and consequently by (1.4) and (1.3)

$$I_k = \beta_k^{(k)} = \beta_k^{(k+1)} - \sum_{i=0}^{k-1} \gamma_i \beta_{k-i-1}^{(k)} = a_0^{(k+1)} - \sum_{i=0}^{k-1} i!\,\gamma_i a_i^{(k)}$$

which is (1.2).

Finally (1.5) follows from (2.3) and Euler's multinominal formula [2] which states that if $b_0 \neq 0$ and s is any real number, then

$$\Big(\sum_{n=0}^{\infty} b_n (z-a)^n\Big)^s = \sum_{n=0}^{\infty} B_n^{(s)} (z-a)^n$$

where

$$B_0^{(s)} = b_0^s \quad \text{and} \quad B_n^{(s)} = \frac{1}{nb_0} \sum_{i=1}^{n} \big(i(s+1)-n\big) b_i B_{n-i}^{(s)} \quad \text{for} \quad n \geqslant 1.$$

Remark. We note that the numbers $B_n^{(k)}$ and $\beta_n^{(k)}$ are related by

$$\beta_n^{(k)} = (-1)^n \sum_{i=0}^{n} (-1)^i B_i^{(k)}.$$

and that $B_n^{(k)}$'s satisfy the recurrence formula

$$B_n^{(k)} = \sum_{i=0}^{n} \alpha_i B_{n-i}^{(k-1)} = B_n^{(k-1)} + \sum_{i=0}^{n-1} \gamma_i B_{n-i-1}^{(k-1)}.$$

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On sum-free sequences

by

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A sequence A: $a_1 < a_2 < a_3 \ldots$ of positive integers is said to be *sum-free* if no member of A is the sum of two or more other members of A. P. Erdös [1] proved a number of results concerning sum-free sequences. One of these is that for any such sequence

$$\sum (1/a_i) < 103.$$

This leads one to define ϱ by

$$\varrho = \sup_A \{\sum_{a\in A} 1/a\}$$

where the supremum is taken over all sum-free sequences A. The powers of 2 form a sum-free sequence so that $2 \leqslant \varrho < 103$. Levine and O'Sullivan [2] considerably improved on Erdös' upper bound by showing that $\varrho < 3.97$ and they constructed an example which shows $\varrho > 2.0351$.

The object of this note is to exhibit an example of a sum-free sequence which establishes $\varrho > 2.0648$. The construction is fairly elaborate. The relatively modest improvement over the result of Levine and O'Sullivan can perhaps be considered as evidence supporting their conjecture that ϱ is much closer to 2 than to 4. The construction is given in the following theorem.

Theorem. *Let A be a (finite) sum-free set. Let $s = \sum_{a\in A} a$ and let t be an integer exceeding s. Define integers l, m, n, r and p as follows:*

$$l = \binom{t-s+2}{2}, \quad m = \binom{t-s+1}{2},$$

$$n = \left[\frac{l-1+s}{t}\right], \quad r = l-nt-1,$$

$$p = \binom{l+1}{2} - \binom{r+1}{2} + n.$$

Suppose that A and t are chosen so that $r > 0$. Define sets B and C as follows:

$$B = \{\mu t+1:\ \mu = 1, 2, \ldots, l\}, \qquad C = \{(p+\nu)t+1:\ \nu = 1, 2, \ldots, m+1\}.$$

Then $S = A \cup B \cup C$ is a sum-free set.

Proof. Suppose $S = A \cup B \cup C$ is not sum-free. Elements of A, B or C are denoted by the corresponding lower case letters.

Case 1. Some element of B is a sum of elements of S. Since the least member of B exceeds the sum of all elements of A, we must have

$$b_0 = b_1+b_2+\ldots+b_k+a_1+a_2+\ldots+a_j, \qquad k \geqslant 1,\ k+j \geqslant 2.$$

Since $b_i = \mu_i t+1$, we get

$$(1) \qquad (\mu_0-\mu_1-\mu_2-\ldots-\mu_k)t = k-1+a_1+a_2+\ldots+a_j.$$

Since the right side of (1) is positive we must have

$$\mu_0-\mu_1-\mu_2-\ldots-\mu_k \geqslant 1,$$

so that

$$k \geqslant t+1-a_1-a_2-\ldots-a_j \geqslant t+1-s.$$

Thus

$$\mu_0 \geqslant 1+\mu_1+\mu_2+\ldots+\mu_k \geqslant 1+\frac{k(k+1)}{2} \geqslant \binom{t-s+2}{2}+1 > l,$$

contrary to the definition of B.

Case 2. Some element of C is a sum of elements of $A \cup B$. We have

$$c_0 = b_1+b_2+\ldots+b_k+a_1+a_2+\ldots+a_j, \qquad k \geqslant 1,\ k+j \geqslant 2.$$

Since $c_0 = (p+\nu_0)t+1$ and $b_i = \mu_i t+1$, we get

$$(2) \qquad (p+\nu_0-\mu_1-\mu_2-\ldots-\mu_k)t = k-1+a_1+a_2+\ldots+a_j.$$

We need to distinguish three subcases.

Case 2.1. $p+\nu_0-\mu_1-\mu_2-\ldots-\mu_k = n$. We then have, from (2)

$$(3) \qquad nt = k-1+a_1+a_2+\ldots+a_j,$$

so that

$$k \leqslant nt+1 = l-r.$$

We also have, from the definition of p,

$$\binom{l+1}{2} = \binom{r+1}{2}+\mu_1+\mu_2+\ldots+\mu_k-\nu_0.$$

In order for this to hold we must have $\nu_0 = 0$, $k = l-r$ and the numbers $\mu_1, \mu_2, \ldots, \mu_k$ must be the numbers $r+1, r+2, \ldots, l$. We then get, from (3),

$$nt = l-r-1+a_1+a_2+\ldots+a_j = nt+a_1+a_2+\ldots+a_j.$$

It follows that $j = 0$ and thus that

$$c_0 = b_1+b_2+\ldots+b_k = (\mu_1+\mu_2+\ldots+\mu_k)t+k$$

$$= \left(\binom{l+1}{2}-\binom{r+1}{2}\right)t+l-r = (p-n)t+l-r = pt+1.$$

However, the least member of C is $(p+1)t+1$. This disposes of case 2.1.

Case 2.2. $p+\nu_0-\mu_1-\mu_2-\ldots-\mu_k \geqslant n+1$. Then we have, from (2),

$$(n+1)t \leqslant k-1+a_1+a_2+\ldots+a_j \leqslant k-1+s \leqslant l-1+s.$$

This gives $n \leqslant \left[\dfrac{l-1+s}{t}\right]-1$, a contradiction.

Case 2.3. $p+\nu_0-\mu_1-\mu_2-\ldots-\mu_k \leqslant n-1$. Then we have, from (2),

$$k \leqslant (n-1)t+1-a_1-a_2-\ldots-a_j \leqslant (n-1)t+1 = l-r-t < l-r.$$

Thus

$$\mu_1+\mu_2+\ldots+\mu_k < l+(l-1)+(l-2)+\ldots+(r+1) = \binom{l+1}{2}-\binom{r+1}{2}.$$

But then

$$p+\nu_0-\mu_1-\mu_2-\ldots-\mu_k > p+\nu_0-\binom{l+1}{2}+\binom{r+1}{2} = n+\nu_0 \geqslant n,$$

another contradiction.

Case 3. Some member of C is the sum of at least one element of C and some elements of $A \cup B$. It is easy to check that the sum of the smallest two members of C exceeds the largest member. Thus we must have

$$c_0 = c_1+b_1+b_2+\ldots+b_k+a_1+a_2+\ldots+a_j, \qquad k+j \geqslant 1.$$

This gives, on setting $c_i = (p+\nu_i)t+1$, $b_i = \mu_i t+1$,

$$(\nu_0-\nu_1-\mu_1-\mu_2-\ldots-\mu_k)t = k+a_1+a_2+\ldots+a_j.$$

We must have

$$\nu_0-\nu_1 \geqslant \mu_1+\mu_2+\ldots+\mu_k+1 \geqslant \frac{k(k+1)}{2}+1.$$

Now $k+s \geqslant k+a_1+a_2+\ldots+a_j \geqslant t$ so that

$$v_0 - v_1 \geqslant \frac{(t-s)(t-s+1)}{2} + 1 = m+1,$$

contrary to the definition of C. This completes the proof of the theore

A computer program was written to compute $\sum_{a \in S} (1/a)$ for various set

and various choices of t. It was found that if $A = \{1, 2, 4, 8\}$ and $t = 24$, (gets

$$\sum_{a \in S} (1/a) > 2.0648.$$

An infinite sum-free set may now be obtained by adjoining to S sufficiently large powers of 2.

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