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Bounds for solutions of additive equations in an algebraic number field I

by

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1. Introduction. Let k be a rational integer $\geqslant 1$. Similar to Waring's problem, one can show by the Hardy–Littlewood's method that an equation

$$a_1 x_1^k + \ldots + a_s x_s^k = 0,$$

where $a_1, \ldots, a_s$ are given rational integers but not all of the same sign, has a nontrivial solution in nonnegative rational integers $x_1, \ldots, x_s$, provided only that $s \geqslant c_1(k)$. (See, e.g., H. Davenport [3]). Here we use $c(f, \ldots, g)$ to denote a positive constant depending on $f, \ldots, g$. As for a bound of these solutions, it was shown by J. Pitman [10] that if $s \geqslant c_2(k)$, then there exists a nontrivial solution in nonnegative integers such that

$$\max_i x_i < c_3(k) \max(1, |a_1|, \ldots, |a_s|)^{c_4(k)} \tag{1}$$

where c_2 and c_4 are explicit. Under suitable conditions and if s is very large, the estimation can be considerably improved. (See, B. J. Birch [2] and W. M. Schmidt [11], [12].) In particular, Schmidt proved that if $s \geqslant c_5(k, \varepsilon)$, the equation

$$a_1 x_1^k + \ldots + a_s x_s^k = b_1 y_1^k + \ldots + b_s y_s^k$$

with positive rational integer coefficients has a nontrivial solution in nonnegative rational integers $x_1, \ldots, x_s, y_1, \ldots, y_s$ such that

$$\max_{i,j}(x_i, y_j) \leqslant \max_{i,j}(a_i, b_j)^{1/k+\varepsilon}. \tag{2}$$

We use hereafter $\varepsilon, \varepsilon_1, \ldots$ to denote arbitrary preassigned positive numbers < 1. The number $1/k$ in (2) is best possible. Although the circle method is still used in the proof of (2), the treatment of the minor arcs is completely distinct from that in Waring's problem.

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It was Siegel ([13], [14]) who succeeded in dealing with Waring's problem in an arbitrary algebraic number field by his generalized circle method, and he obtained the result corresponding to Hardy–Littlewood's estimation on $G(k)$. Siegel's result was improved by R. G. Ayoub [1], Y. Eda [4], O. Körner [8], R. M. Stemmler [15] and T. Tatuzawa [16], [17] in various aspects.

By the combination of the methods of Schmidt and Siegel, we can generalize Schmidt's theorem to an arbitrary algebraic number field.

Let K be an algebraic number field of degree n. Let $K^{(p)}$ $(1 \leqslant p \leqslant r_1)$ be the real conjugates of K and let $K^{(q)}$ and $K^{(q+r_2)}$ $(r_1+1 \leqslant q \leqslant r_1+r_2)$ denote the complex conjugates of K, where $r_1+2r_2 = n$. Throughout this paper, the indices p and q are over the sets of integers cited above. For $\gamma \in K$, we denote by $\gamma^{(i)}$ $(1 \leqslant i \leqslant n)$ the conjugates of γ and by $N(\gamma) = \prod_{i=1}^{n} \gamma^{(i)}$ the norm of γ. Let γ_j $(1 \leqslant j \leqslant n)$ be numbers of K and x_j $(1 \leqslant j \leqslant n)$ be real numbers. We set $\xi = \sum_{j=1}^{n} x_j \gamma_j$ and define $\xi^{(i)} = \sum_{j=1}^{n} x_j \gamma_j^{(i)}$ $(1 \leqslant i \leqslant n)$. We use the notations

$$\|\xi\| = \max_i |\xi^{(i)}|, \quad S(\xi) = \sum_{i=1}^{n} \xi^{(i)} \quad \text{and} \quad E(\xi) = \exp(2\pi i S(\xi)),$$

where $\exp(x) = e^x$. A number γ of K is called *totally nonnegative* if $\gamma^{(p)} \geqslant 0$.

Let $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_s$ be $2s$ nonzero totally nonnegative integers of K. Consider the equation of the type

$$\alpha_1 \lambda_1^k + \ldots + \alpha_s \lambda_s^k = \beta_1 \mu_1^k + \ldots + \beta_s \mu_s^k. \tag{3}$$

A set of numbers $\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_s$ satisfying (3) is called a *nontrivial solution of* (3) if $\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_s$ are totally nonnegative integers of K, not all zero. Set

$$m = \max_{i,j} (N(\alpha_i), N(\beta_j)). \tag{4}$$

In this paper, we shall prove the following

THEOREM. *Suppose $s \geqslant c_6(k, n, \varepsilon)$. Then the equation* (3) *has a nontrivial solution such that*

$$\max_{i,j} (N(\lambda_i), N(\mu_j)) \ll m^{1/k+\varepsilon}. \tag{5}$$

Here and below the constants implicit in $\ll$ or O may depend on k, K, $\varepsilon, \ldots$, but not on m.

If $k = 1$, then $\lambda_i = \beta_i$, $\mu_i = \alpha_i$ $(1 \leqslant i \leqslant s)$ is a nontrivial solution of (3) with (5). So we suppose $k \geqslant 2$ throughout this paper.

Suppose that $\alpha_1, \ldots, \alpha_s$ are given integers of K. In the second part of

this investigation we will show that if $s \geqslant c_7(k, n, \varepsilon)$, the equations

$$\alpha_1 a_1 \lambda_1^k + \ldots + \alpha_s a_s \lambda_s^k = 0$$

has a solution in $a_1, \ldots, a_s, \lambda_1, \ldots, \lambda_s$, where each a_i is 1 or -1 and where λ_i $(1 \leqslant i \leqslant s)$ are totally nonnegative integers, not all zero, with

$$\max_i N(\lambda_i) \ll \max(1, |N(\alpha_1)|, \ldots, |N(\alpha_s)|)^{\varepsilon}.$$

2. Several lemmas.

LEMMA 1. *Let $t_1, \ldots, t_{r_1+r_2}$ be a set of real numbers satisfying*

$$\sum_{p=1}^{r_1} t_p + 2 \sum_{q=r_1+1}^{r_1+r_2} t_q = 0. \tag{6}$$

Then there exists a totally nonnegative unit σ of K such that

$$c_8^{-1} e^{t_p} < \sigma^{(p)} < c_8 e^{t_p}, \quad c_8^{-1} e^{t_q} < |\sigma^{(q)}| < c_8 e^{t_q},$$

where $c_8 = c_8(K)$.

See, e.g. Lemma 1.1 in Hua Loo Keng and Wang Yuan [6]. (Put $\sigma = \eta^2$.)

LEMMA 2. *There exists a rational integer $c_9 = c_9(K)$ such that for any integers α, β of K, where $\beta \neq 0$, there exist a rational integer l and an integer ω of K such that $1 \leqslant l \leqslant c_9$ and $|N(l\alpha - \omega\beta)| < |N(\beta)|$.*

See, e.g., K. Ireland and M. Rosen [7], p. 178.

LEMMA 3. *For any t integers $\gamma_1, \ldots, \gamma_t$ of K, not all zero, let γ be a nonzero element of the integral ideal $\mathfrak{a} = (\gamma_1, \ldots, \gamma_t)$ with the least norm in absolute value. Then*

$$c_9!\,\gamma_i/\gamma, \quad 1 \leqslant i \leqslant t$$

are integers.

Proof. Set $\alpha = \gamma_i$ and $\beta = \gamma$ in Lemma 2. Then there exist a rational integer l_i and an integer ω_i such that

$$|N(l_i \gamma_i - \omega_i \gamma)| < |N(\gamma)|, \quad 1 \leqslant l_i \leqslant c_9.$$

Since $l_i \gamma_i - \omega_i \gamma \in \mathfrak{a}$, it follows that $N(l_i \gamma_i - \omega_i \gamma) = 0$. Therefore $l_i \gamma_i - \omega_i \gamma = 0$, and $l_i \gamma_i/\gamma = \omega_i$ is an integer. Since $l_i | c_9!$,

$$\frac{c_9!\,\gamma_i}{\gamma} = \frac{c_9!}{l_i} \left(\frac{l_i \gamma_i}{\gamma}\right), \quad 1 \leqslant i \leqslant t,$$

are integers. The lemma is proved.

LEMMA 4. *For any t integral vectors (γ_i, δ_i) $(1 \leqslant i \leqslant t)$ of K^2, where $\gamma_i \neq 0$*

$(1 \leqslant i \leqslant t)$, *if*

$$\frac{\delta_1}{\gamma_1} = \ldots = \frac{\delta_t}{\gamma_t},$$

then

$$(\gamma_i, \delta_i) = \frac{1}{c_9!}\chi_i(\gamma, \delta), \quad 1 \leqslant i \leqslant t,$$

where γ *is defined in Lemma* 3, *and where* δ *and* χ_i $(1 \leqslant i \leqslant t)$ *are integers.*

Proof. By Lemma 3, $\chi_i = c_9!\,\gamma_i/\gamma$ $(1 \leqslant i \leqslant t)$ are integers. Let

$$\frac{\delta_1}{\gamma_1} = \ldots = \frac{\delta_t}{\gamma_t} = \alpha.$$

Then $\delta_i = \alpha\gamma_i$ $(1 \leqslant i \leqslant t)$. Since $(\delta_1, \ldots, \delta_t) = \alpha(\gamma_1, \ldots, \gamma_t) = \alpha\mathfrak{a}$ is an integral ideal and $\gamma \in \mathfrak{a}$, $\delta = \alpha\gamma$ is an integer. Therefore

$$\frac{1}{c_9!}\chi_i\delta = \frac{1}{c_9!}\chi_i\alpha\gamma = \frac{1}{c_9!}\chi_i\frac{\delta_i}{\gamma_i}\gamma = \frac{1}{c_9!}\chi_i\gamma\delta_i\left(\frac{\chi_i\gamma}{c_9!}\right)^{-1} = \delta_i, \quad 1 \leqslant i \leqslant t.$$

The lemma is proved.

Lemma 5. *For any nonzero integer* σ, *there exists a nonzero integer* γ *such that* $\|\gamma\| \leqslant c_{10}(K)$ *and* $\gamma\sigma$ *is totally nonnegative.*

Proof. If $r_1 = 0$, the lemma holds clearly. Now suppose that $r_1 > 0$. Let $\omega_1, \ldots, \omega_n$ be an integral basis of K. Let

$$c_{10} = 4\max_i \sum_{j=1}^{n} |\omega_j^{(i)}| \quad \text{and} \quad N_p = \frac{\sigma^{(p)}}{2|\sigma^{(p)}|}c_{10}, \quad 1 \leqslant p \leqslant r_1.$$

Since the matrix $(\omega_j^{(p)})$ $(1 \leqslant p \leqslant r_1,\ 1 \leqslant j \leqslant n)$ has rank r_1, we may suppose $\det(\omega_j^{(p)}) \neq 0$ $(1 \leqslant p,\ j \leqslant r_1)$. The system of linear equations

$$\sum_{j=1}^{r_1} \omega_j^{(p)} x_j = N_p, \quad 1 \leqslant p \leqslant r_1$$

has a unique solution. Set $a_j = [x_j]$ $(1 \leqslant j \leqslant r_1)$, where $[x]$ denotes the integral part of x. Then we have an integer $\gamma = \sum_{j=1}^{r_1} a_j\omega_j$ satisfying $\gamma^{(p)}\sigma^{(p)} > 0$ and $\|\gamma\| \leqslant c_{10}$. The lemma is proved.

3. Reductions.

Proposition 1. *Suppose that* $x \geqslant 1/k$ *and* $s \geqslant c_{11}(k, n, x, \varepsilon)$. *Then* (3) *has a nontrivial solution with*

$$\max_{i,j}\big(N(\lambda_i), N(\mu_j)\big) \ll m^{x+\varepsilon}.$$

The case $x = 1/k$ is the theorem.

One can prove by Siegel's method that if $s \geqslant c_{12}(k, n)$, then the equation of the type

$$\alpha_1\lambda_1^k + \ldots + \alpha_t\lambda_t^k - \alpha_{t+1}\lambda_{t+1}^k - \ldots - \alpha_s\lambda_s^k = 0$$

has a nontrivial solution in totally nonnegative integers $\lambda_1, \ldots, \lambda_s$ such that

(7) $$\max_i N(\lambda_i) \ll \max N(\alpha_i)^{c_{13}(k,n)},$$

where $\alpha_1, \ldots, \alpha_s$ are given nonzero totally nonnegative integers and $1 \leqslant t \leqslant s-1$.

It will suffice to prove Proposition 1 when m is large, say $m \geqslant c_{14}(k, K, x, \varepsilon)$. In fact, if $m < c_{14}$ and $s \geqslant c_{12}$, then it follows by (7) that (3) has a nontrivial solution such that

$$\max_{i,j}\big(N(\lambda_i), N(\mu_j)\big) \ll m^{c_{13}} \ll c_{14}^{c_{13}} \ll m^{x+\varepsilon}.$$

Let X be the set of x such that Proposition 1 holds. Then (7) shows that X is not empty. It is clear that X is a closed set. Hence the proof of Proposition 1 is reduced to proving that if $x > 1/k$ and $x \in X$, then there exists an $x' \in X$, where $x' < x$.

For $1 \leqslant j \leqslant s$, set

$$t_i = k^{-1}\big(\log N(\alpha_j)^{1/n} + \log|\alpha_j^{(i)}|^{-1}\big), \quad 1 \leqslant i \leqslant n.$$

Then (6) holds, and therefore there exists a set of totally nonnegative units σ_j $(1 \leqslant j \leqslant s)$ such that

$$c_8^{-1}N(\alpha_j)^{1/nk}(\alpha_j^{(p)})^{-1/k} \leqslant \sigma_j^{(p)} \leqslant c_8 N(\alpha_j)^{1/nk}(\alpha_j^{(p)})^{-1/k},$$

$$c_8^{-1}N(\alpha_j)^{1/nk}|\alpha_j^{(q)}|^{-1/k} \leqslant |\sigma_j^{(q)}| \leqslant c_8 N(\alpha_j)^{1/nq}|\alpha_j^{(q)}|^{-1/k},$$

i.e.,

$$c_8^{-k}N(\alpha_j)^{1/n} \leqslant \alpha_j^{(p)}\sigma_j^{(p)k} \leqslant c_8^k N(\alpha_j)^{1/n},$$

$$c_8^{-k}N(\alpha_j)^{1/n} \leqslant |\alpha_j^{(q)}\sigma_j^{(q)k}| \leqslant c_8^k N(\alpha_j)^{1/n}, \quad 1 \leqslant j \leqslant s.$$

Similarly, there exists a set of units τ_j $(1 \leqslant j \leqslant s)$ such that

$$c_8^{-k}N(\beta_j)^{1/n} \leqslant \beta_j^{(p)}\tau_j^{(p)k} \leqslant c_8^k N(\beta_j)^{1/n},$$

$$c_8^{-k}N(\beta_j)^{1/n} \leqslant |\beta_j^{(q)}\tau_j^{(q)k}| \leqslant c_8^k N(\beta_j)^{1/n}, \quad 1 \leqslant j \leqslant s.$$

Let

$$\alpha_i' = \alpha_i\sigma_i^k, \quad \beta_i' = \beta_i\tau_i^k, \quad \lambda_i = \sigma_i\lambda_i', \quad \mu_i = \tau_i\mu_i' \quad (1 \leqslant i \leqslant s).$$

Then (3) becomes

(3)′ $$\alpha_1'\lambda_1'^k + \ldots + \alpha_s'\lambda_s'^k = \beta_1'\mu_1'^k + \ldots + \beta_s'\mu_s'^k.$$

If Proposition 1 holds for x' and for the particular equation (3)$'$, then we have a nontrivial solution of (3)$'$ such that

$$\max_{i,j}\left(N(\lambda_i'),\, N(\mu_j')\right) \ll \max_{i,j}\left(N(\alpha_i'),\, N(\beta_j')\right)^{x'+\varepsilon}.$$

Since $N(\lambda_i') = N(\lambda_i)$, $N(\mu_i') = N(\mu_i)$, $N(\alpha_i') = N(\alpha_i)$, $N(\beta_i') = N(\beta_i)$ $(1 \leqslant i \leqslant s)$, we have a nontrivial solution of (3) with

$$\max_{i,j}\left(N(\lambda_i),\, N(\mu_j)\right) \ll m^{x'+\varepsilon}.$$

i.e., Proposition 1 holds for x' and for (3). Hence we may suppose without loss of generality that α_i and β_i satisfy

$$(8)\qquad \begin{gathered} c_{15}^{-1} N(\alpha_i)^{1/n} < \alpha_i^{(p)} < c_{15} N(\alpha_i)^{1/n}, \qquad c_{15}^{-1} N(\alpha_i)^{1/n} < |\alpha_i^{(q)}| < c_{15} N(\alpha_i)^{1/n}, \\ c_{15}^{-1} N(\beta_i)^{1/n} < \beta_i^{(p)} < c_{15} N(\beta_i)^{1/n}, \qquad c_{15}^{-1} N(\beta_i)^{1/n} < |\beta_i^{(q)}| < c_{15} N(\beta_i)^{1/n}, \\ 1 \leqslant i \leqslant s, \end{gathered}$$

where $c_{15} = c_{15}(k, K)$.

In what follows, x will be a fixed number $> 1/k$ for which Proposition 1 holds. Take y sufficiently small such that

$$(9)\qquad 1/k + 6c_{13}ny + 20ny < x \quad \text{and} \quad 22kny < 1,$$

and put

$$(10)\qquad x' = \max\left(x(1-\tfrac{1}{2}y) + y/2kn,\ 1/k + 6c_{13}ny + 20ny\right),$$

so that $x' < x$. We proceed to prove that Proposition 1 holds for x'.

Let $\varepsilon_1 = \min(\varepsilon/8x', \varepsilon/4)$ and divide the interval $[0, 1]$ into a finite number of intervals $\{I\}$ of length $\leqslant \varepsilon_1$. If s is large, one of these intervals I will be such that many of the coefficients $\alpha_1, \ldots, \alpha_s$ are of the type

$$N(\alpha_i) = m^{a_i}, \qquad a_i \in I.$$

We may therefore suppose without loss of generality that

$$\frac{N(\alpha_i)}{N(\alpha_j)} \leqslant m^{\varepsilon_1}, \qquad 1 \leqslant i, j \leqslant s.$$

Similarly, we may suppose

$$\frac{N(\beta_i)}{N(\beta_j)} \leqslant m^{\varepsilon_1}, \qquad 1 \leqslant i, j \leqslant s.$$

Let $a^n = m^{\varepsilon_1} \max_i N(\alpha_i)$ and $b^n = m^{\varepsilon_1} \max_i N(\beta_i)$. Let p_i and q_i be the largest rational integers such that

$$N(\alpha_i)\, p_i^{kn} \leqslant a^n \quad \text{and} \quad N(\beta_i)\, q_i^{kn} \leqslant b^n, \qquad 1 \leqslant i \leqslant s.$$

Since $m \geqslant c_{14}$, $a^n/N(\alpha_i) \geqslant m^{\varepsilon_1}$ and $b^n/N(\beta_i) \geqslant m^{\varepsilon_1}$, we may suppose

$$p_i \geqslant 2^{-1/kn}\left(\frac{a^n}{N(\alpha_i)}\right)^{1/kn} \quad \text{and} \quad q_i \geqslant 2^{-1/kn}\left(\frac{b^n}{N(\beta_i)}\right)^{1/kn}, \qquad 1 \leqslant i \leqslant s.$$

Hence

$$N(\alpha_i)\, p_i^{kn} \geqslant \tfrac{1}{2} a^n \quad \text{and} \quad N(\beta_i)\, q_i^{kn} \geqslant \tfrac{1}{2} b^n, \qquad 1 \leqslant i \leqslant s.$$

Set $\alpha_i' = \alpha_i p_i^k$, $\beta_i' = \beta_i q_i^k$, $\lambda_i = p_i \lambda_i'$, $\mu_i = q_i \mu_i'$ $(1 \leqslant i \leqslant s)$. Then (3) becomes (3)$'$, and by (8), α_i' and β_i' satisfy

$$(2c_{15})^{-1} a < \alpha_i'^{(p)} < c_{15} a, \qquad (2c_{15})^{-1} a < |\alpha_i'^{(q)}| < c_{15} a,$$

$$(2c_{15})^{-1} a < \beta_i'^{(p)} < c_{15} b, \qquad (2c_{15})^{-1} b < |\beta_i'^{(q)}| < c_{15} b, \qquad 1 \leqslant i \leqslant s.$$

Suppose that Proposition 1 holds for x' and for the particular equation (3)$'$. Then there exists a nontrivial solution of (3)$'$ with

$$\max_{i,j}\left(N(\lambda_i'),\, N(\mu_j')\right) \ll \max(a^n, b^n)^{x'+\varepsilon/4} \ll m^{(1+\varepsilon_1)(x'+\varepsilon/4)} \ll m^{x'+\varepsilon/2}.$$

Since

$$N(\alpha_i) = m^{\varepsilon_1} N(\alpha_i) \max_j N(\alpha_j)/m^{\varepsilon_1} \max_l N(\alpha_l)$$

$$= a^n m^{-\varepsilon_1} N(\alpha_i)/\max_l N(\alpha_l) \geqslant a^n m^{-2\varepsilon_1}, \qquad 1 \leqslant i \leqslant s,$$

we have

$$p_i^n \leqslant p_i^{kn} \leqslant a^n/N(\alpha_i) \leqslant m^{2\varepsilon_1} \leqslant m^{\varepsilon/2}, \qquad 1 \leqslant i \leqslant s,$$

and therefore

$$N(\lambda_i) \ll p_i^n N(\lambda_i') \ll m^{x'+\varepsilon}, \qquad 1 \leqslant i \leqslant s.$$

Similarly

$$N(\mu_i) \ll m^{x'+\varepsilon}, \qquad 1 \leqslant i \leqslant s,$$

i.e., Proposition 1 holds for x' and for (3). Thus in proving Proposition 1 for x', we may suppose that

$$(11)\qquad c_{16} a < \alpha_i^{(p)} < c_{17} a, \quad c_{16} a < |\alpha_i^{(q)}| < c_{17} a, \quad c_{16} b < \beta_i^{(p)} < c_{17} b, \quad c_{16} b < |\beta_i^{(q)}| < c_{17} b, \quad 1 \leqslant i \leqslant s$$

for certain positive numbers a, b, where $c_{16} = c_{16}(k, K)$ and $c_{17} = c_{17}(k, K)$.

4. Continuation. In what follows, h will be the integer $c_{11}(k, n, x, \varepsilon)$ occurring in Proposition 1, and $s > h$. Set

$$(12)\qquad z = y/2kn^2.$$

We distinguish two cases.

A. There is a subset of h elements among $\alpha_1, \ldots, \alpha_s$, say $\alpha_1, \ldots, \alpha_h$ and there is a subset of h elements among $\beta_1, \ldots, \beta_s$, say $\beta_1, \ldots, \beta_h$, and there are totally nonnegative integers $\sigma_1, \ldots, \sigma_h, \tau_1, \ldots, \tau_h$ such that

$$(13) \qquad 0 < \|\sigma_i\| \leqslant m^z, \quad 0 < \|\tau_i\| \leqslant m^z, \quad 1 \leqslant i \leqslant h$$

and

$$|N(\sigma)| \geqslant m^y,$$

where σ is a nonzero element in the integral ideal $(\alpha_1\sigma_1, \ldots, \alpha_h\sigma_h, \beta_1\tau_1, \ldots, \beta_h\tau_h)$ with the least norm in absolute value.

By Lemma 5, we may choose a nonzero integer γ such that $\|\gamma\| \leqslant c_{10}$ and $\gamma\sigma$ is totally nonnegative. By Lemma 3,

$$\alpha_i' = \frac{c_9!\,\alpha_i\sigma_i^k\gamma^2}{\gamma\sigma} \quad \text{and} \quad \beta_i' = \frac{c_9!\,\beta_i\tau_i^k\gamma^2}{\gamma\sigma}, \quad 1 \leqslant i \leqslant h$$

are all nonzero totally nonnegative integers. Therefore it follows from the case x of the Proposition 1 that the equation

$$\alpha_1'\lambda_1'^k + \ldots + \alpha_h'\lambda_h'^k = \beta_1'\mu_1'^k + \ldots + \beta_h'\mu_h'^k$$

has a nontrivial solution satisfying

$$\max_{i,j}(N(\lambda_i'), N(\mu_j')) \ll \max_{i,j}(N(\alpha_i'), N(\beta_j'))^{x+\varepsilon} \ll m^{(1+knz-y)(x+\varepsilon)}.$$

Let $\lambda_i = \sigma_i\lambda_i'$, $\mu_i = \tau_i\mu_i'$ $(1 \leqslant i \leqslant h)$ and $\lambda_i = \mu_i = 0$ $(h < i \leqslant s)$. Then by (10) and (12), the equation (3) has a nontrivial solution with

$$\max_{i,j}(N(\lambda_i), N(\mu_j)) \ll m^{(1+knz-y)(x+\varepsilon)+nz} \ll m^{(1-y/2)(x+\varepsilon)+nz} \ll m^{x'+\varepsilon}.$$

We are thus reduced to case

B. For any h elements, say $\alpha_1, \ldots, \alpha_h$, among $\alpha_1, \ldots, \alpha_s$, and for any h elements, say $\beta_1, \ldots, \beta_h$, among $\beta_1, \ldots, \beta_s$, and given any totally nonnegative integers $\sigma_1, \ldots, \sigma_h, \tau_1, \ldots, \tau_h$ satisfying (13), the integer σ defined as in the case A satisfies $|N(\sigma)| < m^y$.

Condition B depends on k, n, h, m, y, and it is denoted by $B(k, n, h, m, y)$.

PROPOSITION 2. *Let $q = 1$ or -1. Let*

$$(14) \qquad m = \max(a^n, b^n)$$

and let $\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_s$ be nonzero totally nonnegative integers satisfying (11) *and $B(k, n, h, m, y)$. Then if $s \geqslant c_{18}(k, n, h, y)$, the equation*

$$\alpha_1\lambda_1^k + \ldots + \alpha_s\lambda_s^k - \beta_1\mu_1^k - \ldots - \beta_s\mu_s^k = q\chi$$

has a solution in totally nonnegative integers $\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_s, \chi$, not all zero, with

$$\max_{i,j}(N(\lambda_i), N(\mu_j)) \ll m^{1/k+20ny}, \quad \|\chi\| \leqslant m^{6y}.$$

Now we proceed to show that Proposition 2 implies that Proposition 1 is true for x'. Let x, x', y, z, h be as above. Suppose that c_{12} and c_{18} are integers. Let $s = uv$, where $u = c_{18}$ and $v = c_{12}$. Replace the indices $1 \leqslant l \leqslant s$ by double indices $1 \leqslant i \leqslant v$, $1 \leqslant j \leqslant u$. Then the equation (3) can be written as

$$(15) \qquad \sum_{i=1}^{v}(\alpha_{i1}\lambda_{i1}^k + \ldots + \alpha_{iu}^k\lambda_{iu}^k - \beta_{i1}\mu_{i1}^k - \ldots - \beta_{iu}\mu_{iu}^k) = 0.$$

For each i, $1 \leqslant i \leqslant v$, the coefficients $\alpha_{i1}, \ldots, \alpha_{iu}, \beta_{i1}, \ldots, \beta_{iu}$ satisfy the conditions in Proposition 2. Hence there are totally nonnegative integers $\lambda_{i1}', \ldots, \lambda_{iu}', \mu_{i1}', \ldots, \mu_{iu}', \chi_i$, not all zero, such that

$$\alpha_{i1}\lambda_{i1}'^k + \ldots + \alpha_{iu}\lambda_{iu}'^k - \beta_{i1}\mu_{i1}'^k - \ldots - \beta_{iu}\mu_{iu}'^k = q_i\chi_i$$

with

$$\max_{j,l}(N(\lambda_{ij}'), N(\mu_{il}')) \ll m^{1/k+20ny}, \quad \|\chi_i\| \leqslant m^{6y}.$$

We may suppose that $\chi_i \neq 0$ $(1 \leqslant i \leqslant v)$. Otherwise we get a small solution straightaway. Take $q_1 = \ldots = q_{v-1} = 1$ and $q_v = -1$. Then by (7), the equation

$$\chi_1\gamma_1^k + \ldots + \chi_{v-1}\gamma_{v-1}^k - \chi_v\gamma_v^k = 0$$

has a nontrivial solution satisfying

$$\max_i N(\gamma_i) \ll m^{6c_{13}ny}.$$

Let $\lambda_{ij} = \gamma_i\lambda_{ij}'$, $\mu_{ij} = \gamma_i\mu_{ij}'$ $(1 \leqslant i \leqslant v,\ 1 \leqslant j \leqslant u)$. Then we have a nontrivial solution of (15) having

$$\max_{i,j,t,l}(N(\lambda_{ij}), N(\mu_{tl})) \ll m^{1/k+6c_{13}ny+20ny} \ll m^{x'}.$$

Thus Proposition 1 holds for x'.

5. Weyl's inequality. Let $\omega_1, \ldots, \omega_n$ be an integral basis of K, $\mathfrak{d}$ the different and D the absolute value of the discriminant of K. We can choose a basis $\varrho_1, \ldots, \varrho_n$ of $\mathfrak{d}^{-1}$ such that

$$S(\varrho_i\omega_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise}. \end{cases}$$

Set

$$\xi = x_1 \varrho_1 + \ldots + x_n \varrho_n \quad \text{and} \quad \eta = y_1 \omega_1 + \ldots + y_n \omega_n,$$

where x_i and y_i $(1 \leqslant i \leqslant n)$ are real numbers. We denote by $P(T)$ the set of $(y_1, \ldots, y_n)$ satisfying

$$0 \leqslant \eta^{(p)} \leqslant T, \quad |\eta^{(q)}| \leqslant T;$$

$\sum_{\lambda \in P(T)}$ a sum where λ runs over all integers such that $0 \leqslant \lambda^{(p)} \leqslant T$, $|\lambda^{(q)}| \leqslant T$, and $\sum_{|\mu| \in P(T)}$ a sum of integers μ satisfying $\|\mu\| \leqslant T$.

LEMMA 6 (Siegel). *Let $h \geqslant 1$. Then for any ξ, there exist an integer α and a number β of $\mathfrak{d}^{-1}$ such that*

$$\|\alpha\xi - \beta\| < h^{-1}, \quad 0 < \|\alpha\| \leqslant h,$$

$$\max\left(h|\alpha^{(i)} \xi^{(i)} - \beta^{(i)}|, |\alpha^{(i)}|\right) \geqslant D^{-1/2}, \quad 1 \leqslant i \leqslant n$$

and

$$N\left((\alpha, \beta\mathfrak{d})\right) \leqslant D^{1/2}.$$

See Lemma 6 in Siegel [14]. Notice that the property of ξ belonging to supplementary domain is only used in the proof of his formula (41).

LEMMA 7 (Mitsui). *Let A, B, h be positive numbers satisfying $A \geqslant 1$, $h > 2^{5+r_2} D$ and $1 \leqslant B < 2^{-4-r_2} D^{-1/n} h$. Then for any ξ*

$$\sum_{|\mu| \in P(B)} \min\left(A, |1 - E(\xi\mu\omega_j)|^{-1} \ (1 \leqslant j \leqslant n)\right)$$

$$= O\left(AB^n \left(\frac{1}{\|\alpha\|} + \frac{1}{B} + \frac{h \log h}{AB} + \frac{\log h}{A}\right)\right),$$

here and also in Lemma 8, α denotes an integer satisfying the conditions in Lemma 6.

See Theorem 3.1 in Mitsui [9]. Notice that in the proof of his formula (3.42), we may use the estimation $|N(\alpha)| \geqslant c\|\alpha\|$ instead of $|N(\alpha)| \geqslant cT$ with $c = c(K)$.

LEMMA 8 (Weyl's inequality). *Let*

$$G = 2^{k-1} \quad \text{and} \quad L(\xi) = \sum_{\lambda \in P(T)} E(\lambda^k \xi), \quad \text{where} \quad T > k!\, 2^{5+k+r_2} D.$$

Let h be a number satisfying

$$k!\, 2^{4+k+r_2} D T^{k-1} < h \leqslant T^k.$$

Then

$$L(\xi) \ll T^{n+\varepsilon_2} \left(\frac{1}{\|\alpha\|} + \frac{1}{T} + \frac{h}{T^k}\right)^{1/G}.$$

Proof. By Hölder's inequality

$$|L(\xi)|^G = \Big|\sum_{\lambda} \sum_{\lambda_1} E\left((\lambda+\lambda_1)^k \xi - \lambda^k \xi\right)\Big|^{2^{k-2}}$$

$$\leqslant \Big(\sum_{\lambda_1} \Big|\sum_{\lambda} E(k\lambda_1 \lambda^{k-1} \xi + \ldots)\Big|\Big)^{2^{k-2}}$$

$$\ll T^{n(2^{k-2}-1)} \sum_{\lambda_1} \Big|\sum_{\lambda} E(k\lambda_1 \lambda^{k-1} \xi + \ldots)\Big|^{2^{k-2}}$$

$$\ll T^{n(2^{k-2}-1)} \sum_{\lambda_1} T^{n(2^{k-3}-1)} \sum_{\lambda_2} \Big|\sum_{\lambda} E\left(k(k-1)\lambda_1 \lambda_2 \lambda^{k-2} \xi + \ldots\right)\Big|^{2^{k-3}}$$

$$\ll \ldots$$

$$\ll T^{n(G-k)} \sum_{\lambda_1} \sum_{\lambda_2} \ldots \sum_{\lambda_{k-1}} \Big|\sum_{\lambda} E(\mu\lambda\xi)\Big|,$$

where

(16) $$\mu = k!\, \lambda_1 \ldots \lambda_{k-1}, \quad |\lambda_i| \in P(2T) \quad (1 \leqslant i \leqslant k-1),$$

and λ runs over all solutions of the conditions

$$\lambda + \lambda_{i_1} + \ldots + \lambda_{i_g} \in P(T) \quad (1 \leqslant i_1 < \ldots < i_g \leqslant k-1, \ 0 \leqslant g \leqslant k-1).$$

Let $A(\mu)$ denote the number of solutions of (16). Then by the well-known properties of the divisor function, we have

$$A(\mu) = \begin{cases} O(T^{n(k-2)}), & \text{if } \mu = 0, \\ O(T^{\varepsilon_2}), & \text{otherwise.} \end{cases}$$

Hence

$$|L(\xi)|^G \ll T^{n(G-2)} + T^{n(G-k)+\varepsilon_2} \sum_{\mu} \Big|\sum_{\lambda} E(\mu\lambda\xi)\Big|,$$

where the summation is extended over all μ, λ satisfying

$$\mu \in P(k!\, 2^{k-1} T^{k-1}) \quad \text{and} \quad \lambda \in P(T).$$

Since

$$\sum_{\lambda \in P(T)} E(\mu\lambda\xi) = O\left(T^{n-1} \min\left(T, |1 - E(\mu\omega_j \xi)|^{-1} \ (1 \leqslant j \leqslant n)\right)\right)$$

(cf. Siegel [14], p. 332), we have

$$|L(\xi)|^G \ll T^{n(G-2)} + T^{n(G-k)+\varepsilon_2} \sum_{\mu} T^{n-1} \min\left(T, |1 - E(\mu\omega_j \xi)|^{-1} \ (1 \leqslant j \leqslant n)\right).$$

Let $A = T$ and $B = k!\, 2^{k-1} T^{k-1}$. Then by Lemma 7, we have

$$|L(\xi)|^G \ll T^{n(G-k)+\varepsilon_2+n-1+1+n(k-1)} \left(\frac{1}{\|\alpha\|} + \frac{1}{T^{k-1}} + \frac{h \log h}{T^k} + \frac{\log h}{T}\right)$$

$$\ll T^{nG+2\varepsilon_2} \left(\frac{1}{\|\alpha\|} + \frac{1}{T} + \frac{h}{T^k}\right).$$

The lemma follows.

6. Schmidt's lemma. In this section, we shall generalize Schmidt's lemma to an arbitrary algebraic number field.

LEMMA 9 (Schmidt). *Suppose that* $T \geqslant c_{19}(k, K, \varepsilon_3)$, $C \geqslant T^{n-1/G+\varepsilon_3}$ *and* $|L(\xi)| \geqslant C$. *Then there exist a totally nonnegative integer* α *and an integer* β *such that*

$$\|\alpha\xi-\beta\| \ll \left(\frac{T^n}{C}\right)^G T^{-k+\varepsilon_3}$$

and

$$0 < \|\alpha\| \ll \left(\frac{T^n}{C}\right)^G T^{\varepsilon_3},$$

where $\alpha = \alpha'\gamma$ *and* $\beta = \beta'\gamma$ *in which* γ *is an integer satisfying* $\|\gamma\| \leqslant c_{20}(K)$ *and* α', β' *satisfy the conditions of Lemma 6 with* $h = T^{k-\varepsilon_3}(C/T^n)^G$.

Proof. We have

$$T^{k-\varepsilon_3}\left(\frac{C}{T^n}\right)^G \geqslant T^{k-\varepsilon_3}\left(\frac{T^{n-1/G+\varepsilon_3}}{T^n}\right)^G \geqslant T^{k-1+\varepsilon_3}$$

and

$$T^{k-\varepsilon_3}\left(\frac{C}{T^n}\right)^G \leqslant T^{k-\varepsilon_3}.$$

Let

$$h = T^{k-\varepsilon_3}\left(\frac{C}{T^n}\right)^G.$$

Then h satisfies the condition of Lemma 8 for $T \geqslant c_{19}$. By Lemma 6, there exist an integer α' and a number β' of $\mathfrak{d}^{-1}$ satisfying

$$\|\alpha'\xi-\beta'\| < h^{-1}, \quad 0 < \|\alpha'\| \leqslant h$$

and the other conclusions in Lemma 6. Take $\varepsilon_2 = \varepsilon_3/2G$. Since

$$T^{n+\varepsilon_2}\left(\frac{h}{T^k}\right)^{1/G} = T^{n+\varepsilon_2-\varepsilon_3/G}\frac{C}{T^n} = CT^{-\varepsilon_2}$$

and

$$T^{n-1/G+\varepsilon_2} \leqslant CT^{\varepsilon_2-\varepsilon_3} < CT^{-\varepsilon_2},$$

we have by Lemma 8 that

$$C \leqslant |L(\xi)| \ll T^{n+\varepsilon_2}\|\alpha'\|^{-1/G},$$

i.e.,

$$0 < \|\alpha'\| \ll \left(\frac{T^n}{C}\right)^G T^{\varepsilon_2 G} < \left(\frac{T^n}{C}\right)^G T^{\varepsilon_3}.$$

There exists a nonzero integer γ_1 such that $\|\gamma_1\| \leqslant c_{21}(K)$ and $\gamma_1\beta'$ is an integer for any $\beta' \in \mathfrak{d}^{-1}$. (See, e.g., Hecke [5], p. 100.) By Lemma 5, there is a nonzero integer γ_2 with $\|\gamma_2\| < c_{10}$ such that $\gamma_1\gamma_2\alpha'$ is totally nonnegative. Let $\alpha = \gamma_1\gamma_2\alpha'$ and $\beta = \gamma_1\gamma_2\beta'$. The lemma follows.

7. The circle method. We denote by G_n the unit cube $\{(x_1, \ldots, x_n): 0 \leqslant x_i < 1 \ (1 \leqslant i \leqslant n)\}$. For any $\gamma \in K$, we can determine uniquely integral ideals $\mathfrak{a}$, $\mathfrak{b}$ such that

$$\gamma\mathfrak{d} = \mathfrak{b}/\mathfrak{a}, \quad (\mathfrak{a}, \mathfrak{b}) = 1.$$

We write $\gamma \to \mathfrak{a}$. Let $t > 1$ and $\Gamma(t)$ be the set consisting of $\gamma = x_1\varrho_1 + \ldots$ $\ldots + x_n\varrho_n$ satisfying

$$(x_1, \ldots, x_n) \in G_n, \quad x_i \ (1 \leqslant i \leqslant n) \text{ rational numbers,}$$

$$\gamma \to \mathfrak{a} \quad \text{and} \quad N(\mathfrak{a}) \leqslant t^n.$$

Let

$$h = abm^{20ky-y/n} \quad \text{and} \quad t = m^{y/n}. \tag{17}$$

For any $\gamma \in \Gamma(t)$, subject to $\gamma \to \mathfrak{a}$, we define the basic domain B_γ by

$$\{(x_1, \ldots, x_n): (x_1, \ldots, x_n) \in G_n, \ \xi = x_1\varrho_1 + \ldots + x_n\varrho_n \text{ such that } h\|\xi-\gamma_0\| < 1 \text{ for some } \gamma_0 \equiv \gamma \,(\mathrm{mod}\ \mathfrak{d}^{-1})\}. \tag{18}$$

We may prove that if $\gamma_1 \neq \gamma_2$, then $B_{\gamma_1} \cap B_{\gamma_2} = \emptyset$. In fact, suppose there is a $\xi \in B_{\gamma_1} \cap B_{\gamma_2}$, i.e., $h\|\xi-\gamma_{0i}\| < 1$, where $\gamma_{0i} \equiv \gamma_i \,(\mathrm{mod}\ \mathfrak{d}^{-1})$ $(i = 1, 2)$. For simplicity, we set $\gamma_{0i} = \gamma_i$ $(i = 1, 2)$. Write

$$\max(h|\xi^{(i)}-\gamma_j^{(i)}|, t^{-1}) = \sigma_j^{(i)}, \quad 1 \leqslant i \leqslant n, \ 1 \leqslant j \leqslant 2.$$

Then

$$\prod_{i=1}^{n}\sigma_j^{(i)} < 1, \quad \max(\sigma_j^{(i)})^{-1} \leqslant t, \quad j = 1, 2,$$

and thus

$$|\gamma_1^{(i)}-\gamma_2^{(i)}| \leqslant |\xi^{(i)}-\gamma_1^{(i)}| + |\xi^{(i)}-\gamma_2^{(i)}| \leqslant h^{-1}(\sigma_1^{(i)}+\sigma_2^{(i)})$$
$$= h^{-1}\sigma_1^{(i)}\sigma_2^{(i)}\left((\sigma_1^{(i)})^{-1}+(\sigma_2^{(i)})^{-1}\right) \leqslant 2h^{-1}\sigma_1^{(i)}\sigma_2^{(i)}t, \quad 1 \leqslant i \leqslant n.$$

Suppose $\gamma_i \to \mathfrak{a}_i$ $(i = 1, 2)$. We have

$$N(\mathfrak{a}_1\mathfrak{a}_2)|N(\gamma_1-\gamma_2)| \leqslant (2h^{-1}t^3)^n < D^{-1},$$

since $m \geqslant c_{14}$. On the other hand, $\mathfrak{a}_1 \mathfrak{a}_2 (\gamma_1 - \gamma_2)\mathfrak{d}$ is an integral ideal, and thus

$$N(\mathfrak{a}_1 \mathfrak{a}_2)\,|N(\gamma_1-\gamma_2)| \geqslant |N(\mathfrak{d}^{-1})| = D^{-1}.$$

This gives a contradiction, and therefore the assertion follows.

We define the supplementary domain E by

$$E = G_n - \bigcup_{\gamma \in \Gamma(t)} B_\gamma. \tag{19}$$

We use the notations

$$\xi = x_1 \varrho_1 + \ldots + x_n \varrho_n, \quad dx = dx_1 \ldots dx_n,$$
$$A = b^{1/k} m^{20y}, \quad B = a^{1/k} m^{20y}, \quad H = m^{6y};$$

$$S_i(\xi) = \sum_{\lambda \in P(A)} E(\alpha_i \lambda^k \xi), \quad T_i(\xi) = \sum_{\mu \in P(B)} E(-\beta_i \mu^k \xi), \quad 1 \leqslant i \leqslant s, \tag{20}$$

$$S(\xi) = \prod_{i=1}^{s} S_i(\xi), \quad T(\xi) = \prod_{i=1}^{s} T_i(\xi)$$

and

$$F(\xi) = \sum_{\chi \in P(H)} S(\xi)\,T(\xi)\,E(-q\chi\xi),$$

where $q = 1$ or -1. Let Z denote the number of solutions of the equation

$$\alpha_1 \lambda_1^k + \ldots + \alpha_s \lambda_s^k - \beta_1 \mu_1^k - \ldots - \beta_s \mu_s^k = q\chi$$

in totally nonnegative integers $\lambda_1, \ldots, \lambda_s, \mu_1, \ldots, \mu_s, \chi$ satisfying

$$\lambda_i \in P(A), \quad \mu_i \in P(B) \quad (1 \leqslant i \leqslant s), \quad \chi \in P(H).$$

Then

$$Z = \sum_{\gamma \in \Gamma(t)} \int_B F(\xi)\,dx + \int_E F(\xi)\,dx. \tag{21}$$

We shall show that under the assumption made in Proposition 2, $Z > 1$.

8. Supplementary domain. Take ε_3 such that

$$\varepsilon_3 < 1/2G, \quad \varepsilon_3(1+20y) < \tfrac{1}{2}z, \tag{22}$$

and s so large that

$$s > \frac{10G}{z} + h. \tag{23}$$

Lemma 10. *Suppose that* $(x_1, \ldots, x_n) \in G_n$ *and*

$$|F(\xi)| \geqslant H^n (AB)^{ns} m^{-4}. \tag{24}$$

Then ξ *lies in a basic domain.*

Proof. We may suppose that

$$|S_1(\xi)| \geqslant \ldots \geqslant |S_s(\xi)|.$$

Then

$$F(\xi) \ll H^n A^{n(h-1)} B^{ns} |S_h(\xi)|^{s-h+1},$$

and thus by (24) and $m \geqslant c_{14}$, we have

$$|S_i(\xi)| \geqslant |S_h(\xi)| \geqslant A^n m^{-5/(s-h+1)} = C, \quad \text{say for } 1 \leqslant i \leqslant h.$$

By (20), (22) and (23), we have

$$m^{5/(s-h+1)} \leqslant A^{1/4y(s-h+1)} \leqslant A^{1/2G} < A^{1/G-\varepsilon_3},$$

and therefore

$$C \geqslant A^{n-1/G+\varepsilon_3}.$$

It follows by Lemma 9 that there are totally nonnegative integers σ_i $(1 \leqslant i \leqslant h)$ and integers φ_i $(1 \leqslant i \leqslant h)$ such that

$$0 < \|\sigma_i\| \ll m^{5G/(s-h+1)} A^{\varepsilon_3} < m^{z/2+z/2} = m^z$$

and

$$\|\xi \alpha_i \sigma_i - \varphi_i\| \ll m^{5G/(s-h+1)} A^{\varepsilon_3 - k} < m^z A^{-k}, \quad 1 \leqslant i \leqslant h,$$

since $m \geqslant c_{14}$. After a recording of $\beta_1, \ldots, \beta_s$, we may also suppose that

$$|T_1(\xi)| \geqslant \ldots \geqslant |T_s(\xi)|.$$

Similarly, there are totally nonnegative integers τ_i $(1 \leqslant i \leqslant h)$ and integers ψ_i $(1 \leqslant i \leqslant h)$ having

$$0 < \|\tau_i\| < m^z \quad \text{and} \quad \|\xi \beta_i \tau_i - \psi_i\| < m^z B^{-k}, \quad 1 \leqslant i \leqslant h.$$

Hence by (11), (12), (20) and $m \geqslant c_{14}$, we have

$$\begin{aligned}
\|\varphi_i \beta_j \tau_j - \psi_j \alpha_i \sigma_i\| &= \|\varphi_i \beta_j \tau_j - \xi \alpha_i \sigma_i \beta_j \tau_j + \xi \alpha_i \sigma_i \beta_j \tau_j - \psi_j \alpha_i \sigma_i\| \\
&\leqslant \|\beta_j \tau_j\|\,\|\xi \alpha_i \sigma_i - \varphi_i\| + \|\alpha_i \sigma_i\|\,\|\xi \beta_j \tau_j - \psi_j\| \\
&\ll b m^{2z} A^{-k} + a m^{2z} B^{-k} \ll m^{2z - 20ky} < 1,
\end{aligned}$$

and thus

$$N(\varphi_i \beta_j \tau_j - \psi_j \alpha_i \sigma_i) = 0.$$

Since $\varphi_i \beta_j \tau_j - \psi_j \alpha_i \sigma_i$ is an integer, we have

$$\varphi_i \beta_j \tau_j - \psi_j \alpha_i \sigma_i = 0.$$

Thus by Lemma 4, the $2h$ integral vectors $c_9!(\alpha_i \sigma_i, \varphi_i)$ and $c_9!(\beta_i \tau_i, \psi_i)$

$(1 \leqslant i \leqslant h)$ are all integral multiples of an integral vector (σ, τ), where σ is a nonzero element of the integral ideal $(\alpha_1 \sigma_1, \ldots, \alpha_h \sigma_h, \beta_1 \tau_1, \ldots, \beta_h \tau_h)$ with the least norm in absolute value. Therefore the condition in case B yields that

$$0 < |N(\sigma)| < m^y.$$

Let

$$\sigma^{-1} \tau \mathfrak{d} = \mathfrak{b}/\mathfrak{a}, \quad (\mathfrak{a}, \mathfrak{b}) = 1.$$

Then $\mathfrak{a}|\sigma$, and thus

$$N(\mathfrak{a}) \leqslant |N(\sigma)| < m^y = t^n.$$

Since $\|\sigma_1\| < m^z$, we have

$$m^{(n-1)z} |\sigma_1^{(i)}| \geqslant N(\sigma_1) \geqslant 1, \quad 1 \leqslant i \leqslant n,$$

and by (11), (12), (17), (20) and $m \geqslant c_{14}$,

$$\begin{aligned} |\xi^{(i)} - (\sigma^{(i)})^{-1} \tau^{(i)}| &= \frac{1}{|\alpha_1^{(i)} \sigma_1^{(i)}|} |\xi^{(i)} \alpha_1^{(i)} \sigma_1^{(i)} - \varphi_1^{(i)}| \\ &\leqslant a^{-1} m^{nz} A^{-k} = a^{-1} b^{-1} m^{-20ky+nz} \\ &= a^{-1} b^{-1} m^{-20ky+y/2nk} < h^{-1}, \quad 1 \leqslant i \leqslant n. \end{aligned}$$

Therefore $\xi \in B_\gamma$, where $\gamma \equiv \sigma^{-1} \tau \pmod{\mathfrak{d}^{-1}}$. The lemma is proved.

9. Basic domain. We use the notations

$$\xi - \gamma = \zeta, \quad \eta = y_1 \omega_1 + \ldots + y_n \omega_n, \quad dy = dy_1 \ldots dy_n,$$

$$G_i(\gamma) = N(\mathfrak{a})^{-1} \sum_{\lambda (\mathrm{mod}\, \mathfrak{a})} E(\alpha_i \lambda^k \gamma), \quad H_i(\gamma) = N(\mathfrak{a})^{-1} \sum_{\mu (\mathrm{mod}\, \mathfrak{a})} E(-\beta_i \mu^k \gamma),$$

$$I_i(\zeta, A) = \int_{P(A)} E(\alpha_i \eta^k \zeta)\, dy, \quad J_i = (\zeta, B) = \int_{P(B)} E(-\beta_i \eta^k \zeta)\, dy, \quad 1 \leqslant i \leqslant s,$$

(25) $$G(\gamma) = \prod_{i=1}^{s} G_i(\gamma), \quad H(\gamma) = \prod_{i=1}^{s} H_i(\gamma), \quad I(\zeta, A) = \prod_{i=1}^{s} I_i(\zeta, A) \quad \text{and}$$

$$J(\zeta, B) = \prod_{i=1}^{s} J_i(\zeta, B),$$

where $\gamma \to \mathfrak{a}$.

Lemma 11. *Let $\mathfrak{a}$ be an integral ideal. Let $N(\mathfrak{a}, T)$ be the number of elements ν of $\mathfrak{a}$ satisfying*

$$0 \leqslant \nu^{(p)} \leqslant T, \quad |\nu^{(q)}| \leqslant T.$$

Then

$$N(\mathfrak{a}, T) = \frac{(2\pi)^{r_2} T^n}{\sqrt{D}\, N(\mathfrak{a})} + O\left(\frac{T_0^{n-1}}{N(\mathfrak{a})^{1-1/n}}\right),$$

where $T_0 = \max(N(\mathfrak{a})^{1/n}, T)$.

See, e.g., Lemma 3.2 in Mitsui [9]. Notice that the conclusion is still true for the number of ν satisfying $\nu + \mu \in \mathfrak{a}$, $0 \leqslant \nu^{(p)} + \mu^{(p)} \leqslant T$ and $|\nu^{(q)} + \mu^{(q)}| \leqslant T$, where μ is a given number in a residue class mod $\mathfrak{a}$.

Now we can prove the following lemma by the Siegel argument (see [14], pp. 328–330).

Lemma 12. *Suppose that $\xi \in B_\gamma$. Then*

(26) $$S_i(\xi) = G_i(\gamma) I_i(\zeta, A) + O(t^2 A^{n-1})$$

and

(27) $$T_i(\xi) = H_i(\gamma) J_i(\zeta, B) + O(t^2 B^{n-1}), \quad 1 \leqslant i \leqslant s.$$

Proof. Determine positive numbers $\theta^{(i)}$ $(1 \leqslant i \leqslant n)$, with $\theta^{(q)} = \theta^{(q+r_2)}$, such that

$$\theta^{(i)} \max\left(h|\zeta^{(i)}|, t^{-1} N(\mathfrak{a})^{1/n}\right) = D^{1/2n} \prod_{j=1}^{n} \max\left(h|\zeta^{(j)}|, t^{-1} N(\mathfrak{a})^{1/n}\right)^{1/n} N(\mathfrak{a})^{1/n}, \quad 1 \leqslant i \leqslant n,$$

Then

$$\prod_{i=1}^{n} \theta^{(i)} = D^{1/2} N(\mathfrak{a}),$$

and it follows by Minkowski's linear form theorem that there exists $\alpha \in \mathfrak{a}$ such that $0 < |\alpha^{(i)}| \leqslant \theta^{(i)}$, $1 \leqslant i \leqslant n$. Hence $\alpha \mathfrak{a}^{-1} = \mathfrak{b}$ is an integral ideal and

$$N(\mathfrak{b}) = |N(\alpha)| N(\mathfrak{a})^{-1} \leqslant \left(\prod_{i=1}^{n} \theta^{(i)}\right) N(\mathfrak{a})^{-1} = \sqrt{D};$$

hence $\mathfrak{b}$ belongs to a finite set depending on K only. Let $\sigma_1, \ldots, \sigma_n$ be a basis of $\mathfrak{b}^{-1}$. Then $\mathfrak{a} = \alpha \mathfrak{b}^{-1}$ has a basis

$$\tau_i = \alpha \sigma_i, \quad 1 \leqslant i \leqslant n$$

satisfying

$$\|\tau_i\| = O(\|\alpha\|) = O(\max_i \theta^{(i)}) = O(t).$$

Let μ run over a complete residue system modulo $\mathfrak{a}$, and λ over all numbers in $\mathfrak{a}$ such that $\lambda+\mu\in P(A)$. Then

(28) $$S_i(\xi)=\sum_{\mu(\mathrm{mod}\,\mathfrak{a})}E(\alpha_i\mu^k\gamma)\sum_{\substack{\mathfrak{a}|\lambda\\ \lambda+\mu\in P(A)}}E(\alpha_i(\lambda+\mu)^k\zeta).$$

Expressing λ in terms of τ_i $(1\leqslant i\leqslant n)$, we obtain

$$\lambda=g_1\tau_1+\ldots+g_n\tau_n,$$

where $g_1,\ldots,g_n$ are rational integers. Let $G(\lambda)$ denote the cube

$$(s_1,\ldots,s_n):\ \sigma=s_1\tau_1+\ldots+s_n\tau_n,\quad g_i\leqslant s_i<g_i+1\quad(1\leqslant i\leqslant n).$$

Then

$$\|\sigma-\lambda\|=O(t),$$

$$\|(\sigma+\mu)^k\zeta-(\lambda+\mu)^k\zeta\|\ll\|\sigma-\lambda\|\,\|\zeta\|\,(\|\sigma+\mu\|^{k-1}+\|\lambda+\mu\|^{k-1})\ll th^{-1}A^{k-1},$$

and therefore by (11),

$$E(\alpha_i(\lambda+\mu)^k\zeta)=\int_{G(\lambda)}E(\alpha_i(\sigma+\mu)^k\zeta)\,ds+O(ath^{-1}A^{k-1}),$$

where $ds=ds_1\ldots ds_n$. Since $N(\mathfrak{a})\leqslant t^n\leqslant A$ by (17) and (20), it follows by Lemma 11 that the number of λ with $\mathfrak{a}|\lambda$ and $\lambda+\mu\in P(A)$ is $O(N(\mathfrak{a})^{-1}A^n)$. Therefore

$$\sum_{\substack{\mathfrak{a}|\lambda\\ \lambda+\mu\in P(A)}}E(\alpha_i(\lambda+\mu)^k\zeta)=\sum_{\substack{\mathfrak{a}|\lambda\\ \lambda+\mu\in P(A)}}\int_{G(\lambda)}E(\alpha_i(\sigma+\mu)^k\zeta)\,ds+O(N(\mathfrak{a})^{-1}ath^{-1}A^{n+k-1}).$$

Let F denote the domain in the s-space defined by

$$0\leqslant\sigma^{(p)}+\mu^{(p)}\leqslant A,\quad |\sigma^{(q)}+\mu^{(q)}|\leqslant A.$$

Then the volume of the area belonging to exactly one of $\bigcup_{\substack{\mathfrak{a}|\lambda\\ \lambda+\mu\in P(A)}}G(\lambda)$ and F is dominated by $O(N(\mathfrak{a})^{-1}tA^{n-1})$. (See, Siegel [14], p. 329.) Therefore by (17) and (20), we have

$$\sum_{\substack{\mathfrak{a}|\lambda\\ \lambda+\mu\in P(A)}}E(\alpha_i(\lambda+\mu)^k\zeta)=\int_F E(\alpha_i(\sigma+\mu)^k\zeta)\,ds+O(N(\mathfrak{a})^{-1}t^2A^{n-1}).$$

Let $\sigma+\mu=\eta$. Since the Jacobian of $s_1,\ldots,s_n$ with respect to $y_1,\ldots,y_n$ is equal to

$$D^{1/2}|\det(\tau_j^{(i)})|^{-1}=N(\mathfrak{a})^{-1},$$

we have

$$\sum_{\substack{\mathfrak{a}|\lambda\\ \lambda+\mu\in P(A)}}E(\alpha_i(\lambda+\mu)^k\zeta)=N(\mathfrak{a})^{-1}I_i(\zeta,A)+O(N(\mathfrak{a})^{-1}t^2A^{n-1}).$$

Substituting into (28), we have (26). The proof of (27) is similar.

10. Continuation. We use E_n to denote the whole n-dimensional Euclidean space.

LEMMA 13. *We have*

(29) $$I_i(\zeta,A)\ll\prod_{i=1}^{n}\min(A,a^{-1/k}|\zeta^{(i)}|^{-1/k})$$

and

(30) $$J_i(\zeta,B)\ll\prod_{i=1}^{n}\min(B,b^{-1/k}|\zeta^{(i)}|^{-1/k}).$$

See, Siegel [14], p. 335. The only difference between the proofs of (29) and the corresponding formula of Siegel is that we use $\alpha_i^{(p)}\tau^{(p)}$ and $|\alpha_i^{(q)}\tau^{(q)}|$ instead of his $\tau^{(p)}$ and $|\tau^{(q)}|$.

LEMMA 14.

$$\int_{B_\gamma}S(\xi)T(\xi)E(-q\chi\xi)\,dx$$
$$=G(\gamma)H(\gamma)E(-q\chi\gamma)\int_{E_n}I(\zeta,A)J(\zeta,B)\,dx+O((AB)^{ns}(ab)^{-n}m^{-20kn\gamma-17\gamma}).$$

Proof. By Lemmas 12 and 13, we have

$$S(\xi)T(\xi)=G(\gamma)H(\gamma)I(\zeta,A)J(\zeta,B)+O((AB)^{ns}t^2\max(A^{-1},B^{-1})).$$

Let

(31) $$\zeta^{(p)}=u_p,\quad \zeta^{(q)}=u_qe^{i\varphi_q}.$$

The Jacobian of $x_1,\ldots,x_n$ with respect to u_p, u_q, φ_q is equal to the product of the Jacobian of $x_1,\ldots,x_n$ with respect to $\zeta^{(p)}$, $\zeta^{(q)}$ and the Jacobian of $\zeta^{(p)}$, $\zeta^{(q)}$ with respect to u_p, u_q, φ_q, i.e., it is equal to

$$2^{r_2}D^{1/2}\prod_q u_q.$$

It follows by (17) and (20) that

$$\int_{B_\gamma}dx\ll\prod_p\Big(\int_0^{h^{-1}}du_p\Big)\prod_q\Big(\int_{-\pi}^{\pi}\int_0^{h^{-1}}u_q\,du_q\,d\varphi_q\Big)\ll h^{-n}=(ab)^{-n}m^{-20kn\gamma+\gamma}$$

and

$$\max(A^{-1}, B^{-1}) \ll m^{-20y}.$$

Therefore

$$(32)\qquad \int_{B_\gamma} S(\xi)\,T(\xi)\,E(-q\chi\xi)\,dx$$

$$= G(\gamma)\,H(\gamma)\,E(-q\chi\gamma)\int_{B_\gamma} I(\zeta, A)\,J(\zeta, B)\,E(-q\chi\zeta)\,dx +$$

$$+\,O\big((AB)^{ns}(ab)^{-n}m^{-20kny-17y}\big).$$

In the integral in the right-hand side of (32) we replace $E(-q\chi\zeta)$ by 1. Then by (20) and Lemma 13, the error is

$$(AB)^{ns}\int_{B_\gamma}\|\chi\zeta\|\,dx \ll (AB)^{ns}Hh^{-n-1} \ll (AB)^{ns}(ab)^{-n}m^{-20kny-17y}.$$

Hence

$$(33)\qquad \int_{B_\gamma} S(\xi)\,T(\xi)\,E(-q\chi\xi)\,dx = G(\gamma)\,H(\gamma)\,E(-q\chi\gamma)\int_{B_\gamma} I(\zeta, A)\,J(\zeta, B)\,dx +$$

$$+\,O\big((AB)^{ns}(ab)^{-n}m^{-20kny-17y}\big).$$

If $(x_1, \ldots, x_n)$ is a point in $E_n - B_\gamma$, then $h|\zeta^{(i)}| \geqslant 1$ is true for at least one index i. By Lemma 13 and (31), we have

$$\int_{E_n-B_\gamma} I(\zeta, A)\,J(\zeta, B)\,dx$$

$$\ll \int_{E_n-B_\gamma}\Big(\prod_{i=1}^{n}\min(A, a^{-1/k}|\zeta^{(i)}|^{-1/k})\prod_{j=1}^{n}\min(B, b^{-1/k}|\zeta^{(j)}|^{-1/k})\Big)^s dx$$

$$\ll \Big(\int_{h^{-1}}^{\infty}(ab)^{-s/k}u^{-2s/k}\,du\Big)\Big(\int_0^{\infty}\min(A^2, a^{-s/k}v^{-s/k})\min(B^s, b^{-s/k}v^{-s/k})\,dv\Big)^{r_1-1}\times$$

$$\times\Big(\int_{-\pi}^{\pi}\int_0^{\infty}\min(A^{2s}, a^{-2s/k}w^{-2s/k})\min(B^{2s}, b^{-2s/k}w^{-2s/k})\,w\,dw\,d\varphi\Big)^{r_2}+$$

$$+\Big(\int_0^{\infty}\min(A^s, a^{-s/k}u^{-s/k})\min(B^s, b^{-s/k}u^{-s/k})\,du\Big)^{r_1}\times$$

$$\times\Big(\int_{-\pi}^{\pi}\int_{h^{-1}}^{\infty}(ab)^{-2s/k}v^{-4s/k+1}\,dv\,d\varphi\Big)\Big(\int_{-\pi}^{\pi}\int_0^{\infty}\min(A^{2s}, a^{-2s/k}w^{-2s/k})\times$$

$$\times\min(B^{2s}, b^{-2s/k}w^{-2s/k})\,w\,dw\,d\varphi\Big)^{r_2-1}.$$

Since

$$\int_0^{\infty}\min(A^s, a^{-s/k}u^{-s/k})\min(B^s, b^{-s/k}u^{-s/k})\,du$$

$$\ll A^s\int_0^{\infty}\min(B^s, b^{-s/k}u^{-s/k})\,du$$

$$\ll A^s\Big(\int_0^{B^{-k}b^{-1}}B^s\,du + \int_{B^{-k}b^{-1}}^{\infty}b^{-s/k}u^{-s/k}\,du\Big) \ll A^sB^{s-k}b^{-1}$$

and

$$\int_0^{\infty}\min(A^{2s}, a^{-2s/k}w^{-2s/k})\min(B^{2s}, b^{-2s/k}w^{-2s/k})\,w\,dw \ll A^{2s}B^{2(s-k)}b^{-2},$$

we have by (9), (12), (17), (20) and (23),

$$\int_{E_n-B_\gamma} I(\zeta, A)\,J(\zeta, B)\,dx$$

$$\ll h^{2s/k-1}(ab)^{-s/k}A^{(r_1-1)s}b^{-(r_1-1)}B^{(r_1-1)(s-k)}A^{2r_2s}b^{-2r_2}B^{2r_2(s-k)}+$$

$$+A^{r_1s}b^{-r_1}B^{r_1(s-k)}h^{4s/k-2}(ab)^{-2s/k}A^{2(r_2-1)s}b^{-2(r_2-1)}B^{2(r_2-1)(s-k)}$$

$$\ll h^{2s/k-1}(ab)^{-s/k}b^{-n+1}A^{(n-1)s}B^{(n-1)(s-k)}+$$

$$+h^{4s/k-2}(ab)^{-2s/k}b^{-n+2}B^{(n-2)(s-k)}$$

$$\ll (AB)^{ns}(ab)^{-n}m^{-20kny}\big(m^{\frac{-2sy}{kn}+\frac{y}{n}}+m^{\frac{-4sy}{kn}+\frac{2y}{n}}\big)$$

$$\ll (AB)^{ns}(ab)^{-n}m^{-20kny-17y}.$$

The lemma follows by substitution into (33).

11. The singular integral. Let $\eta' = y_1'\omega_1 + \ldots + y_n'\omega_n$, $\zeta' = x_1'\varrho_1 + \ldots + x_n'\varrho_n$, $dy' = dy_1'\ldots dy_n'$, $dx' = dx_1'\ldots dx_n'$, $\eta = A\eta'$ and $\zeta = a^{-1}b^{-1}m^{-20ky}\zeta'$. The Jacobians of $y_1, \ldots, y_n$ and $x_1, \ldots, x_n$ with respect to $y_1', \ldots, y_n'$ and $x_1', \ldots, x_n'$ are A^n and $(a^{-1}b^{-1}m^{-20ky})^n$ respectively. Set $\gamma_i = \alpha_i/a$ $(1 \leqslant i \leqslant s)$. Then

$$\alpha_i\eta^k\zeta = \gamma_i\eta'^k\zeta',$$

where by (11), γ_i $(1 \leqslant i \leqslant s)$ satisfy

$$(34)\qquad c_{16} < \gamma_i^{(p)} < c_{17},\qquad c_{16} < |\gamma_i^{(q)}| < c_{17},\qquad 1 \leqslant i \leqslant s.$$

Let us write η' and ζ' as η and ζ again and let

$$I_i(\zeta) = \int_P E(\gamma_i\eta^k\zeta)\,dy,$$

where $P = P(1)$. Then

$$I_i(\zeta, A) = A^nI_i(\zeta),\qquad 1 \leqslant i \leqslant s.$$

Similarly, we have

$$J_i(\zeta, B) = B^n J_i(\zeta), \quad 1 \leqslant i \leqslant s,$$

where

$$J_i(\zeta) = \int_P E(-\gamma_{s+i}\eta^k\zeta)\,dy, \quad 1 \leqslant i \leqslant s,$$

and $\gamma_{s+i} = \beta_i/b$ $(1 \leqslant i \leqslant s)$ which satisfy

(35) $$c_{16} < \gamma_{s+i}^{(p)} < c_{17}, \quad c_{16} < |\gamma_{s+i}^{(q)}| < c_{17}, \quad 1 \leqslant i \leqslant s.$$

Set

$$I(\zeta) = \prod_{i=1}^{s} I_i(\zeta), \quad J(\zeta) = \prod_{i=1}^{s} J_i(\zeta) \quad \text{and} \quad \Phi = \int_{E_n} I(\zeta)J(\zeta)\,dx.$$

Then we have

(36) $$\int_{E_n} I(\zeta, A)J(\zeta, B)\,dx = (AB)^{ns}(ab)^{-n}m^{-20kny}\Phi.$$

Now we shall treat the integral Φ by Tatuzawa's method. (See [17].)

If $F(x_1, \ldots, x_t)$ is nondecreasing for variables $x_{g_1}, \ldots, x_{g_r}$ and nonincreasing for other variables $x_{h_1}, \ldots, x_{h_s}$ $(r+s=t)$ over the rectangle

$$I = \{(x_1, \ldots, x_t):\ a_i \leqslant x_i \leqslant b_i\ (1 \leqslant i \leqslant t)\},$$

then F is said to be *monotonic over* I.

LEMMA 15 (Tatuzawa). *Let $F(x_1, \ldots, x_t)$ be a finite product of positive bounded monotonic functions over the rectangle*

$$\{(x_1, \ldots, x_t):\ 0 \leqslant x_i \leqslant c_i\ (1 \leqslant i \leqslant t)\}.$$

If we write

$$\chi_\lambda(x) = \frac{\sin 2\pi\lambda x}{\pi x},$$

then

$$\lim_{\substack{\lambda_i \to \infty \\ (1 \leqslant i \leqslant t)}} \int_0^{c_1} \ldots \int_0^{c_t} F(x_1, \ldots, x_t)\chi_{\lambda_1}(x_1)\ldots\chi_{\lambda_t}(x_t)\,dx_1\ldots dx_t = (\tfrac{1}{2})^t F(+0, \ldots, +0).$$

See Tatuzawa [17], pp. 47–49.

LEMMA 16.

$$\Phi = D^{(1-2s)/2}k^{-2ns}N(\gamma_1\ldots\gamma_s)^{-1/k}\prod_p F_p \prod_q H_q,$$

where

$$F_p = \int_{U_p} \prod_{i=1}^{2s} w_i^{1/k-1}\,dw_1\ldots dw_{2s-1}$$

in which U_p denotes the domain

$$0 \leqslant w_i \leqslant \gamma_i^{(p)}, \quad 1 \leqslant i \leqslant 2s, \quad w_{2s} = w_1 + \ldots + w_s - w_{s+1} - \ldots - w_{2s-1},$$

and where

$$H_q = \int_{V_q} \prod_{i=1}^{2s} w_i^{1/k-1}\,dw_1\ldots dw_{2s-1}\,d\varphi_1\ldots d\varphi_{2s-1}$$

in which V_q denotes the domain

$$0 \leqslant w_i \leqslant |\gamma_i^{(q)}|^2, \quad 1 \leqslant i \leqslant 2s, \quad -\pi \leqslant \varphi_j \leqslant \pi, \quad 1 \leqslant j \leqslant 2s-1,$$
$$w_{2s} = |w_1^{1/2}e^{i\varphi_1} + \ldots + w_{2s-1}^{1/2}e^{i\varphi_{2s-1}}|^2.$$

Proof. By Lemma 13, we have

$$I_i(\zeta) \ll \prod_{j=1}^{n} \min(1, |\zeta^{(j)}|^{-1/k}), \quad 1 \leqslant i \leqslant s,$$

and $J_i(\zeta)$ $(1 \leqslant i \leqslant s)$ satisfy the same inequality. Then by the transformation (31), we have

$$\int_{E_n} |I(\zeta)J(\zeta)|\,dx \ll \prod_p \Big(\int_0^{\infty} \min(1, u_p^{-2s/k})\,du_p\Big) \prod_q \Big(\int_{-\pi}^{\pi}\int_0^{\infty} \min(1, u_q^{-4s/k})\,u_q\,du_q\,d\varphi_q\Big)$$

which converges for $s > k$. Therefore

$$\Phi = \lim_{\lambda_p, \lambda_q, \lambda_q' \to \infty} \Phi(\Omega), \quad \Phi(\Omega) = \int_\Omega I(\zeta)J(\zeta)\,dx,$$

where Ω denotes the closed region of x defined by

$$|v_p| \leqslant \lambda_p, \quad |v_q| \leqslant \lambda_q, \quad |v_q'| \leqslant \lambda_q'$$

in which

$$v_p = \zeta^{(p)}, \quad v_q = \frac{\zeta^{(q)} + \zeta^{(q+r_2)}}{\sqrt{2}}, \quad v_q' = \frac{\zeta^{(q)} - \zeta^{(q+r_2)}}{\sqrt{2}i}.$$

Consider $2ns$ real variables y_{ij} $(1 \leqslant i \leqslant 2s,\ 1 \leqslant j \leqslant n)$. Let

$$\eta_i = y_{i1}\omega_1 + \ldots + y_{in}\omega_n, \quad dY_i = dy_{i1}\ldots dy_{in}$$

and let P_i be the domain

$$0 \leqslant \eta_i^{(p)} \leqslant 1, \quad |\eta_i^{(q)}| \leqslant 1, \quad 1 \leqslant i \leqslant 2s.$$

Let

$$\gamma_1^{(i)}\eta_1^{(i)k}+\ldots+\gamma_s^{(i)}\eta_s^{(i)k}-\gamma_{s+1}^{(i)}\eta_{s+1}^{(i)k}-\ldots-\gamma_{2s}^{(i)}\eta_{2s}^{(i)k}=z_i,\qquad 1\leqslant i\leqslant n,$$

and

$$u_p=z_p,\qquad u_q=\frac{z_q+z_{q+r_2}}{\sqrt{2}},\qquad u'_q=-\frac{z_q-z_{q+r_2}}{\sqrt{2}\,i}.$$

Since

$$\zeta^{(q)}=\frac{v_q+iv'_q}{\sqrt{2}},\quad \zeta^{(q+r_2)}=\frac{v_q-iv'_q}{\sqrt{2}},\quad z_q=\frac{u_q-iu'_q}{\sqrt{2}},\quad z_{q+r_2}=\frac{u_q+iu'_q}{\sqrt{2}},$$

we have

$$\sum_{i=1}^{n}\zeta^{(i)}z_i=\sum_p u_p v_p+\sum_q u_q v_q+\sum_q u'_q v'_q.$$

The Jacobian of $x_1,\ldots,x_n$ with respect to v_p, v_q, v'_q is equal to

$$|\det(\varrho_i^{(j)})|^{-1}|-i|^{r_2}=D^{1/2}.$$

Set

$$dv=dv_1\ldots dv_{r_1}\,dv_{r_1+1}\ldots dv_{r_1+r_2}\,dv'_{r_1+1}\ldots dv'_{r_1+r_2}.$$

We have

$$\begin{aligned}\Phi(\Omega)&=\int_{P_1}\ldots\int_{P_{2s}}dY_1\ldots dY_{2s}\int_{\Omega}\exp\Big(2\pi i\sum_{j=1}^{n}\zeta^{(j)}z_j\Big)dx\\&=D^{1/2}\int_{P_1}\ldots\int_{P_{2s}}dY_1\ldots dY_{2s}\int_{\Omega}\exp\Big(2\pi i\Big(\sum_p u_p v_p+\sum_q u_q v_q+\sum_q u'_q v'_q\Big)\Big)dv\\&=D^{1/2}\int_{P_1}\ldots\int_{P_{2s}}\prod_p\chi_{\lambda_p}(u_p)\prod_q\big(\chi_{\lambda_q}(u_q)\chi_{\lambda'_q}(u'_q)\big)dY_1\ldots dY_{2s}.\end{aligned}$$

Let

$$z_i=t_1\omega_1^{(i)}+\ldots+t_n\omega_n^{(i)},\qquad 1\leqslant i\leqslant n.$$

Then

$$-\gamma_{2s}^{(i)}\eta_{2s}^{(i)k}=z_i-(\gamma_1^{(i)}\eta_1^{(i)k}+\ldots+\gamma_s^{(i)}\eta_s^{(i)k}-\gamma_{s+1}^{(i)}\eta_{s+1}^{(i)k}-\ldots-\gamma_{2s-1}^{(i)}\eta_{2s-1}^{(i)k}).$$

The Jacobian of $y_{2s,1},\ldots,y_{2s,n}$ with respect to $t_1,\ldots,t_n$ is equal to

$$|\det(k\gamma_{2s}^{(i)}\eta_{2s}^{(i)k-1}\omega_j^{(i)})|^{-1}|\det(\omega_r^{(l)})|=N(k^{-1}|\gamma_{2s}^{-1}\eta_{2s}^{1-k}|),$$

and the Jacobian of $t_1,\ldots,t_n$ with respect to u_p, u_q, u'_q is

$$|\det(\omega_r^{(l)})|^{-1}|i|^{r_2}=D^{-1/2}.$$

Therefore

$$\Phi(\Omega)=\int_Q\prod_p\chi_{\lambda_p}(u_p)\prod_q\big(\chi_{\lambda_q}(u_q)\chi_{\lambda'_q}(u'_q)\big)du\times$$
$$\times\int_{P_1}\ldots\int_{P_{2s-1}}N(k^{-1}|\gamma_{2s}^{-1}\eta_{2s}^{1-k}|)\,dY_1\ldots dY_{2s-1},$$

where Q is a closed region containing the origin of u in its interior and

$$du=\prod_p du_p\prod_q(du_q\,du'_q).$$

Let

$$\eta_j^{(p)}=y_{j1}\omega_1^{(p)}+\ldots+y_{jn}\omega_n^{(p)}=u_{jp}^{1/k},$$
$$\eta_j^{(q)}=y_{j1}\omega_1^{(q)}+\ldots+y_{jn}\omega_n^{(q)}=u_{jq}^{1/2k}e^{i\psi_{jq}/k},\qquad 1\leqslant j\leqslant 2s.$$

The Jacobian of $y_{j1},\ldots,y_{jn}$ with respect to u_{jp}, u_{jq}, u'_{jq} is

$$|\det(\omega_r^{(l)})|^{-1}N(k^{-1}|\eta_j^{1-k}|)=D^{-1/2}(k^{-1}|\eta_j^{1-k}|).$$

Then we have

$$\Phi(\Omega)=\int_Q\prod_p\chi_{\lambda_p}(u_p)\prod_q\big(\chi_{\lambda_q}(u_q)\chi_{\lambda'_q}(u'_q)\big)du\times$$
$$\times D^{(1-2s)/2}\int_R N(k^{-1}|\gamma_{2s}^{-1}\eta_{2s}^{1-k}|)\prod_{j=1}^{2s-1}N(k^{-1}|\eta_j^{1-k}|)\times$$
$$\times\prod_{j=1}^{2s-1}(du_{j1}\ldots du_{j,r_1+r_2}\,d\psi_{j,r_1+1}\ldots d\psi_{j,r_1+r_2}),$$

where R denotes the region

$$0\leqslant u_{jl}\leqslant 1\qquad(1\leqslant j\leqslant 2s-1,\ 1\leqslant l\leqslant r_1+r_2),$$
$$-\pi\leqslant\psi_{rt}\leqslant\pi\qquad(1\leqslant r\leqslant 2s-1,\ r_1+1\leqslant t\leqslant r_1+r_2)$$
$$-\gamma_{2s}^{(p)}\eta_{2s}^{(p)k}=z_p-(\gamma_1^{(p)}u_{1p}+\ldots-\gamma_{2s-1}^{(p)}u_{2s-1,p}),$$
$$|\gamma_{2s}^{(q)}\eta_{2s}^{(q)k}|=|z_p-(\gamma_1^{(q)}u_{1q}^{1/2}e^{i\psi_{1q}}+\ldots-\gamma_{2s-1}^{(q)}u_{2s-1,q}^{1/2}e^{i\psi_{2s-1,q}})|,$$
$$0\leqslant\eta_{2s}^{(p)}\leqslant 1,\ |\eta_{2s}^{(q)}|\leqslant 1.$$

Therefore

$$\Phi=\lim_{\lambda_p,\lambda_q,\lambda'_q\to\infty}\Phi(\Omega)=D^{(1-2s)/2}\prod_p F'_p\prod_q H'_q,$$

where

$$F'_p=\lim_{\lambda_p\to\infty}k^{-2s}\gamma_{2s}^{(p)-1}\int_{Q_p}\chi_{\lambda_p}(u_p)\,du_p\int_{U'_p}\prod_{i=1}^{2s}w_i'^{1/k-1}\,dw'_1\ldots dw'_{2s-1}$$

in which w_i' is used instead of u_{ip}, Q_p denotes the range of u_p in Q and U_p' the domain

$$0 \leqslant w_i' \leqslant 1 \quad (1 \leqslant i \leqslant 2s), \qquad \gamma_1^{(p)} w_1' + \ldots - \gamma_{2s}^{(p)} w_{2s}' = z_p,$$

and where

$$H_q' = \lim_{\lambda_q, \lambda_q' \to \infty} k^{-4s} |\gamma_{2s}^{(q)}|^{-2} \int_{Q_q} \chi_{\lambda_q}(u_q) \chi_{\lambda_q'}(u_q') \, du_q \, du_q' \times$$
$$\times \int_{V_q'} \prod_{i=1}^{2s} w_i'^{1/k-1} dw_1' \ldots dw_{2s-1}' \, d\psi_1 \ldots d\psi_{2s-1}$$

in which w_i' stands for u_{iq}, ψ_i for ψ_{iq}, Q_q denotes the region of u_q and u_q' in Q and V_q' the domain

$$0 \leqslant w_i' \leqslant 1 \quad (1 \leqslant i \leqslant 2s), \qquad -\pi \leqslant \psi_j \leqslant \pi \quad (1 \leqslant j \leqslant 2s-1),$$
$$|\gamma_{2s}^{(q)}|^2 w_{2s}' = |z_q - (\gamma_1^{(q)} w_1'^{1/2} e^{i\psi_1} + \ldots - \gamma_{2s-1}^{(q)} w_{2s-1}'^{1/2} e^{i\psi_{2s-1}})|^2.$$

By Lemma 15 and the transformations $\gamma_i^{(p)} w_i' = w_i$ $(1 \leqslant i \leqslant 2s)$ for the integral F_p' and $|\gamma_i^{(q)}|^2 w_i' = w_i$ $(1 \leqslant i \leqslant 2s)$, $\varphi_j = \theta_j + \psi_j$ $(1 \leqslant j \leqslant s)$, $\varphi_l = \theta_l + \psi_l + \pi$ $(s+1 \leqslant l \leqslant 2s-1)$ for H_q', where $\theta_t = \arg \gamma_t^{(q)}$ $(1 \leqslant t \leqslant 2s-1)$, we have

$$F_p' = k^{-2s} \prod_{i=1}^{2s} \gamma_i^{(p)-1/k} F_p, \qquad H_q' = k^{-4s} \prod_{i=1}^{2s} |\gamma_i^{(q)}|^{-2/k} H_q,$$

and the lemma follows.

12. The proof of theorem. By Lemma 11, we have

(37) $$\sum_{\chi \in P(H)} 1 = \frac{(2\pi)^{r_2}}{\sqrt{D}} H^n + O(H^{n-1}).$$

Therefore by (9), (14), (21) and Lemma 10, we have

$$Z = \sum_{\gamma \in \Gamma(t)} \int_{B_\gamma} F(\xi)\, dx + O\big(H^n (AB)^{ns} (ab)^{-n} m^{-20kny-17y}\big).$$

For a given $\mathfrak{a}$, the number of γ in Γ, subject to $\gamma \to \mathfrak{a}$, is $O(N(\mathfrak{a}))$. By Theorems 35 and 76 in Hecke [5], it follows that the number of $\mathfrak{a}$ with $N(\mathfrak{a}) = d$ is $O\big(\sum_{d_1 \ldots d_n = d} 1\big) = O(d^{\varepsilon_4})$. Therefore

$$\sum_{\gamma \in \Gamma(t)} 1 \ll \sum_{N(\mathfrak{a}) \leqslant t^n} N(\mathfrak{a}) \ll \sum_{d \leqslant t^n} d^2 \ll t^{3n} = m^{3y},$$

and by (20), (36) and Lemmas 14 and 16, we have

$$Z = J_0 \,\mathfrak{S}(t, H) (AB)^{ns} (ab)^{-n} m^{-20kny} + O\big(H^n (AB)^{ns} (ab)^{-n} m^{-20kny-14y}\big),$$

where

$$J_0 = D^{(1-2s)/2} k^{-2ns} N(\gamma_1 \ldots \gamma_{2s})^{-1/k} \prod_p F_p \prod_q H_q$$

and

$$\mathfrak{S} = \mathfrak{S}(t, H) = \sum_{\chi \in P(H)} \sum_{\gamma \in \Gamma(t)} G(\gamma) H(\gamma) E(-q\chi\gamma).$$

Let $\sum_\gamma^*$ denote a sum, where γ runs over a reduced residue system of $(\mathfrak{a}\mathfrak{d})^{-1}$, mod $\mathfrak{d}^{-1}$. Thus $\gamma \in (\mathfrak{a}\mathfrak{d})^{-1}$, $(\gamma, \mathfrak{d}^{-1}) = (\mathfrak{a}, \mathfrak{d}^{-1})$, and we take only one γ in each class modulo $\mathfrak{d}^{-1}$. Then

$$\mathfrak{S} = \sum_{N(\mathfrak{a})=1} \sum_\gamma^* G(\gamma) H(\gamma) \sum_{\chi \in P(H)} E(-q\chi\gamma) + \sum_{1 < N(\mathfrak{a}) \leqslant t^n} \sum_\gamma^* G(\gamma) H(\gamma) \sum_{\chi \in P(H)} E(-q\chi\gamma)$$
$$= \mathfrak{S}_1 + \mathfrak{S}_2, \quad \text{say}.$$

By (37) we have

$$\mathfrak{S}_1 = \sum_{\chi \in P(H)} 1 = \frac{(2\pi)^{r_2}}{\sqrt{D}} H^n + O(H^{n-1}).$$

If $N(\mathfrak{a}) > 1$, then

$$\sum_{\chi (\mathrm{mod}\, \mathfrak{a})} E(-q\chi\gamma) = 0.$$

(See, e.g., Hecke [5], p. 197.) For any given integer μ, it follows by (17), (20) and Lemma 11 that the number of $\nu \in \mathfrak{a}$ and $\nu + \mu \in P(H)$ is

$$\frac{(2\pi)^{r_2}}{\sqrt{D}\, N(\mathfrak{a})} H^n + O\left(\frac{H^{n-1}}{N(\mathfrak{a})^{1-1/n}}\right).$$

Hence if the domain $\chi \in P(H)$ is split up into a union of complete residue sets (mod $\mathfrak{a}$), plus a few others, remaining elements, say R elements, then

$$R \ll N(\mathfrak{a}) \frac{H^{n-1}}{N(\mathfrak{a})^{1-1/n}} = N(\mathfrak{a})^n H^{n-1},$$

and therefore by (17) and (20)

$$\mathfrak{S}_2 \ll \sum_{N(\mathfrak{a}) \leqslant t^n} \sum_\gamma^* R \ll H^{n-1} \sum_{N(\mathfrak{a}) \leqslant t^n} \sum_\gamma^* N(\mathfrak{a})^{1/n}$$
$$\ll H^{n-1} \sum_{N(\mathfrak{a}) \leqslant t^n} N(\mathfrak{a})^{1+1/n} \ll H^{n-1} \sum_{d \leqslant t^n} d^3 \ll H^{n-1} t^{4n} \ll H^n m^{-2y}.$$

Consequently, we have

$$\mathfrak{S} \geqslant c_{22}(K) H^n.$$

It follows by (23), (34) and (35) that $J_0 \geqslant c_{23}(k, K, h, y)$. Therefore

$$Z > c_{24}(k, K, h, y) H^n (AB)^{ns} (ab)^{-n} m^{-20kny} > 1$$

if $m \geqslant c_{14}(k, K, x', \varepsilon)$. The theorem is proved.

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Théorèmes de densité dans $F_q[X]$

par

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Introduction. Soit F_q le corps fini à q éléments. Soit $\mathscr{U}$ l'ensemble des polynômes unitaires de l'anneau $F_q[X]$. Soit I un ensemble de polynômes irréductibles unitaires de $F_q[X]$ et $\mathscr{U}(I)$ l'ensemble des polynômes de $\mathscr{U}$ dont tous les facteurs irréductibles sont dans I. Soit $a(n, I)$ le nombre de polynômes de degré n de $\mathscr{U}(I)$. Dans [6] on démontre que lorsque l'ensemble I vérifie certaines conditions de régularité, on a une estimation asymptotique du nombre $a(n, I)$. Ces conditions de régularité sont par exemple réalisées lorsque I est l'ensemble des polynômes irréductibles de degré congru à r modulo un entier h. Nous imposons maintenant des conditions de régularité d'un autre type. L'ensemble I sera l'ensemble des polynômes irréductibles de degré au plus d (ou au moins d). Nous obtenons des résultats analogues aux résultats connus sur les nombres $\Psi(x, y)$, resp. $\Phi(x, y)$ d'entiers $n \leqslant x$ n'ayant aucun facteur premier $p > y$, resp. $p < y$. On trouvera une démonstration de ces résultats dans [7], [3], [4], [2]. L'estimation des nombres $a(n, I)$ s'exprimera à l'aide de la fonction ϱ de Dickman [7], [1], et de la fonction ω de Buchstab [5]. Nous étudierons les nombres $a(n, I)$ lorsque I est l'un des deux ensembles suivants:

ensemble des polynômes irréductibles de degré inférieur à un nombre y donné,

ensemble des polynômes irréductibles de degré supérieur à un nombre y donné.

Nous indiquerons sans démonstration les résultats que l'on peut obtenir lorsque I est l'ensemble des polynômes irréductibles de degré appartenant à un intervalle (x, y) donné ou lorsque I est le complémentaire d'un tel ensemble et une généralisation possible de certains résultats.

I. Notations et conventions. On désigne par $\mathscr{U}_n$ l'ensemble des polynômes unitaires de degré n de $F_q[X]$. Remarquons que

$$\text{(I.1)} \qquad \operatorname{Card}(\mathscr{U}_n) = q^n.$$

On note Π_n le nombre de polynômes irréductibles appartenant à $\mathscr{U}_n$. On a la