). Therefore, with (8.3), we obtain

\[
|1 - e^{-1/k}| > |e^\xi - e^{\xi/2k}|.\]

Here we quote a result of M. Waldschmidt on the estimation of transcendental measure of the transcendental number \( e^\xi \).

**Lemma 12** (Waldschmidt [10]). If \( \alpha \) is an algebraic number with degree \( D \) and height \( H \), then

\[
|e^\alpha - 1| > \exp(-CD^2(D + \log H)(D + \log H))^2
\]

for some absolute constant \( C > 0 \).

In our case, \( \alpha = e^{\xi/2k} \) is a real root of the equation \( X^{2k} - p_1 \cdots p_N = 0 \). Hence we can apply the above lemma to the right-hand side of (8.5) with \( D \leq 2k \) and \( H \leq \exp(2\pi k) \). So we have

\[
|1 - e^{-1/k}| > \exp(-Ck^3(\log(k + 1))^2).
\]

By using the estimate

\[
k < |h_1| \log p_1 + \cdots + |h_N| \log p_N \ll |h| \sum_{n=1}^N \log p_n
\]

\[
\ll |h| N \cdot \log N \ll mN \cdot \log N,
\]

we obtain

\[
|1 - e^{-1/k}| > \exp(-C(mN \cdot \log N)^3(\log(mN))^2).
\]

If \( k = 0 \), then \( |1 - e^{-1/k}| > |A| = |e^{-1} - 1| \). We write \( e^\xi = q_1 \cdots q_N = q/g_2 \) with co-prime rational integers \( q_1 \) and \( q_2 \). Since \( e^\xi \neq 1 \), we have

\[
|e^{\xi - 1}| \geq q_2^{-1} \geq (p_1^{\xi_1} \cdots p_N^{\xi_N})^{-1} = \exp(-N|h_1| \log p_1 + \cdots + |h_N| \log p_N).
\]

Hence, with (8.6), we see that the estimate (8.7) also holds in case \( k = 0 \).

Here we note that the author's original proof of (8.7) included an error, which the referee has pointed out. The simple argument to deduce (8.5) in case \( k \geq 1 \) is also suggested by the referee.

By (8.1) and (8.7), we have

\[
D_T(t) \ll N^2 (m^{-1} + T^{-1}) \left( \sum_{0 < |h| \leq n} r(h)^{-1} \times \exp\left( C(mN \cdot \log N)^3(\log(mN))^2 \right) \right).
\]

Finally we evaluate the sum \( \sum r(h)^{-1} \). We define the mapping \( \phi: \mathbb{Z}^N \to \mathbb{Z}^{N-1} \) by \( \phi: h = (h_1, \ldots, h_{N-1}, h_N) \mapsto \phi(h) = h' = (h_1, \ldots, h_{N-1}) \), and put \( ||h'|| = \max_{1 \leq i < N-1} |h_j| \).

We see that if \( ||h'|| \leq m \), then \( ||\phi(h)|| \leq m \). Conversely, for any \( h' = (h_1, \ldots, h_{N-1}) \in \mathbb{Z}^{N-1} \) with the condition \( ||h'|| \leq m \), the number of the elements of the set \( \phi^{-1}(h') \cap \{ h \in \mathbb{Z}^N \; ||h|| \leq m \} \) is \( 2m + 1 \); they are the integer vectors \( h = (h_1, \ldots, h_{N-1}, h_N) \) \((-m \leq h_N \leq m)\). And it is easy to see that

\[
r(h_0)/r(h) = \begin{cases} k & \text{if } 1 \leq |h| \leq m, \\ 1 & \text{if } k = 0. \end{cases}
\]

The above discussion leads to the following relation:

\[
\sum_{||h|| \leq m} r(h)^{-1} = \sum_{||h'|| \leq m} r(h')^{-1} (1 + 2(1 + 2^{-1} + 3^{-1} + \cdots + m^{-1})) 
\]

\[
\ll (\sum_{||h'|| \leq m} r(h')^{-1}) (3 + 2 \cdot \log m).
\]

We apply this argument repeatedly, and get

\[
\sum_{||h|| \leq m} r(h)^{-1} \ll (3 + 2 \cdot \log m)^N.
\]

Substituting this inequality in (8.8), we obtain the result of (7.3). This completes the proof of Proposition 1.

Remark. The referee pointed out to the author that the inequality (8.9) can be proved in a few lines. See [4], p. 155.

**9. Proof of Proposition 2.** Now the only task remaining to us is to prove Proposition 2. Let \( A = R - R_1 \), and we evaluate the measure of \( \Omega_N(A) = \{ (\theta_1, \ldots, \theta_N) \in Q \mid S_N(\theta_1, \ldots, \theta_N) \in A \} \).

Let \( \Sigma_{2,N} = \{ z_1 + \ldots + z_N \mid z_1 \in \Gamma_1, \ldots, z_N \in \Gamma_N \} \). Then

\[
\Sigma_N = \Gamma_1 + \Sigma_{2,N} = \{ z_1 + z \mid z_1 \in \Gamma_1, z \in \Sigma_{2,N} \}.
\]

For any \( z \in \Sigma_{2,N} \), we put

\[
\Theta(z) = m \{ \theta_1 \mid 0 \leq \theta_1 < 1, z_1(\theta_1) + z \in A \}.
\]

Then it is obvious that

\[
m(\Omega_N(A)) = \int_0^1 \ldots \int_0^1 \Theta(z_1(\theta_2) + \ldots + z_N(\theta_N)) d\theta_2 \ldots d\theta_N.
\]
The set \( z + \Gamma_1 \) is a closed convex curve, and it intersects with \( \Delta \) only several times. We consider one of such intersections of \( z + \Gamma_1 \) and \( \Delta \). This intersection is an arc of the curve \( z + \Gamma_1 \). We denote the end points of this arc \( \Gamma' \) by \( z + z_1(\theta_0) \) and \( z + z_1(\theta_1) \); \( \Gamma' = [z + z_1(\theta_1), z + z_1(\theta_0)] \). We can easily show that \( \text{dist}(z + z_1(\theta_0), z + z_1(\theta_1)) \leq \delta_{\Gamma'}^2 \) in a similar (in fact, simpler) way as in the proof of Lemma 8. Hence, by Lemma 9, we have \( |\theta_1 - \theta_0| \leq \delta_{\Gamma'}^2 \) and therefore, \( \Theta(z) \leq \delta_{\Gamma'}^2 \). Hence, by (9.1), we obtain

\[
|W_{\nu}(R_0) - W_{\nu}(R)| \leq \delta_{\Gamma'}^2 \leq N^{(1 - \sigma_0)/2} (\log N)^{-\sigma_0^2/2}.
\]

This implies Proposition 2, since the same estimate holds for \( |W_{\nu}(R_0) - W_{\nu}(R)| \). The proof of our theorem is thus completed.

References


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Pisot sequences which satisfy no linear recurrence II

by

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Introduction. In this paper we continue our study of Pisot sequences begun in [1]. Recall that the Pisot sequence \( E(a_0, a_1) \) is the sequence of integers defined for \( 0 < a_0 < a_1 \) by

\[
-1/2 < a_{n+1} - a_n^2/a_{n-1} \leq 1/2.
\]

In [1] we proved that there are Pisot sequences satisfying no linear recurrence relation. Our proof made use of an inequality from Pisot’s thesis [5]. We recently discovered that the constant in this inequality is incorrect. Since it is used at three separate points in [1], it would appear that many of the details in [1] need to be modified.

Our first purpose here is to show that all the theorems of [1] are correct as stated and to indicate the necessary changes in the proofs. To do this, we prove a number of new inequalities for Pisot sequences. Since these should be useful in other investigations we give more general versions than needed to simply repair the proofs of [1].

Our second purpose is to sketch a simplified proof of the main Theorem 4 of [1], avoiding the use of the Kronecker–Weyl theorem. Combining this proof with results from [2] shows, for example, that none of \( E(10899, 1782) \) satisfy a linear recurrence for any odd \( m \).

1. The new inequalities. The notation will be as in [1].

If \( \{a_n\} = E(a_0, a_1) \), write \( \theta_n = a_{n+1} - a_n/a_n \) and \( \phi_n = \inf \{\theta_m; m \geq n\} \) (misprinted in [1] as "sup"). We write \( \theta(a_0, a_1) = \theta = \lim \theta_n \) which always exists. We are interested only in \( \theta > 1 \) for which it is necessary and sufficient that \( a_{n+1} > a_n + \sqrt{a_n^2} \), according to results of Pisot [5] and Flor [4]. Let \( \lambda = \lim a_n / a_n^{1/2} > 0 \), and define \( a_n = \lambda^a - a_n \).

**Lemma 1.** For all \( n \geq 0 \),

\[
|\theta - \lambda| < 1/(2a_n(\phi_n - 1)), \quad \phi_n > 1.
\]

\[
|\phi_n| < 1/(2(\theta - 1)(\phi_n - 1)).
\]

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