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Discrepancy estimates for the value-distribution of the Riemann zeta-function I

by

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1. Introduction. The purpose of this paper is to refine a theorem of Bohr–Jessen related with the value-distribution of the Riemann zeta-function.

Let $s = \sigma + it$ be a complex variable, and $\zeta(s)$ the Riemann zeta-function. Bohr–Jessen [1] discussed the value-distribution of $\zeta(s)$ on the fixed line $\sigma = \sigma_0$ (> 1). Let R be any closed rectangle in the complex z-plane with the edges parallel to the axes, and $L(T, R)$ the (Jordan) measure of the set $\{t \in [0, T] \mid \log \zeta(\sigma_0 + it) \in R\}$. Then, Bohr–Jessen proved that there exists the limit

$$W(R) = \lim_{T \to \infty} L(T, R)/T, \tag{1.1}$$

which depends only on σ_0 and R. We can consider $W(R)$ as a "probability" of how many values of $\log \zeta(s)$ on the line $\sigma = \sigma_0$ belong to the rectangle R.

We cannot obtain any quantitative version of (1.1) by Bohr–Jessen's method only. Hence, we will introduce the method of discrepancies, and will prove the following sharpening of (1.1):

Theorem. *For any $\varepsilon > 0$,*

$$L(T, R) = W(R)\,T + O\left((m(R) + \varepsilon)\,T(\log\log T)^{-(\sigma_0 - 1)/7 + \varepsilon}\right) \tag{1.2}$$

where $m(R)$ is the measure of the rectangle R, and the O-constant depends only on σ_0 and ε.

In [1], Bohr–Jessen used the celebrated Kronecker–Weyl theorem on the uniform distribution of sequences. We put

$$f_N(t) = -\sum_{n=1}^{N} \log\left(1 - p_n^{-\sigma_0} \exp(-it \cdot \log p_n)\right),$$

where p_n is the nth prime number. Of course, $f_N(t)$ tends to the function $\log \zeta(\sigma_0 + it)$ as N tends to infinity. Then, by using the Kronecker–Weyl theorem, Bohr–Jessen proved that there exists the limit

$$W_N(R) = \lim_{T\to\infty} L_N(T, R)/T, \tag{1.3}$$

where $L_N(T, R)$ is the measure of the set $\{t \in [0, T] \mid f_N(t) \in R\}$. Hence, our refinement will require essentially the information on discrepancies – the information on how rapidly $L_N(T, R)/T$ converges to the value $W_N(R)$. We use the multi-dimensional version of the Erdös–Turán inequality, which was proved by J. F. Koksma [3] and P. Szüsz [9]. (See Kuipers–Niederreiter [4], Chap. 2. A further generalization is proved by Niederreiter–Philipp [7].) Combining this inequality with a result on transcendental number theory (Waldschmidt [10]), we can deduce the estimate of

$$E(T, N, R) = L_N(T, R)/T - W_N(R).$$

We note that a geometric lemma is indispensable in order to apply the Koksma–Szüsz inequality to our case. Such a lemma was already shown by T. Miyazaki and the author [5].

Furthermore, we shall prove an upper-bound estimate of $|W_N(R) - W(R)|$. To prove this estimate, we must refine Bohr–Jessen's theory on the sums of convex curves [2], and the details of our argument are rather complicated. These estimates of $E(T, N, R)$ and $|W_N(R) - W(R)|$ lead to the result of our theorem.

Throughout this paper, the rectangles we consider are closed and have the edges parallel to the axes. For any subset X of (real or complex) Euclidean space, we denote the boundary of X by ∂X. By $m(X)$ we mean the Jordan measure of X. The symbol dist(,) signifies the usual Euclidean metric. The O-constants depend only on σ_0 in the following sections. Also, C denotes a positive constant, which depends only on σ_0 and is not necessarily the same in each occurrence.

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2. Outline of the proof. Let R be the given rectangle, δ_N a small positive number, and $A_p + iB_q$ ($1 \leqslant p, q \leqslant 2$, $i = \sqrt{-1}$, $A_1 < A_2$, $B_1 < B_2$) the four vertices of R:

$$R = \{z \mid A_1 \leqslant \operatorname{Re} z \leqslant A_2,\ B_1 \leqslant \operatorname{Im} z \leqslant B_2\}.$$

We consider two rectangles R_i and R_y defined by

$$R_i = \{z \mid A_1 + \delta_N \leqslant \operatorname{Re} z \leqslant A_2 - \delta_N,\ B_1 + \delta_N \leqslant \operatorname{Im} z \leqslant B_2 - \delta_N\}$$

and

$$R_y = \{z \mid A_1 - \delta_N \leqslant \operatorname{Re} z \leqslant A_2 + \delta_N,\ B_1 - \delta_N \leqslant \operatorname{Im} z \leqslant B_2 + \delta_N\}.$$

If δ_N satisfies

$$|\log \zeta(\sigma_0 + it) - f_N(t)| < \delta_N \tag{2.1}$$

for any real t, then we see that, if $\log \zeta(\sigma_0 + it) \in R$ then $f_N(t) \in R_y$, and, if $f_N(t) \in R_i$ then $\log \zeta(\sigma_0 + it) \in R$. Hence we obtain

$$L_N(T, R_i) \leqslant L(T, R) \leqslant L_N(T, R_y). \tag{2.2}$$

Now we specify the value of δ_N. We have

$$\begin{aligned} &|\log \zeta(\sigma_0 + it) - f_N(t)| \\ &\quad = \Big|\sum_{n=N+1}^{\infty} \log\big(1 - p_n^{-\sigma_0} \exp(-it \cdot \log p_n)\big)\Big| \\ &\quad \ll \sum_{n=N+1}^{\infty} p_n^{-\sigma_0} \ll \sum_{n=N+1}^{\infty} (n \cdot \log n)^{-\sigma_0} \\ &\quad \ll (\log N)^{-\sigma_0} \int_N^{\infty} x^{-\sigma_0}\, dx \ll N^{1-\sigma_0} (\log N)^{-\sigma_0}. \end{aligned} \tag{2.3}$$

So we can choose $\delta_N = CN^{1-\sigma_0}(\log N)^{-\sigma_0}$ for some constant C. Under this choice of δ_N, the inequalities (2.1) and (2.2) hold.

We already mentioned in Section 1 that we shall obtain an estimate for $E(T, N, R)$, by using the Koksma–Szüsz inequality. In fact, we obtain later the following estimate:

Proposition 1. *For any large positive integers m and r, we have*

$$E(T, N, R) \ll N^2 (3r)^N (m^{-1} + D_T') + r^{-N/(N+1)} N^{(3/2)+2\sigma_0} + T^{-1} \tag{2.4}$$

$$(= A_1 + A_2 + T^{-1},\ \textit{say})$$

where

$$D_T' = T^{-1} (3 + 2 \cdot \log m)^N \exp\big(C (mN \cdot \log N)^3 (\log(mN))^2\big).$$

Since the right-hand side of (2.4) is independent of R, we can apply this inequality to R_i and R_y, and get

$$L_N(T, R_i)/T - W_N(R_i) \ll A_1 + A_2 + T^{-1}, \tag{2.5}$$

$$L_N(T, R_y)/T - W_N(R_y) \ll A_1 + A_2 + T^{-1}. \tag{2.6}$$

Next we treat the evaluation of $|W_N(R_i) - W_N(R)|$ and $|W_N(R_y) - W_N(R)|$. We first introduce some notations. Let Q_N be the real N-dimensional unit cube: $Q_N = \{(\theta_1, \ldots, \theta_N) \mid 0 \leqslant \theta_n < 1\}$. We define the mapping S_N from Q_N to the complex z-plane by

$$S_N\colon (\theta_1, \ldots, \theta_N) \mapsto -\sum_{n=1}^{N} \log(1 - p_n^{-\sigma_0} \exp(2\pi i \theta_n)),$$

and we put

$$\Omega_N(R) = \{(\theta_1, \ldots, \theta_N) \in Q_N \mid S_N(\theta_1, \ldots, \theta_N) \in R\}.$$

Applying Kronecker–Weyl's theorem, we can show that $L_N(T, R)/T$ tends to $m(\Omega_N(R))$ as T tends to infinity. This, so far, is an outline of Bohr–Jessen's proof of (1.3). In particular, we see $W_N(R) = m(\Omega_N(R))$. Hence, our present problem is reduced to a geometric one. We will compare the volume of $\Omega_N(R_i)$, $\Omega_N(R_y)$ and $\Omega_N(R)$, and will show the following result:

PROPOSITION 2. *If we choose* $\delta_N = CN^{1-\sigma_0}(\log N)^{-\sigma_0}$, *then*

$$W_N(R_i) - W_N(R) \ll N^{(1-\sigma_0)/2}(\log N)^{-\sigma_0/2} \quad (= A_3,\ say),$$

and the same estimate holds for $W_N(R_y) - W_N(R)$.

Combining (2.2), (2.5) and (2.6) with the above proposition, we have

$$L(T, R)/T - W_N(R) \ll A_1 + A_2 + A_3 + T^{-1}. \tag{2.7}$$

Now we proceed to the estimation of $|W_N(R) - W(R)|$. Bohr–Jessen [2] showed the following results:

(1) For any sufficiently large N, there is a function $F_N(z)$, continuous in the whole plane, for which

$$W_N(R) = \iint_R F_N(z)\,dx\,dy \quad (z = x + iy) \tag{2.8}$$

holds for any rectangle R.

(2) $F_N(z)$ converges uniformly to a continuous function $F(z)$ (as $N \to \infty$), and

$$W(R) = \iint_R F(z)\,dx\,dy. \tag{2.9}$$

In [2], Bohr–Jessen constructed concretely the functions $F_N(z)$. In this paper, we will investigate their construction in detail, and will obtain the following

PROPOSITION 3.

$$F_N(z) - F(z) \ll N^{(1-\sigma_0)/7}(\log N)^{1-(\sigma_0/7)} \quad (= A_4,\ say)$$

holds uniformly in the whole plane.

Hence we have

$$|W_N(R) - W(R)| \leqslant \iint_R |F_N(z) - F(z)|\,dx\,dy \ll m(R) A_4.$$

This inequality and (2.7) show that

$$L(T, R)/T - W(R) \ll A_1 + A_2 + A_3 + m(R) A_4 + T^{-1}.$$

Lastly we choose the values of the parameters m, r and N, and obtain the result of the theorem. We first note that it is obvious that $A_3 \ll A_4$. We can decide the value of m by the equation $m^{-1} = D'_T$, then decide r by requiring $A_1 = A_2$, and finally decide N by requiring $A_2 = A_4$. This is the author's original method, but it requires rather complicated calculations. Here we show the following simple choice of the parameters, which is due to Mr. S. Egami (by a private communication to the author, April 20, 1984).

Let $a = \frac{1}{2} + 3\sigma_0$, $m = [\frac{1}{2}(\log T)^{1/4}]$ (where $[x]$ is the integer part of x), $N = [(6a \cdot \log\log m)^{-1} \log m]$ and $r = [N^a]$. Then we have

$$D'_T \ll T^{-1} \exp(C \cdot \log m + C(m \cdot \log m)^3 (\log m)^2) \ll T^{-1} \exp(m^4) \ll T^{-15/16},$$

$$N^2 (3r)^N \ll \exp(2aN \cdot \log N) \ll \exp(\tfrac{1}{2} \cdot \log m) \ll (\log T)^{1/8}.$$

So,

$$A_1 \ll (\log T)^{1/8}((\log T)^{-1/4} + T^{-15/16}) \ll (\log T)^{-1/8}.$$

Next we have

$$A_2 \ll N^{-a+(3/2)+2\sigma_0} \ll (\log\log\log T/\log\log T)^{\sigma_0 - 1},$$

$$A_3 \ll (\log\log T)^{-(\sigma_0-1)/2} (\log\log\log T)^{-1/2},$$

$$A_4 \ll (\log\log\log T)^{6/7} (\log\log T)^{-(\sigma_0-1)/7},$$

and these estimates lead to the theorem. In fact, the above estimate yields a slightly stronger result stated as follows:

$$\begin{aligned} L(T, R) = W(R)T + O\big(m(R)\,T \cdot \exp\{&-((\sigma_0 - 1)/7)\log\log\log T + \\ &+ (6/7)\log\log\log\log T + C\} + \\ &+ T(\log\log T)^{-(\sigma_0-1)/2}(\log\log\log T)^{-1/2}\big). \end{aligned} \tag{2.10}$$

We remark that if we follow the original proof of the author, we can write down explicitly the value of the constant C, appearing in the right-hand side of (2.10). An elementary (but complicated) calculation shows that

$$C \leqslant ((\sigma_0 - 1)/7) \cdot \log((90\sigma_0 + 57)/14) + O(\log\log\log\log T/\log\log\log T).$$

Now our remaining task is to prove the Propositions 1, 2 and 3.

Before starting the proof of these propositions, we must sketch Bohr–Jessen's theory developed in [2]. Then we will start with the proof of Proposition 3, which will be completed in Sections 5 and 6. Next, Sections 7 and 8 will be devoted to the proof of Proposition 1. And in the last section we shall prove Proposition 2.

3. Sketch of Bohr–Jessen's theory on the sums of convex curves. In this and the next section we sketch Bohr–Jessen's theory [2] on the sums of convex curves, and quote some of the lemmas proved in [2], which we shall use later.

Let

$$\Gamma_n = \{z_n = z_n(\theta_n) = -\log(1-q_n^{-\sigma_0}\exp(2\pi i\theta_n)) \mid 0 \leq \theta_n < 1\}$$

for some positive integer q_n. Then, Γ_n describes a closed convex Jordan curve in z-plane. We can consider θ_n as a function of z_n defined on Γ_n. If we put $f_n(z_n) = |d\theta_n(z_n)/dz_n|$, then Bohr–Jessen proved that

$$f_n(z_n) = |1-q_n^{-\sigma_0}\exp(2\pi i\theta_n)|/2\pi q_n^{-\sigma_0}. \tag{3.1}$$

We remark that

$$d\theta_n = f_n(z_n)\cdot|dz_n| \tag{3.2}$$

on the curve Γ_n.

To study the geometric properties of Γ_n, we introduce some general notions on closed convex curves. At first we define the "inner radius" and the "outer radius" of a closed convex Jordan curve Γ as follows. For any point z on Γ, we denote by L_z the tangent line of Γ at z. The curve Γ separates the z-plane into three parts: A bounded open set $I(\Gamma)$, another open set $Y(\Gamma)$, and Γ itself. Let

$$C_i = \left\{C\colon \text{circle} \;\middle|\; \begin{array}{l} z\in C, \text{ the tangent line to } C \text{ at} \\ z \text{ is } L_z, \text{ and } C\subset I(\Gamma)\cup\Gamma \end{array}\right\},$$

$$C_y = \left\{C\colon \text{circle} \;\middle|\; \begin{array}{l} z\in C, \text{ the tangent line to } C \text{ at} \\ z \text{ is } L_z, \text{ and } \Gamma\subset I(C)\cup C \end{array}\right\}.$$

By $\varrho(C)$ we mean the radius of the circle C. Now we define the inner radius $r_i(\Gamma)$ and the outer radius $r_y(\Gamma)$ by

$$r_i(\Gamma) = \inf_{z\in\Gamma}\sup_{C\in C_i}\varrho(C), \qquad r_y(\Gamma) = \sup_{z\in\Gamma}\inf_{C\in C_y}\varrho(C).$$

Then Bohr–Jessen showed that

$$r_i(\Gamma_n) = q_n^{-\sigma_0}, \quad r_y(\Gamma_n) = q_n^{-\sigma_0}/(1-q_n^{-2\sigma_0})^{1/2}. \tag{3.3}$$

Next we define the "parallel curves" of the convex curve Γ. Let $z\in\Gamma$, and $\boldsymbol{n}(z)$ be the unit normal vector of Γ at z, oriented to the outside of Γ. We put $z(\delta) = z+\delta\cdot\boldsymbol{n}$ for any real δ. Then, we define the parallel curve of Γ with the distance δ by $\Gamma(\delta) = \{z(\delta) \mid z\in\Gamma\}$. Then, $\Gamma(\delta)$ is again a closed convex curve, and it can be shown that

$$r_i(\Gamma(\delta)) = r_i(\Gamma)+\delta, \quad r_y(\Gamma(\delta)) = r_y(\Gamma)+\delta. \tag{3.4}$$

Now we discuss the "sum" Σ of two convex curves Γ and Γ^*: $\Sigma = \{z+z^* \mid z\in\Gamma,\ z^*\in\Gamma^*\}$. We assume that

$$r_y(\Gamma) \geq r_i(\Gamma) > r_y(\Gamma^*) \geq r_i(\Gamma^*). \tag{3.5}$$

Then we have that $\partial\Sigma$ consists of two convex curves Γ_i and Γ_y, satisfying $\Gamma_i\subset I(\Gamma_y)$, and $\Sigma = \Gamma_i\cup(I(\Gamma_y)\cap Y(\Gamma_i))\cup\Gamma_y$. For any point z, we denote the set $\{z-w \mid w\in\Gamma\}$ by $z-\Gamma$. The convex curves Γ_i and Γ_y have the following properties:

$$I(\Gamma_i)\cup Y(\Gamma_y) = \{z \mid \Gamma\cap(z-\Gamma^*) = \emptyset\},$$

$$\Gamma_i\cup\Gamma_y = \{z \mid \Gamma\cap(z-\Gamma^*) \text{ consists of one point}\},$$

and

$$Y(\Gamma_i)\cap I(\Gamma_y) = \{z \mid \Gamma\cap(z-\Gamma^*) \text{ consists of two points}\}.$$

And also we have

Lemma 1 (Bohr–Jessen [2], § 18).

$$r_i(\Gamma)-r_y(\Gamma^*) \leq r_i(\Gamma_i), \quad r_y(\Gamma_i) \leq r_y(\Gamma)-r_i(\Gamma^*),$$
$$r_i(\Gamma)+r_i(\Gamma^*) \leq r_i(\Gamma_y), \quad r_y(\Gamma_y) \leq r_y(\Gamma)+r_y(\Gamma^*).$$

Lemma 2 (Bohr–Jessen [2], § 19). *Let $z\in Y(\Gamma_i)\cap I(\Gamma_y)$, and we put*

$$d = \min(\operatorname{dist}(z,\Gamma_i),\ \operatorname{dist}(z,\Gamma_y))$$

(where $\operatorname{dist}(z,\Gamma) = \inf\{\operatorname{dist}(z,w) \mid w\in\Gamma\}$). The set $\Gamma\cap(z-\Gamma^*)$ consists of two points z', z''. Let L, L^* be the tangent lines to Γ and $z-\Gamma^*$ respectively, at z'. If L and L^* cross each other at z' with angle θ', then we call θ' the cross angle of Γ and $z-\Gamma^*$ at z'. Similarly, we denote by θ'' the cross angle of Γ and $z-\Gamma^*$ at z''. Then

$$d^{1/2}/\sin\theta' < \left(2r_i(\Gamma)r_y(\Gamma^*)/(r_i(\Gamma)-r_y(\Gamma^*))\right)^{1/2}$$

and

$$d^{1/2}/\sin\theta'' < \left(2r_i(\Gamma)r_y(\Gamma^*)/(r_i(\Gamma)-r_y(\Gamma^*))\right)^{1/2}.$$

This Lemma 2 is quite useful in the sections below.

Now we return to the convex curves Γ_n, introduced at the beginning of this section. We put

$$\Sigma_N = \{z_1+z_2+\ldots+z_N \mid z_j\in\Gamma_j\ (1\leq j\leq N)\}.$$

Then, for any N, one of the following two cases happens:

Case I. The boundary $\partial\Sigma_N$ consists of one convex curve $\Gamma_{y,N}$ and

$$\Sigma_N = I(\Gamma_{y,N})\cup\Gamma_{y,N}.$$

Case II. The boundary $\partial\Sigma_N$ consists of two convex curves $\Gamma_{y,N}$ and $\Gamma_{i,N}$ and

$$\Sigma_N = \Gamma_{i,N} \cup (I(\Gamma_{y,N}) \cap Y(\Gamma_{i,N})) \cup \Gamma_{y,N}.$$

Now we decide the values of q_n. We can choose the primes $p_{m(1)}, p_{m(2)}, p_{m(3)}$ and $p_{m(4)}$ for which the inequality

$$r_i(\Gamma_k) \geqslant 2r_y(\Gamma_{k+1}) \tag{3.6}$$

holds for $k = 1, 2, 3$, where $q_k = p_{m(k)}$ $(1 \leqslant k \leqslant 4)$. We note that the choices of the values of $m(k)$ depend only on σ_0. (See (3.3).) Further we define $q_n = p_{n-3}$ for $5 \leqslant n \leqslant m(2)+2$, $q_n = p_{n-2}$ for $m(2)+3 \leqslant n \leqslant m(3)+1$, $q_n = p_{n-1}$ for $m(3)+2 \leqslant n \leqslant m(4)$, and $q_n = p_n$ for $n \geqslant m(4)+1$ $(= N_0$, say). Then, for any $N \geqslant N_0$, Σ_N is the range of the values of the mapping S_N; $\Sigma_N = S_N(Q_N)$. Under the condition (3.6), Case II holds for Σ_2, Σ_3 and Σ_4.

Our aim is to construct the continuous function F_N which satisfies the relation (2.8). We put

$$S'_N\colon (\theta_1, \ldots, \theta_N) \mapsto -\sum_{n=1}^{N} \log(1-q_n^{-\sigma_0}\exp(2\pi i\theta_n))$$

and

$$\Omega'_N(R) = \{(\theta_1, \ldots, \theta_N) \in Q^N \mid S'_N(\theta_1, \ldots, \theta_N) \in R\}.$$

Then it is obvious that $m(\Omega'_N(R)) = W_N(R)$ for $N \geqslant N_0$. Bohr–Jessen constructed concretely the continuous function $F_4(z)$, which satisfies

$$m(\Omega'_N(R)) = \iint_R F_N(z)\,dx\,dy \tag{3.7}$$

for $N = 4$. Then we define inductively

$$F_N(z) = \int_0^1 F_{N-1}(z - z_N(\theta_N))\,d\theta_N \tag{3.8}$$

for $N \geqslant 5$. We can easily show that each of these F_N's again satisfies (3.7), and therefore, satisfies (2.8) as well for $N \geqslant N_0$. In the next section, we sketch Bohr–Jessen's method to construct the function F_4.

4. Bohr–Jessen's construction of the functions F_2, F_3 and F_4. We remark first of all that our F_2, F_3, F_4 are written as F_1, F_2, F_3 in Bohr–Jessen's paper.

Let $\Gamma_i = \Gamma_{i,2}$, $\Gamma_y = \Gamma_{y,2}$. For any $z \in Y(\Gamma_i) \cap I(\Gamma_y)$, the set $\Gamma_1 \cap (z - \Gamma_2)$ consists of two points z'_1 and z''_1. Then, $z'_2 = z - z'_1$ and $z''_2 = z - z''_1$ are on Γ_2. We denote the cross angles of Γ_1 and $z - \Gamma_2$ at z'_1 and z''_1 by θ' and θ'', respectively. (We can assume that $0 \leqslant \theta' \leqslant \pi/2$, $0 \leqslant \theta'' \leqslant \pi/2$. We always keep this assumption in the following.) We define the function $F_2(z)$ as follows:

$$F_2(z) = \begin{cases} 0 & \text{if } z \in I(\Gamma_i) \cup Y(\Gamma_y), \\ \infty & \text{if } z \in \Gamma_i \cup \Gamma_y, \\ (\sin\theta')^{-1} f_1(z'_1) f_2(z'_2) + (\sin\theta'')^{-1} f_1(z''_1) f_2(z''_2) & \\ \qquad\qquad \text{if } z \in Y(\Gamma_i) \cap I(\Gamma_y). \end{cases}$$

Then we have

Lemma 3 (Bohr–Jessen [2], § 31). *For any $z \notin \Gamma_i \cup \Gamma_y$,*

$$F_2(z) \ll d^{-1/2}$$

holds, where $d = \min(\mathrm{dist}(z, \Gamma_i), \mathrm{dist}(z, \Gamma_y))$.

We "cut off" the values of $F_2(z)$ if z is close to the curve Γ_i or Γ_y, and define the function $F_2^d(z)$. For small positive d, we define

$$F_2^d(z) = \begin{cases} F_2(z) & \text{if } \mathrm{dist}(z, \Gamma_i) > d \text{ and } \mathrm{dist}(z, \Gamma_y) > d, \\ 0 & \text{otherwise}. \end{cases}$$

And further, we define

$$F_3^d(z) = \int_0^1 F_2^d(z - z_3(\theta_3))\,d\theta_3.$$

Then, $F_3^d(z)$ tends to the function

$$F_3(z) = \int_0^1 F_2(z - z_3(\theta_3))\,d\theta_3$$

as d tends to 0. To estimate the speed of this convergence, we first investigate the shape of the set Σ_3. The boundary $\partial\Sigma_3$ consists of two convex curves $\Gamma_{ii} = \Gamma_{i,3}$ and $\Gamma_{yy} = \Gamma_{y,3}$, and Σ_3 contains other two convex curves Γ_{iy} and Γ_{yi}. ($\Gamma_{ii} \subset I(\Gamma_{iy})$, $\Gamma_{iy} \subset I(\Gamma_{yi})$, $\Gamma_{yi} \subset I(\Gamma_{yy})$.) These curves have the following properties:

(1) If $z \in I(\Gamma_{ii}) \cup Y(\Gamma_{yy})$, then $\Sigma_2 \cap (z - \Gamma_3) = \emptyset$.

(2) If $z \in \Gamma_{ii}$, then $\Sigma_2 \cap (z - \Gamma_3)$ consists of one point z', and $z' \in \Gamma_i$. Similarly, if $z \in \Gamma_{yy}$, then $\Sigma_2 \cap (z - \Gamma_3)$ consists of one point z', and $z' \in \Gamma_y$.

(3) If $z \in Y(\Gamma_{ii}) \cap I(\Gamma_{iy})$, then $\Gamma_i \cap (z - \Gamma_3)$ consists of two points. Similarly, if $z \in I(\Gamma_{yy}) \cap Y(\Gamma_{yi})$, then $\Gamma_y \cap (z - \Gamma_3)$ consists of two points.

(4) If $z \in \Gamma_{iy}$, then $\Gamma_i \cap (z - \Gamma_3)$ consists of one point z', and similarly, if $z \in \Gamma_{yi}$, then $\Gamma_y \cap (z - \Gamma_3)$ consists of one point z'. In both of these cases, $(z - \Gamma_3) - \{z'\} \subset Y(\Gamma_i) \cap I(\Gamma_y)$.

(5) If $z \in Y(\Gamma_{iy}) \cap I(\Gamma_{yi})$, then $z - \Gamma_3 \subset Y(\Gamma_i) \cap I(\Gamma_y)$.

Let a be a small positive number with the condition $a \geqslant 2d$, and put

$$A = \{z \mid \mathrm{dist}(z, \Gamma_{ii}) \geqslant a, \mathrm{dist}(z, \Gamma_{iy}) \geqslant a, \mathrm{dist}(z, \Gamma_{yi}) \geqslant a, \mathrm{dist}(z, \Gamma_{yy}) \geqslant a\}.$$

Then, for sufficiently small a, the set A consists of five closed subsets

$$A_{ii} = A \cap I(\Gamma_{ii}), \quad A_{iy} = A \cap Y(\Gamma_{ii}) \cap I(\Gamma_{iy}),$$

$$A^* = A \cap Y(\Gamma_{iy}) \cap I(\Gamma_{yi}), \quad A_{yi} = A \cap Y(\Gamma_{yi}) \cap I(\Gamma_{yy}), \quad A_{yy} = A \cap Y(\Gamma_{yy}).$$

Let $z \in A_{iy}$. Then, $\Gamma_i \cap (z-\Gamma_3)$ consists of two points. Furthermore, since $z \in A$, $\Gamma_i(\delta) \cap (z-\Gamma_3)$ consists of two points z' and z'', for any positive $\delta \leqslant d$. We denote the cross angles of $\Gamma_i(\delta)$ and $z-\Gamma_3$ at z' and z'' by $\theta'(\delta)$ and $\theta''(\delta)$, respectively, and put

$$g_{iy} = \min\{\inf_{\substack{z \in A_{iy} \\ 0 \leqslant \delta \leqslant a/2}} (\sin \theta'(\delta)), \inf_{\substack{z \in A_{iy} \\ 0 \leqslant \delta \leqslant a/2}} (\sin \theta''(\delta))\}.$$

Also we define g_{yi} in a similar way. Then we have

Lemma 4 (Bohr–Jessen [2], § 33). *If $z \in A_{iy}$, then*

$$F_3(z) - F_3^d(z) \ll d^{1/2}/g_{iy}.$$

If $z \in A_{yi}$, then

$$F_3(z) - F_3^d(z) \ll d^{1/2}/g_{yi}.$$

If $z \in A_{ii} \cup A^* \cup A_{yy}$, then

$$F_3(z) = F_3^d(z).$$

Now we arrive at the stage to define the function $F_4(z)$. We define

$$F_4^d(z) = \int_0^1 F_3^d(z - z_4(\theta_4))\, d\theta_4$$

and

$$F_4(z) = \int_0^1 F_3(z - z_4(\theta_4))\, d\theta_4.$$

By using the definition of $F_2(z)$, we can easily prove (3.7) for $N = 2$. However, F_2 is not a continuous function, so we integrate F_2 twice to secure the continuity. In fact, we can show that the continuous function $F_4^d(z)$ converges to $F_4(z)$ uniformly in the whole plane as d tends to 0. Hence F_4 is continuous, and it can be proved that this F_4 satisfies (3.7) for $N = 4$. Therefore, this $F_4(z)$ is our desired function.

We quote a lemma of Bohr–Jessen related with the difference between $F_4^d(z)$ and $F_4(z)$. Let

$$\Omega_{ii}(a) = \{z \mid z \in \Sigma_3, \operatorname{dist}(z, \Gamma_{ii}) < a\},$$

$$\Omega_{iy}(a) = \{z \mid \operatorname{dist}(z, \Gamma_{iy}) < a\}, \quad \Omega_{yi}(a) = \{z \mid \operatorname{dist}(z, \Gamma_{yi}) < a\},$$

$$\Omega_{yy}(a) = \{z \mid z \in \Sigma_3, \operatorname{dist}(z, \Gamma_{yy}) < a\},$$

$$\Gamma_4(a) = \{z_4 \in \Gamma_4 \mid z - z_4 \in \Omega_{ii}(a) \cup \Omega_{iy}(a) \cup \Omega_{yi}(a) \cup \Omega_{yy}(a)\},$$

$$\Gamma_4^*(a) = \Gamma_4 - \Gamma_4(a).$$

Then, by (3.1) and (3.2) it follows that

$$F_4(z) - F_4^d(z) \ll \int_{\Gamma_4^*(a)} (F_3(z-z_4) - F_3^d(z-z_4))|dz_4| + \int_{\Gamma_4(a)} F_3(z-z_4)|dz_4| \tag{4.1}$$
$$= U_4^*(z) + U_4(z), \quad \text{say}.$$

Further, if we denote the set $\{z - z_4 \mid z_4 \in \Gamma_4(a)\}$ by $\Gamma_c(z)$, then $\Gamma_c(z)$ belongs to only one of the four sets $\Omega_{ii}(a)$, $\Omega_{iy}(a)$, $\Omega_{yi}(a)$ or $\Omega_{yy}(a)$. In case $\Gamma_c(z) \subset \Omega_{iy}(a)$, we define δ_0 and δ_1 by

$$\delta_0 = \inf\{\delta \mid \Gamma_{iy}(\delta) \cap \Gamma_c(z) \neq \emptyset\}, \quad \delta_1 = \sup\{\delta \mid \Gamma_{iy}(\delta) \cap \Gamma_c(z) \neq \emptyset\},$$

and in case $\Gamma_c(z) \subset \Omega_{yi}(a)$, we define δ_0, δ_1 in a similar way. Then we have

Lemma 5 (Bohr–Jessen [2], § 38). *If $\Gamma_c(z) \subset \Omega_{ii}(a)$ or $\Omega_{yy}(a)$, then*

$$U_4(z) \ll a^{1/2}.$$

If $\Gamma_c(z) \subset \Omega_{iy}(a)$ or $\Omega_{yi}(a)$, then

$$U_4(z) \ll (\delta_1 - \delta_0)^{1/2} \log(1/(\delta_1 - \delta_0)).$$

So we can say that for any z, we have

$$U_4(z) \ll a^{1/2} \log(1/a) \tag{4.2}$$

for small a. It is worthwhile noting that Lemma 2 is used essentially in Bohr–Jessen's proof of Lemmas 3, 4 and 5.

Starting with the above function F_4, we can construct the functions F_N ($N \geqslant 5$) by (3.7). We can show that as N tends to infinity, F_N converges to a continuous function F. Let

$$\Sigma_{N,N+m} = \{z_{N+1} + z_{N+2} + \ldots + z_{N+m} \mid z_{N+j} \in \Gamma_{N+j} \ (1 \leqslant j \leqslant m)\},$$

$$\delta_{N,N+m} = \sup_{z \in \Sigma_{N,N+m}} \operatorname{dist}(0, z),$$

and

$$\delta(N) = \sup_m \delta_{N,N+m}. \tag{4.3}$$

Then we have

Lemma 6 (Bohr–Jessen [2], § 41). *For any $N \geqslant N_0$,*

$$\sup_{m,z} |F_N(z) - F_{N+m}(z)| \leqslant \sup_{\operatorname{dist}(z,w) \leqslant \delta(N)} |F_{N_0}(z) - F_{N_0}(w)|.$$

This implies the uniform convergence of $\{F_N\}$. The function $F = \lim F_N$ satisfies the relation (2.9).

5. Outline of the proof of Proposition 3. Now we shall start on the proof of Proposition 3. For any fixed z, by $w = w(z)$ we mean any point satisfying the condition $\operatorname{dist}(w, z) \leqslant \delta(N)$. From Lemma 6 it follows that

$$(5.1)\qquad |F_N(z)-F(z)| \ll \sup_{z,w} |F_{N_0}(z)-F_{N_0}(w)|.$$

By (3.8) we see that

$$(5.2)\qquad |F_n(z)-F_n(w)| \leqslant \int_0^1 \left|F_{n-1}\big(z-z_n(\theta_n)\big)-F_{n-1}\big(w-z_n(\theta_n)\big)\right| d\theta_n .$$

Hence,

$$\sup_{z,w} |F_n(z)-F_n(w)| \leqslant \sup_{z,w} |F_{n-1}(z)-F_{n-1}(w)|.$$

By using this inequality repeatedly, we have from (5.1) that, for any small positive d,

$$(5.3)\qquad F_N(z)-F(z)$$
$$\ll \sup_{z,w} |F_4(z)-F_4(w)|$$
$$\leqslant \sup_{z} |F_4(z)-F_4^d(z)| + \sup_{z,w} |F_4^d(z)-F_4^d(w)| + \sup_{w} |F_4(w)-F_4^d(w)|$$
$$= X_1(z)+X_2+X_1(w), \qquad \text{say}.$$

The estimation of X_2 is the most essential part of the proof of Proposition 3, and we devote the next section to the discussion of this point. The result obtained there states as follows:

$$(5.4)\qquad X_2 \ll d^{-3/2}\,\delta(N)+d^{-1/2}\,\delta(N)^{1/2}.$$

On the other hand, we shall prove in this section that

$$(5.5)\qquad X_1(z),\ X_1(w) \ll d^{1/4}\log(1/d).$$

Hence, if we choose $d = \delta(N)^{4/7}$, then by (5.3), (5.4) and (5.5) we have

$$(5.6)\qquad F_N(z)-F(z) \ll \delta(N)^{1/7}\log\big(1/\delta(N)\big).$$

Now we estimate the value of $\delta(N)$. By (4.3) and (3.3) we have

$$\delta(N) \ll \sum_{n=N+1}^{\infty} r_y(\Gamma_n) \ll \sum_{n=N+1}^{\infty} p_n^{-\sigma_0} \ll N^{1-\sigma_0}(\log N)^{-\sigma_0}$$

for $N \geqslant N_0$. (See (2.3).) Substituting this result in (5.6), we arrive at the result of Proposition 3.

We shall estimate $X_1(z)$ in the rest of this section, by using several lemmas prepared by Bohr and Jessen. The same argument, of course, holds for $X_1(w)$.

Our starting point is the inequality (4.1). This inequality, with (4.2), reduces our problem to the estimation of $U_4^*(z)$. And the integrand of the

term $U_4^*(z)$ was already estimated in Lemma 4. To use the result of Lemma 4, we need a lower bound of g_{iy} and g_{yi}. We can get such a bound by virtue of Lemma 2; we apply Lemma 2 in the case of $\Gamma = \Gamma_i(\delta)$ and $\Gamma^* = \Gamma_3$, where δ is the parameter appearing in the definition of g_{iy}. We remark that $\partial(\Gamma_i+\Gamma_3)$ consists of two convex curves Γ_{ii} and Γ_{iy}. The boundary of the set $\Gamma_i(\delta)+\Gamma_3$ also consists of two convex curves Γ_{ii}^* and Γ_{iy}^*. ($\Gamma_{ii}^* \subset I(\Gamma_{iy}^*)$.) Then we have

LEMMA 7. $\Gamma_{ii}^* = \Gamma_{ii}(\delta)$, $\Gamma_{iy}^* = \Gamma_{iy}(\delta)$.

Proof. We only prove the former; the proof of the latter is similar. Let $z \in \Gamma_{ii}$. Then $\Gamma_i \cap (z-\Gamma_3)$ consists of one point z'. So the curves Γ_i and $z-\Gamma_3$ are tangent to each other at z', and we denote by L' the common tangent line of Γ_i and $z-\Gamma_3$ at z'. Then, L' is parallel to the tangent line L of Γ_{ii} at z. (See Bohr–Jessen [2], § 17.) Hence, for the point $z(\delta) = z+\delta\cdot \boldsymbol{n}(z) \in \Gamma_{ii}(\delta)$, the curves $\Gamma_i(\delta)$ and $z(\delta)-\Gamma_3$ are tangent to each other at $z'(\delta) = z'+\delta\cdot\boldsymbol{n}'(z')$ (where $\boldsymbol{n}$ and $\boldsymbol{n}'$ are the unit normal vector of Γ_{ii} and Γ_i, respectively, see Fig. 1). This implies that $\Gamma_{ii}(\delta) \subset \Gamma_{ii}^*$. Both $\Gamma_{ii}(\delta)$ and Γ_{ii}^* are closed convex curves, so we have $\Gamma_{ii}(\delta) = \Gamma_{ii}^*$. This proves the lemma.

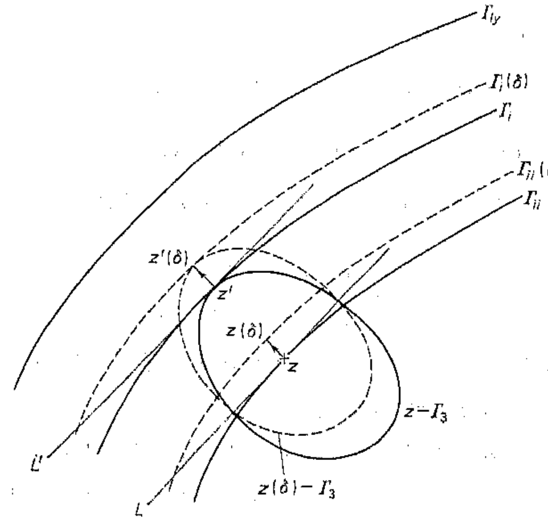

Fig. 1

Now we deduce the lower bound of g_{iy}. If $z \in A_{iy}$, then by Lemma 7, we have $\operatorname{dist}(z, \Gamma_{ii}^*) \geqslant a-\delta \geqslant \frac{1}{2}a$, and similarly $\operatorname{dist}(z, \Gamma_{iy}^*) \geqslant \frac{1}{2}a$. Applying Lemma 2 in this case, we have

$$(5.7)\qquad g_{iy} > \left(\tfrac{1}{2}a\big(r_i(\Gamma_i(\delta))-r_y(\Gamma_3)\big)/2r_i(\Gamma_i(\delta))\,r_y(\Gamma_3)\right)^{1/2}.$$

By (3.4) and Lemma 1, we have

$$r_i(\Gamma_1)-r_y(\Gamma_2)+\delta \leqslant r_i(\Gamma_i(\delta)) \leqslant r_y(\Gamma_1)-r_i(\Gamma_2)+\delta.$$

Since we can assume $r_i(\Gamma_1)-r_y(\Gamma_2)-r_y(\Gamma_3)=C=C(\sigma_0)>0$, by (5.7) we can say that

$$g_{iy} \gg a^{1/2}.$$

And similarly, we can show the estimate $g_{yi} \gg a^{1/2}$. Hence, by using Lemma 4, we have

$$U_4^*(z) \ll a^{-1/2} d^{1/2} \int_{\Gamma_4^*(a)} |dz_4| \ll a^{-1/2} d^{1/2}.$$

From this result and (4.2), we obtain

$$F_4(z)-F_4^d(z) \ll a^{-1/2} d^{1/2} + a^{1/2} \log(1/a). \tag{5.8}$$

Now we choose $a = d^{1/2}$. We remark that this choice satisfies the condition $a \geqslant 2d$ for sufficiently small d. Under this choice of a, (5.8) leads to the estimate $F_4(z)-F_4^d(z) \ll d^{1/4}\log(1/d)$. This completes the proof of (5.5).

6. Estimation of X_2. Our aim in this section is to estimate $|F_4^d(z) - F_4^d(w)|$ under the condition $\operatorname{dist}(z, w) \leqslant \delta(N)$. We first note that by using (5.2) again, we have

$$\begin{aligned} &|F_4^d(z)-F_4^d(w)| \\ &\quad \leqslant \int_{\Gamma_4}\int_{\Gamma_3} |F_2^d(z-z_3-z_4)-F_2^d(w-z_3-z_4)|\, f_3(z_3)\, f_4(z_4)\, |dz_3|\, |dz_4|, \end{aligned} \tag{6.1}$$

so it is sufficient to estimate $|F_2^d(z)-F_2^d(w)|$. Let

$$\Sigma_2^* = \{z \in \Sigma_2 |\ \operatorname{dist}(z, \Gamma_i) > d,\ \operatorname{dist}(z, \Gamma_y) > d\}.$$

Then

$$F_2^d(z) = \begin{cases} F_2(z) & \text{if } z \in \Sigma_2^*, \\ 0 & \text{otherwise.} \end{cases}$$

We first assume that both z and w belong to Σ_2^*. Then we have

$$\begin{aligned} |F_2^d(z)-F_2^d(w)| &= |F_2(z)-F_2(w)| \\ &\leqslant |(\sin\theta')^{-1} f_1(z_1') f_2(z_2') - (\sin\varphi')^{-1} f_1(w_1') f_2(w_2')| + \\ &\quad + |(\sin\theta'')^{-1} f_1(z_1'') f_2(z_2'') - (\sin\varphi'')^{-1} f_1(w_1'') f_2(w_2'')| \\ &= X_2' + X_2'', \quad \text{say.} \end{aligned} \tag{6.2}$$

Here, z_1', z_1'', z_2', z_2'', θ' and θ'' are the same as those which appeared in the definition of $F_2(z)$, and we define w_1', w_1'', w_2', w_2'', φ' and φ'' in a similar way. Furthermore, for X_2' we have

$$\begin{aligned} X_2' &\leqslant |(\sin\theta')^{-1} f_2(z_2')|\, |f_1(z_1')-f_1(w_1')| + \\ &\quad + |(\sin\theta')^{-1} f_1(w_1')|\, |f_2(z_2')-f_2(w_2')| + \\ &\quad + |(\sin\theta' \sin\varphi')^{-1} f_1(w_1') f_2(w_2')|\, |\sin\theta' - \sin\varphi'| \\ &= Y_1+Y_2+Y_3, \quad \text{say.} \end{aligned} \tag{6.3}$$

To estimate the terms Y_1, Y_2 and Y_3, we first prove the following lemma:

LEMMA 8. *If* $\operatorname{dist}(z, w) \leqslant \delta(N)$, *then*

$$\operatorname{dist}(z_1', w_1') \ll d^{-1/2}\delta(N), \qquad \operatorname{dist}(z_2', w_2') \ll d^{-1/2}\delta(N).$$

Proof. Let $z_3 = z_1' + (w-z) \in w-\Gamma_2$. We define the four circles C_1^i, C_1^y, C_2^i, C_2^y as follows (see Fig. 2):

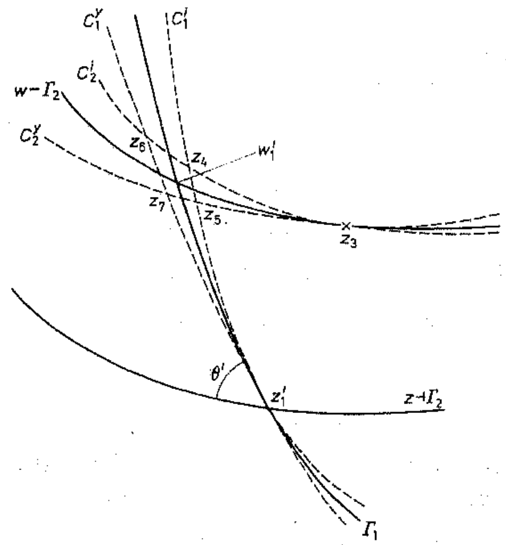

Fig. 2

C_1^i: tangent to Γ_1 at z_1', $C_1^i \subset I(\Gamma_1) \cup \Gamma_1$, and

$$\varrho(C_1^i) = q_1^{-\sigma_0}.$$

C_1^y: tangent to Γ_1 at z_1', $\Gamma_1 \subset I(C_1^y) \cup C_1^y$, and

$$\varrho(C_1^y) = q_1^{-\sigma_0}/(1-q_1^{-2\sigma_0})^{1/2}.$$

C_2^i: tangent to $w-\Gamma_2$ at z_3, $C_2^i \subset I(w-\Gamma_2) \cup (w-\Gamma_2)$, and

$$\varrho(C_2^i) = q_2^{-\sigma_0}.$$

C_2^y: tangent to $w-\Gamma_2$ at z_3, $w-\Gamma_2 \subset I(C_2^y) \cup C_2^y$, and

$$\varrho(C_2^y) = q_2^{-\sigma_0}/(1-q_2^{-2\sigma_0})^{1/2}.$$

We denote then by z_4, z_5, z_6 and z_7 the intersection points of C_1^i and C_2^i, C_1^i and C_2^y, C_1^y and C_2^i, and, C_1^y and C_2^y, respectively. Then, by an elementary calculation, we can show that

$$\operatorname{dist}(z_1', z_j) \ll (\sin\theta')^{-1}\delta(N) \qquad (j = 4, 5, 6, 7).$$

We apply Lemma 2 to $\Gamma = \Gamma_1$ and $\Gamma^* = \Gamma_2$, and obtain that

$$\sin\theta' \gg d^{1/2}, \tag{6.4}$$

so it follows that $\operatorname{dist}(z_1', z_j) \ll d^{-1/2}\delta(N)$. Hence we have that

$$\operatorname{dist}(z_1', w_1') \ll d^{-1/2}\delta(N).$$

And also, since $w_2' - z_2' = \big((w - z_1') - (z - z_1')\big) + (z_1' - w_1')$, it follows that

$$\operatorname{dist}(z_2', w_2') \ll d^{-1/2}\delta(N).$$

This implies the lemma.

Let $z_1' = z_1(\theta)$ and $w_1' = z_1(\varphi)$. Then, by (3.1), we have

$$|f_1(z_1') - f_1(w_1')| = (2\pi q_1^{-\sigma_0})^{-1}\,|\exp(L_z) - \exp(L_w)|,$$

where

$$L_z = \log|1 - q_1^{-\sigma_0}\exp(2\pi i\theta)| \quad \text{and} \quad L_w = \log|1 - q_1^{-\sigma_0}\exp(2\pi i\varphi)|.$$

There exists a real number ξ between L_z and L_w for which

$$|\exp(L_z) - \exp(L_w)| \ll \exp(\xi)\,|L_z - L_w| \ll |L_z - L_w|$$

holds. So, by Lemma 8, it follows that

$$|f_1(z_1') - f_1(w_1')| \ll |L_z - L_w| \leqslant |z_1' - w_1'| \ll d^{-1/2}\delta(N).$$

Hence, by using (6.4), we have that $Y_1 \ll d^{-1}\delta(N)$. Since the same argument holds for Y_2, we now obtain the following estimate:

$$Y_1 + Y_2 \ll d^{-1}\delta(N). \tag{6.5}$$

Next we estimate the term Y_3. Using (6.4) again, we see

$$Y_3 \ll d^{-1}|\sin\theta' - \sin\varphi'| \ll d^{-1}|\theta' - \varphi'|, \tag{6.6}$$

so our problem is reduced to proving the following lemma:

LEMMA 9. *Let $z = z_1(\theta)$ and $w = z_1(\varphi)$ be points on Γ_1, let K_z and K_w be the tangent lines of Γ_1 at z and w, respectively, and let Θ be the cross angle of K_z and K_w. If $\operatorname{dist}(z, w) \leqslant \delta$ for some small positive δ, then it follows that*

$$|\theta - \varphi| \ll \delta \quad \text{and} \quad \Theta \ll \delta.$$

Also, similar results hold for any pair of points z and w on Γ_2.

By Lemma 8 we have $\operatorname{dist}(z_1', w_1') \ll d^{-1/2}\delta(N)$, and also,

$$\operatorname{dist}(z_3, w_1') \leqslant \operatorname{dist}(z_1', w_1') + \operatorname{dist}(z_1', z_3) \ll d^{-1/2}\delta(N).$$

Hence we obtain $|\theta' - \varphi'| \ll d^{-1/2}\delta(N)$ by the above lemma. This result and (6.6) lead to the following estimate:

$$Y_3 \ll d^{-3/2}\delta(N). \tag{6.7}$$

The proof of Lemma 9 is quite elementary, so we omit the details. We only note the fact that we can assume $q_1 = q_1(\sigma_0)$ is sufficiently large.

By (6.3), (6.5) and (6.7) we have

$$X_2' \ll d^{-3/2}\delta(N).$$

The same estimate holds for X_2'', so by (6.2) we obtain

$$|F_2^d(z) - F_2^d(w)| \ll d^{-3/2}\delta(N) \tag{6.8}$$

if both z and w belong to Σ_2^*.

Let

$$z - \Sigma = \{z - z_3 - z_4 \mid z_3 \in \Gamma_3,\ z_4 \in \Gamma_4\},$$
$$\Gamma_i^d = \{z \in \Sigma_2 \mid \operatorname{dist}(z, \Gamma_i) = d\},$$
$$\Gamma_y^d = \{z \in \Sigma_2 \mid \operatorname{dist}(z, \Gamma_y) = d\},$$

and

$$K = \{x \in z - \Sigma \mid \min(\operatorname{dist}(x, \Gamma_i^d), \operatorname{dist}(x, \Gamma_y^d)) \leqslant \delta(N)\}.$$

(See Fig. 3)

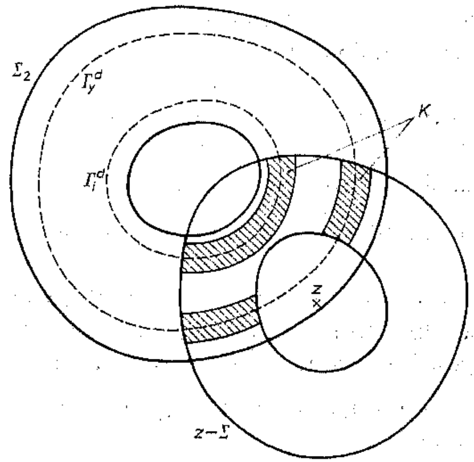

Fig. 3

Let $z_3 \in \Gamma_3$ and $z_4 \in \Gamma_4$. If $z - z_3 - z_4 \in \Sigma_2^* - K$, then $w - z_3 - z_4 \in \Sigma_2^*$, so by (6.8) we have

$$|F_2^d(z - z_3 - z_4) - F_2^d(w - z_3 - z_4)| \ll d^{-3/2}\delta(N).$$

If $z - z_3 - z_4 \notin \Sigma_2^* \cup K$, then $w - z_3 - z_4 \notin \Sigma_2^*$, so

$$F_2^d(z - z_3 - z_4) - F_2^d(w - z_3 - z_4) = 0.$$

Hence, by (6.1) and Lemma 3, it follows that

$$|F_4^d(z)-F_4^d(w)| \leqslant \iint_{z-z_3-z_4\in K} + \iint_{z-z_3-z_4\notin K} \leqslant d^{-1/2}\iint_{z-z_3-z_4\in K} |dz_3|\,|dz_4| + d^{-3/2}\delta(N). \tag{6.9}$$

For any point $u = x+iy \in K$, the set $(z-\Gamma_3)\cap(u+\Gamma_4)$ consists at most of two points. We denote them by $z-z_3'$ and $z-z_3''$. (If $u\in\partial K$, then $z_3' = z_3''$.) Let $\theta'(u)$ and $\theta''(u)$ be the cross angles of $z-\Gamma_3$ and $-\Gamma_4$ at z_3' and z_3'', respectively. Then we have

$$\iint_{z-z_3-z_4\in K} |dz_3|\,|dz_4| \ll \iint_K (\sin\theta'(u))^{-1}\,dx\,dy + \iint_K (\sin\theta''(u))^{-1}\,dx\,dy = I'+I'', \quad \text{say}.$$

We remark that we can choose the values of z_3' and z_3'' for which the functions $\theta'(u)$ and $\theta''(u)$ are continuous in K.

Let

$$\Sigma(\delta) = \{u\in z-\Sigma \mid \operatorname{dist}(u, \partial(z-\Sigma)) \leqslant 2\delta(N)\}.$$

For any $u\in\Sigma(\delta)$, we put $t = \operatorname{dist}(u, \partial(z-\Sigma))$. Then, by Lemma 2, we have $\sin\theta'(u) \gg t^{1/2}$. Hence it follows that

$$\iint_{K\cap\Sigma(\delta)} (\sin\theta'(u))^{-1}\,dx\,dy \ll \int_0^{2\delta(N)} t^{-1/2}\,dt \ll \delta(N)^{1/2}.$$

On the other hand, since $m(K) \ll \delta(N)$, we have

$$\iint_{K-\Sigma(\delta)} (\sin\theta'(u))^{-1}\,dx\,dy \ll \delta(N)^{-1/2}\iint_K dx\,dy \ll \delta(N)^{1/2},$$

by using Lemma 2 again. Hence we have $I' \ll \delta(N)^{1/2}$, and the same estimate holds for I''. Substituting these estimates in (6.9), we obtain the inequality (5.4), so that the proof of Proposition 3 is now completed.

7. Some reductions for the proof of Proposition 1. Our next aim is the estimation of the term

$$E(T, N, R) = L_N(T, R)/T - W_N(R).$$

We first quote the Koksma–Szüsz inequality, which plays the essential role in our proof of Proposition 1. Let $x_1, \ldots, x_k$ be any (given) points of the real N-dimensional Euclidean space $\boldsymbol{R}^N$. A sub-interval J of Q_N is a direct product of one-dimensional sub-intervals $[\alpha_1, \beta_1), \ldots, [\alpha_N, \beta_N)$ of $[0, 1)$:

$$J = \{(\theta_1, \ldots, \theta_N)\in Q_N \mid \alpha_j \leqslant \theta_j < \beta_j \ (1\leqslant j\leqslant N)\}.$$

We denote the set of all sub-intervals of Q_N by $\mathscr{A}$. And we define the N-dimensional discrepancy of the sequence $x_1, \ldots, x_k$ by

$$D(x_1, \ldots, x_k) = \sup_{J\in\mathscr{A}} |A(J)/k - m(J)|,$$

where $A(J)$ is the number of elements of the set $\{j \mid 1\leqslant j\leqslant k,\ x_j\in J\}$. Then, the following inequality holds:

Lemma 10 (Koksma [3], Szüsz [9]). *For any $\boldsymbol{h} = (h_1, \ldots, h_N)\in \boldsymbol{Z}^N$ ($\boldsymbol{Z}$ is the ring of rational integers), we define*

$$\|\boldsymbol{h}\| = \max_{1\leqslant j\leqslant N} |h_j| \quad \text{and} \quad r(\boldsymbol{h}) = \prod_{j=1}^{N} \max(|h_j|, 1).$$

Then, for any positive integer m, we have

$$D(x_1, \ldots, x_k) \leqslant 2N^2 3^{N+1}\Big(m^{-1} + \sum_{0<\|\boldsymbol{h}\|\leqslant m} r(\boldsymbol{h})^{-1}\Big|k^{-1}\sum_{n=1}^{k} \exp(2\pi i\langle \boldsymbol{h}, x_n\rangle)\Big|\Big),$$

where $\langle\ ,\ \rangle$ is the standard inner product of $\boldsymbol{R}^N$.

This lemma is a multi-dimensional generalization of the famous inequality of Erdös and Turán. (See Kuipers–Niederreiter [4].)

Now we start to estimate $E(T, N, R)$. Let $C_R(z)$ be the characteristic function of the rectangle R:

$$C_R(z) = \begin{cases} 1 & \text{if} \quad z\in R, \\ 0 & \text{if} \quad z\notin R. \end{cases}$$

Then we have

$$L_N(T, R) = \int_0^T C_R(f_N(t))\,dt = \sum_{n=0}^{[T]-1}\int_0^1 C_R(f_N(n+t))\,dt + \int_{[T]}^{T} C_R(f_N(t))\,dt.$$

Hence

$$\begin{aligned} E(T, N, R) &= [T]^{-1}\sum_{n=0}^{[T]-1}\int_0^1 C_R(f_N(n+t))\,dt - W_N(R) + \\ &\quad + [T]^{-1}\int_{[T]}^{T} C_R(f_N(t))\,dt + (L_N(T, R)/[T])\cdot(([T]-T)/T) \\ &= \int_0^1\Big([T]^{-1}\sum_{n=0}^{[T]-1} C_R(f_N(n+t)) - W_N(R)\Big)dt + O(T^{-1}). \end{aligned}$$

It is easy to see that, if we put

$$x_n(t) = (\{-((n+t)/2\pi)\log p_1\}, \ldots, \{-((n+t)/2\pi)\log p_N\})$$

(where $\{x\} = x-[x]$), then $f_N(n+t)\in R$ if and only if $x_n(t)\in\Omega_N(R)$. For any subset $S\subset Q_N$, we denote the number of elements of the set $\{n \mid 0\leqslant n\leqslant [T]$

$-1,\ x_n(t)\in S\}$ by $A_t(S)$. Then we have

$$(7.1)\qquad E(T, N, R) = \int_0^1 \big([T]^{-1} A_t(\Omega_N(R)) - m(\Omega_N(R))\big)\,dt + O(T^{-1}).$$

We shall estimate the integrand of the right-hand side of (7.1) by the Koksma–Szüsz inequality. For our purpose we must replace the set $\Omega_N(R)$ by a sum of sub-intervals. We put

$$Q_N(a_1, \ldots, a_N) = \{(\theta_1, \ldots, \theta_N)\in Q_N \mid a_j/r \leqslant \theta_j < (a_j+1)/r\ (1 \leqslant j \leqslant N)\},$$

where a_j is an element of the set $\{0, 1, 2, \ldots, r-1\}$ for any j. Then it is obvious that

$$Q_N = \bigcup_{0\leqslant a_1<r} \cdots \bigcup_{0\leqslant a_N<r} Q_N(a_1, \ldots, a_N).$$

And we define

$$R_1 = \bigcup_{Q_N(a_1,\ldots,a_N)\subset \Omega_N(R)} Q_N(a_1, \ldots, a_N)$$

and

$$R_2 = \bigcup_{Q_N(a_1,\ldots,a_N)\cap \Omega_N(R)\neq\emptyset} Q_N(a_1, \ldots, a_N).$$

Since $R_1 \subset \Omega_N(R) \subset R_2$, we have

$$\begin{aligned}[T]^{-1} A_t(R_1) - m(R_1) + m(R_1) - m(\Omega_N(R)) &\leqslant [T]^{-1} A_t(\Omega_N(R)) - m(\Omega_N(R)) \\ &\leqslant [T]^{-1} A_t(R_2) - m(R_2) + m(R_2) - m(\Omega_N(R)).\end{aligned}$$

So,

$$\begin{aligned}(7.2)\qquad &\big|[T]^{-1} A_t(\Omega_N(R)) - m(\Omega_N(R))\big| \\ &\leqslant \max_{j=1,2}\big(\big|[T]^{-1} A_t(R_j) - m(R_j)\big| + \big|m(R_j) - m(\Omega_N(R))\big|\big) \\ &\leqslant \sum_{0\leqslant a_1,\ldots,a_N<r} \big|[T]^{-1} A_t(Q_N(a_1, \ldots, a_N)) - m(Q_N(a_1, \ldots, a_N))\big| + \\ &\qquad\qquad + \max_{j=1,2}\big|m(R_j) - m(\Omega_N(R))\big| \\ &\leqslant r^N D_T(t) + \max_{j=1,2}\big|m(R_j) - m(\Omega_N(R))\big|,\end{aligned}$$

where $D_T(t) = D(x_0(t), x_1(t), \ldots, x_{[T]-1}(t))$.

For the evaluation of the term $|m(R_j) - m(\Omega_N(R))|$, some geometric investigations of $\Omega_N(R)$ are indispensable. Such a study was already done by Matsumoto–Miyazaki [5]. Since the length of the diagonal of the cube $Q_N(a_1, \ldots, a_N)$ is $r^{-1}N^{1/2}$, we have

LEMMA 11 ([5]). *For $j = 1, 2$, we have*

$$|m(R_j) - m(\Omega_N(R))| \ll r^{-N/(N+1)} N^{(3/2)+2\sigma_0}.$$

On the other hand, in the next section we shall prove the following estimate of $D_T(t)$:

$$(7.3)\qquad D_T(t) \ll N^2 3^N (m^{-1} + D'_T),$$

where D'_T is the term defined in the statement of Proposition 1.

Substituting these results in (7.2), we have

$$\big|[T]^{-1} A_t(\Omega_N(R)) - m(\Omega_N(R))\big| \ll N^2(3r)^N(m^{-1} + D'_T) + r^{-N/(N+1)} N^{(3/2)+2\sigma_0}.$$

Since the right-hand side of the above is independent of t, by combining this result with (7.1), the assertion of Proposition 1 follows.

Remark. Professor H. Niederreiter kindly informed the author of the existence of his paper [8]. (See also [6], p. 982.) We can apply the result in [8], by virtue of the result proved in [5]. If we use the theorem of Niederreiter–Wills, the result of Proposition 1 can be improved slightly. This improvement, however, has no effect on our final result.

8. Estimation of discrepancies. The purpose of this section is to prove (7.3). By Lemma 10 we have

$$D_T(t) \leqslant 2N^2 3^{N+1}\Big(m^{-1} + \sum_{0<\|\boldsymbol{h}\|\leqslant m} r(\boldsymbol{h})^{-1}\Big|[T]^{-1}\sum_{n=0}^{[T]-1}\exp(2\pi i\langle \boldsymbol{h}, x_n(t)\rangle)\Big|\Big).$$

We first evaluate the exponential sum

$$S = \sum_{n=0}^{[T]-1}\exp(2\pi i\langle \boldsymbol{h}, x_n(t)\rangle).$$

We have

$$(8.1)\qquad |S| = \Big|\exp(-iAt)\sum_{n=0}^{[T]-1}\exp(-iAn)\Big| \leqslant 2/|1-\exp(-iA)|,$$

where $A = h_1\log p_1 + \ldots + h_N \log p_N = \log(p_1^{h_1}\ldots p_N^{h_N})$.

We shall obtain a lower bound of $|1-e^{-iA}|$. We first remark that we can assume $A>0$; if $A<0$, then we can use the relation $|1-e^{-iA}| = |1-e^{-i(-A)}|$, and the case $A=0$ is impossible because of the uniqueness of the decomposition of integers into prime divisors.

Let $k \geqslant 0$ be the nearest integer to $A/2\pi$, and put $\eta = (A/2\pi) - k$. Namely,

$$(8.2)\qquad p_1^{h_1}\ldots p_N^{h_N} = e^A = e^{2\pi k}e^{2\pi\eta}.$$

We first assume $k \geqslant 1$. In this case we have

(8.3) $$|1-e^{-iA}| = |1-e^{-2\pi i\eta}| \gg |\eta| \gg k\,|\pi - A/2k| \gg |\pi - A/2k|.$$

On the other hand, there exists a number ξ between π and $A/2k$, for which

(8.4) $$e^{\pi} - e^{A/2k} = (\pi - A/2k)\, e^{\xi}$$

holds. We claim that $\xi \leqslant (3/2)\pi$. In fact, if $k \geqslant 2$, then $A/2\pi \geqslant 3/2$, so we have $A/2k = \pi A/(A-2\pi\eta) \leqslant \pi A/(A-\pi) \leqslant 3\pi^2/(3\pi-\pi) = (3/2)\pi$, and this implies $\xi \leqslant (3/2)\pi$. And if $k = 1$, then our claim follows from the inequality $A/2\pi \leqslant 3/2$.

Hence, from (8.4), we have $|e^{\pi} - e^{A/2k}| \ll |\pi - A/2k|$. Therefore, with (8.3), we obtain

(8.5) $$|1-e^{-iA}| \gg |e^{\pi} - e^{A/2k}|.$$

Here we quote a result of M. Waldschmidt on the estimation of transcendental measure of the transcendental number e^{π}.

LEMMA 12 (Waldschmidt [10]). *If α is an algebraic number with degree D and height H, then*

$$|e^{\pi} - \alpha| > \exp\left(-CD^2(D+\log H)(\log(D+\log H))^2\right)$$

for some absolute constant $C > 0$.

In our case, $\alpha = e^{A/2k}$ is a real root of the equation $X^{2k} - p_1^{h_1}\ldots p_N^{h_N} = 0$. Hence we can apply the above lemma to the right-hand side of (8.5) with $D \leqslant 2k$ and $H \ll \exp(2\pi k)$. So we have

$$|1-e^{-iA}| \gg \exp\left(-Ck^3(\log(k+1))^2\right).$$

By using the estimate

(8.6) $$k \ll |h_1|\log p_1 + \ldots + |h_N|\log p_N \ll \|\boldsymbol{h}\| \sum_{n=1}^{N} \log p_n$$
$$\ll \|\boldsymbol{h}\|\, N \cdot \log N \ll mN \cdot \log N,$$

we obtain

(8.7) $$|1-e^{-iA}| \gg \exp\left(-C(mN\cdot\log N)^3(\log(mN))^2\right).$$

If $k = 0$, then $|1-e^{-iA}| \gg |A| \gg |e^{A}-1|$. We write $e^{A} = p_1^{h_1}\ldots p_N^{h_N} = q_1/q_2$ with co-prime rational integers q_1 and q_2. Since $e^{A} \neq 1$, we have

$$|e^{A}-1| \geqslant q_2^{-1} \geqslant (p_1^{|h_1|}\ldots p_N^{|h_N|})^{-1} = \exp\left(-(|h_1|\log p_1 + \ldots + |h_N|\log p_N)\right).$$

Hence, with (8.6), we see that the estimate (8.7) also holds in case $k = 0$.

Here we note that the author's original proof of (8.7) included an error, which the referee has pointed out. The simple argument to deduce (8.5) in case $k \geqslant 1$ is also suggested by the referee.

By (8.1) and (8.7), we have

(8.8) $$D_T(t) \ll N^2 3^N \Big(m^{-1} + T^{-1}\Big(\sum_{0<\|\boldsymbol{h}\|\leqslant m} r(\boldsymbol{h})^{-1}\Big) \times$$
$$\times \exp\left(C(mN\cdot\log N)^3(\log(mN))^2\right)\Big).$$

Finally we evaluate the sum $\sum r(\boldsymbol{h})^{-1}$. We define the mapping $\varphi\colon \boldsymbol{Z}^N \to \boldsymbol{Z}^{N-1}$ by $\varphi\colon \boldsymbol{h} = (h_1, \ldots, h_{N-1}, h_N) \mapsto \varphi(\boldsymbol{h}) = \boldsymbol{h}' = (h_1, \ldots, h_{N-1})$, and put

$$\|\boldsymbol{h}'\| = \max_{1\leqslant j\leqslant N-1} |h_j|.$$

We see that if $\|\boldsymbol{h}\| \leqslant m$, then $\|\varphi(\boldsymbol{h})\| \leqslant m$. Conversely, for any $\boldsymbol{h}' = (h_1, \ldots, h_{N-1}) \in \boldsymbol{Z}^{N-1}$ with the condition $\|\boldsymbol{h}'\| \leqslant m$, the number of the elements of the set $\varphi^{-1}(\boldsymbol{h}') \cap \{\boldsymbol{h} \in \boldsymbol{Z}^N \mid \|\boldsymbol{h}\| \leqslant m\}$ is $2m+1$; they are the integer vectors $\boldsymbol{h}_k = (h_1, \ldots, h_{N-1}, k)$ $(-m \leqslant k \leqslant m)$. And it is easy to see that

$$r(\boldsymbol{h}_k)/r(\boldsymbol{h}) = \begin{cases} k & \text{if} \quad 1 \leqslant |k| \leqslant m, \\ 1 & \text{if} \quad k = 0. \end{cases}$$

The above discussion leads to the following relation:

$$\sum_{\|\boldsymbol{h}\|\leqslant m} r(\boldsymbol{h})^{-1} = \sum_{\|\boldsymbol{h}'\|\leqslant m} r(\boldsymbol{h}')^{-1}\left(1+2(1+2^{-1}+3^{-1}+\ldots+m^{-1})\right)$$
$$\leqslant \Big(\sum_{\|\boldsymbol{h}'\|\leqslant m} r(\boldsymbol{h}')^{-1}\Big)(3+2\cdot\log m).$$

We apply this argument repeatedly, and get

(8.9) $$\sum_{\|\boldsymbol{h}\|\leqslant m} r(\boldsymbol{h})^{-1} \leqslant (3+2\cdot\log m)^N.$$

Substituting this inequality in (8.8), we obtain the result of (7.3). This completes the proof of Proposition 1.

Remark. The referee pointed out to the author that the inequality (8.9) can be proved in a few lines. See [4], p. 155.

9. Proof of Proposition 2. Now the only task remaining to us is to prove Proposition 2. Let $\Delta = R - R_i$, and we evaluate the measure of

$$\Omega_N(\Delta) = \{(\theta_1, \ldots, \theta_N) \in Q_N \mid S_N(\theta_1, \ldots, \theta_N) \in \Delta\}.$$

Let $\Sigma_{2,N} = \{z_2 + \ldots + z_N \mid z_2 \in \Gamma_2, \ldots, z_N \in \Gamma_N\}$. Then

$$\Sigma_N = \Gamma_1 + \Sigma_{2,N} = \{z_1 + z \mid z_1 \in \Gamma_1,\ z \in \Sigma_{2,N}\}.$$

For any $z \in \Sigma_{2,N}$, we put

$$\Theta(z) = m\{\theta_1 \mid 0 \leqslant \theta_1 < 1,\ z_1(\theta_1) + z \in \Delta\}.$$

Then it is obvious that

(9.1) $$m(\Omega_N(\Delta)) = \int_0^1 \ldots \int_0^1 \Theta\left(z_2(\theta_2) + \ldots + z_N(\theta_N)\right) d\theta_2 \ldots d\theta_N.$$

The set $z+\Gamma_1$ is a closed convex curve, and it intersects with Δ only several times. We consider one of such intersections of $z+\Gamma_1$ and Δ. This intersection is an arc of the curve $z+\Gamma_1$. We denote the end points of this arc $\tilde{\Gamma}$ by $z+z_1(\theta_0)$ and $z+z_1(\theta_0')$: $\tilde{\Gamma} = \{z+z_1(\theta_1)|\ \theta_0 \leqslant \theta_1 \leqslant \theta_0'\}$. We can easily show that $\operatorname{dist}(z+z_1(\theta_0), z+z_1(\theta_0')) \ll \delta_N^{1/2}$ in a similar (in fact, simpler) way as in the proof of Lemma 8. Hence, by Lemma 9, we have $|\theta_0'-\theta_0| \ll \delta_N^{1/2}$, and therefore, $\Theta(z) \ll \delta_N^{1/2}$. Hence, by (9.1), we obtain

$$|W_N(R_i)-W_N(R)| \ll \delta_N^{1/2} \ll N^{(1-\sigma_0)/2}(\log N)^{-\sigma_0/2}.$$

This implies Proposition 2, since the same estimate holds for $|W_N(R_y)-W_N(R)|$. The proof of our theorem is thus completed.

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Pisot sequences which satisfy no linear recurrence II

by

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Introduction. In this paper we continue our study of Pisot sequences begun in [1]. Recall that the Pisot sequence $E(a_0, a_1)$ is the sequence of integers defined for $0 < a_0 < a_1$ by

$$(1)\qquad -1/2 < a_{n+1} - a_n^2/a_{n-1} \leqslant 1/2.$$

In [1] we proved that there are Pisot sequences satisfying no linear recurrence relation. Our proof made use of an inequality from Pisot's thesis [5]. We recently discovered that the constant in this inequality is incorrect. Since it is used at three separate points in [1], it would appear that many of the details in [1] need to be modified.

Our first purpose here is to show that all the theorems of [1] are correct as stated and to indicate the necessary changes in the proofs. To do this, we prove a number of new inequalities for Pisot sequences. Since these should be useful in other investigations we give more general versions than needed to simply repair the proofs of [1].

Our second purpose is to sketch a simplified proof of the main Theorem 4 of [1], avoiding the use of the Kronecker–Weyl theorem. Combining this proof with results from [2] shows, for example, that none of $E(1089m, 1782m)$ satisfy a linear recurrence for any odd m.

1. The new inequalities. The notation will be as in [1].

If $\{a_n\} = E(a_0, a_1)$, write $\theta_n = a_{n+1}/a_n$ and $\varphi_n = \inf\{\theta_m: m \geqslant n\}$ (misprinted in [1] as "sup"). We write $\theta(a_0, a_1) = \theta = \lim \theta_n$ which always exists. We are interested only in $\theta > 1$ for which it is necessary and sufficient that $a_1 \geqslant a_0 + \sqrt{a_0/2}$, according to results of Pisot [5] and Flor [4]. Let $\lambda = \lim a_n/\theta^n > 0$, and define $\varepsilon_n = \lambda\theta^n - a_n$.

LEMMA 1. *For all* $n \geqslant 0$,

$$(2)\qquad |\theta - \theta_n| \leqslant 1/(2a_n(\varphi_n - 1)),$$

$$(3)\qquad |\varepsilon_n| \leqslant 1/(2(\theta-1)(\varphi_n - 1)).$$