

and consequently by (1.4) and (1.3)

$$I_k = \beta_k^{(k)} = \beta_k^{(k+1)} - \sum_{i=0}^{k-1} \gamma_i \beta_{k-i-1}^{(k)} = a_0^{(k+1)} - \sum_{i=0}^{k-1} i! \gamma_i a_i^{(k)}$$

which is (1.2).

Finally (1.5) follows from (2.3) and Euler's multinomial formula [2] which states that if $b_0 \neq 0$ and s is any real number, then

$$\left(\sum_{n=0}^{\infty} b_n (z-a)^n \right)^s = \sum_{n=0}^{\infty} B_n^{(s)} (z-a)^n$$

where

$$B_0^{(s)} = b_0^s \quad \text{and} \quad B_n^{(s)} = \frac{1}{nb_0} \sum_{i=1}^n (i(s+1)-n) b_i B_{n-i}^{(s)} \quad \text{for } n \geq 1.$$

Remark. We note that the numbers $B_n^{(k)}$ and $\beta_n^{(k)}$ are related by

$$\beta_n^{(k)} = (-1)^n \sum_{i=0}^n (-1)^i B_i^{(k)}$$

and that $B_n^{(k)}$'s satisfy the recurrence formula

$$B_n^{(k)} = \sum_{i=0}^n \alpha_i B_{n-i}^{(k-1)} = B_n^{(k-1)} + \sum_{i=0}^{n-1} \gamma_i B_{n-i-1}^{(k-1)}$$

References

- [1] B. Baillaud and H. Bourget, *Correspondence d'Hermite et de Stieltjes*, Tome I, Gauthier-Villars, Paris 1905.
- [2] H. W. Gould, *Coefficient identities for powers of Taylor and Dirichlet series*, Amer. Math. Monthly 8 (1974), pp. 3-14.
- [3] A. F. Lavrik, *On the main term in the problem of divisors and the power series coefficients of Riemann zeta-function in a neighbourhood of its pole* (Russian), Trudy Math. Inst. Acad. Sci. USSR 142 (1976), pp. 165-173.
- [4] A. F. Lavrik, M. I. Israilov and Ž. Edgоров, *On an integral containing the remainder term in divisor problems* (Russian), Acta Arith. 37 (1980), pp. 381-389.
- [5] E. C. Titchmarsh, *The theory of the Riemann zeta function*, Clarendon Press, Oxford 1951.

DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF TOLEDO
Toledo, Ohio 43606
U.S.A.

Received on 26. 7. 1985

On sum-free sequences

by

H. L. ABBOTT (Edmonton, Canada)

A sequence $A: a_1 < a_2 < a_3 \dots$ of positive integers is said to be *sum-free* if no member of A is the sum of two or more other members of A . P. Erdős [1] proved a number of results concerning sum-free sequences. One of these is that for any such sequence

$$\sum (1/a_i) < 103.$$

This leads one to define ϱ by

$$\varrho = \sup_A \left\{ \sum_{a \in A} 1/a \right\}$$

where the supremum is taken over all sum-free sequences A . The powers of 2 form a sum-free sequence so that $2 \leq \varrho < 103$. Levine and O'Sullivan [2] considerably improved on Erdős' upper bound by showing that $\varrho < 3.97$ and they constructed an example which shows $\varrho > 2.0351$.

The object of this note is to exhibit an example of a sum-free sequence which establishes $\varrho > 2.0648$. The construction is fairly elaborate. The relatively modest improvement over the result of Levine and O'Sullivan can perhaps be considered as evidence supporting their conjecture that ϱ is much closer to 2 than to 4. The construction is given in the following theorem.

THEOREM. Let A be a (finite) sum-free set. Let $s = \sum_{a \in A} a$ and let t be an integer exceeding s . Define integers l, m, n, r and p as follows:

$$l = \binom{t-s+2}{2}, \quad m = \binom{t-s+1}{2},$$

$$n = \left\lfloor \frac{l-1+s}{t} \right\rfloor, \quad r = l - nt - 1,$$

$$p = \binom{l+1}{2} - \binom{r+1}{2} + n.$$

Suppose that A and t are chosen so that $r > 0$. Define sets B and C as follows:

$$B = \{\mu t + 1; \mu = 1, 2, \dots, l\}, \quad C = \{(p+v)t + 1; v = 1, 2, \dots, m+1\}.$$

Then $S = A \cup B \cup C$ is a sum-free set.

Proof. Suppose $S = A \cup B \cup C$ is not sum-free. Elements of A , B or C are denoted by the corresponding lower case letters.

Case 1. Some element of B is a sum of elements of S . Since the least member of B exceeds the sum of all elements of A , we must have

$$b_0 = b_1 + b_2 + \dots + b_k + a_1 + a_2 + \dots + a_j, \quad k \geq 1, k+j \geq 2.$$

Since $b_i = \mu_i t + 1$, we get

$$(1) \quad (\mu_0 - \mu_1 - \mu_2 - \dots - \mu_k)t = k-1 + a_1 + a_2 + \dots + a_j.$$

Since the right side of (1) is positive we must have

$$\mu_0 - \mu_1 - \mu_2 - \dots - \mu_k \geq 1,$$

so that

$$k \geq t+1 - a_1 - a_2 - \dots - a_j \geq t+1-s.$$

Thus

$$\mu_0 \geq 1 + \mu_1 + \mu_2 + \dots + \mu_k \geq 1 + \frac{k(k+1)}{2} \geq \binom{t-s+2}{2} + 1 > l,$$

contrary to the definition of B .

Case 2. Some element of C is a sum of elements of $A \cup B$. We have

$$c_0 = b_1 + b_2 + \dots + b_k + a_1 + a_2 + \dots + a_j, \quad k \geq 1, k+j \geq 2.$$

Since $c_0 = (p+v_0)t+1$ and $b_i = \mu_i t + 1$, we get

$$(2) \quad (p+v_0 - \mu_1 - \mu_2 - \dots - \mu_k)t = k-1 + a_1 + a_2 + \dots + a_j.$$

We need to distinguish three subcases.

Case 2.1. $p+v_0 - \mu_1 - \mu_2 - \dots - \mu_k = n$. We then have, from (2)

$$(3) \quad nt = k-1 + a_1 + a_2 + \dots + a_j,$$

so that

$$k \leq nt+1 = l-r.$$

We also have, from the definition of p ,

$$\binom{l+1}{2} = \binom{r+1}{2} + \mu_1 + \mu_2 + \dots + \mu_k - v_0.$$

In order for this to hold we must have $v_0 = 0$, $k = l-r$ and the numbers $\mu_1, \mu_2, \dots, \mu_k$ must be the numbers $r+1, r+2, \dots, l$. We then get, from (3),

$$nt = l-r-1 + a_1 + a_2 + \dots + a_j = nt + a_1 + a_2 + \dots + a_j.$$

It follows that $j = 0$ and thus that

$$c_0 = b_1 + b_2 + \dots + b_k = (\mu_1 + \mu_2 + \dots + \mu_k)t + k$$

$$= \left(\binom{l+1}{2} - \binom{r+1}{2} \right) t + l - r = (p-n)t + l - r = pt + 1.$$

However, the least member of C is $(p+1)t+1$. This disposes of case 2.1.

Case 2.2. $p+v_0 - \mu_1 - \mu_2 - \dots - \mu_k \geq n+1$. Then we have, from (2),

$$(n+1)t \leq k-1 + a_1 + a_2 + \dots + a_j \leq k-1 + s \leq l-1+s.$$

This gives $n \leq \left\lfloor \frac{l-1+s}{t} \right\rfloor - 1$, a contradiction.

Case 2.3. $p+v_0 - \mu_1 - \mu_2 - \dots - \mu_k \leq n-1$. Then we have, from (2),

$$k \leq (n-1)t + 1 - a_1 - a_2 - \dots - a_j \leq (n-1)t + 1 = l-r-t < l-r.$$

Thus

$$\mu_1 + \mu_2 + \dots + \mu_k < l + (l-1) + (l-2) + \dots + (r+1) = \binom{l+1}{2} - \binom{r+1}{2}.$$

But then

$$p+v_0 - \mu_1 - \mu_2 - \dots - \mu_k > p+v_0 - \left(\binom{l+1}{2} \right) + \binom{r+1}{2} = n+v_0 \geq n,$$

another contradiction.

Case 3. Some member of C is the sum of at least one element of C and some elements of $A \cup B$. It is easy to check that the sum of the smallest two members of C exceeds the largest member. Thus we must have

$$c_0 = c_1 + b_1 + b_2 + \dots + b_k + a_1 + a_2 + \dots + a_j, \quad k+j \geq 1.$$

This gives, on setting $c_i = (p+v_i)t+1$, $b_i = \mu_i t + 1$,

$$(v_0 - v_1 - \mu_1 - \mu_2 - \dots - \mu_k)t = k + a_1 + a_2 + \dots + a_j.$$

We must have

$$v_0 - v_1 \geq \mu_1 + \mu_2 + \dots + \mu_k + 1 \geq \frac{k(k+1)}{2} + 1.$$



Now $k+s \geq k+a_1+a_2+\dots+a_j \geq t$ so that

$$v_0 - v_1 \geq \frac{(t-s)(t-s+1)}{2} + 1 = m+1,$$

contrary to the definition of C . This completes the proof of the theorem.

A computer program was written to compute $\sum_{a \in S} (1/a)$ for various sets and various choices of t . It was found that if $A = \{1, 2, 4, 8\}$ and $t = 24$, C gets

$$\sum_{a \in S} (1/a) > 2.0648.$$

An infinite sum-free set may now be obtained by adjoining to S sufficiently large powers of 2.

References

- [1] P. Erdős, *Remarks on number theory III. Some problems in additive number theory*, *Mathematika* 13 (1962), pp. 28-38.
 [2] E. Levine and J. O'Sullivan, *An upper estimate for the reciprocal sum of a sum-free sequence*, *Acta Arith.* 23 (1977), pp. 9-24.

DEPARTMENT OF MATHEMATICS
 UNIVERSITY OF ALBERTA
 Edmonton, Canada
 T6G 2G1

Received on 29. 7. 1985
 and in revised form on 6. 9. 1985

Les volumes IV
 et suivants sont
 à obtenir chez

Volumes from IV
 on are available
 at

Die Bände IV und
 folgende sind zu
 beziehen durch

Томы IV и следую-
 щие можно по-
 лучить через

Ars Polona, Krakowskie Przedmieście 7, 00-068 Warszawa

Les volumes I-III
 sont à obtenir chez

Volumes I-III
 are available at

Die Bände I-III sind
 zu beziehen durch

Томы I-III можно
 получить через

Johnson Reprint Corporation, 111 Fifth Ave., New York, N. Y.

BOOKS PUBLISHED BY THE POLISH ACADEMY OF SCIENCES INSTITUTE OF MATHEMATICS

- S. Banach, *Oeuvres*, vol. II, 1979, 470 pp.
 S. Mazurkiewicz, *Travaux de topologie et ses applications*, 1969, 380 pp.
 W. Sierpiński, *Oeuvres choisies*, vol. I, 1974, 300 pp.; vol. II, 1975, 780 pp.; vol. III, 1976, 688 pp.
 J. P. Schauder, *Oeuvres*, 1978, 487 pp.
 K. Borsuk, *Collected papers*, Parts I, II, 1983, xxiv+1357 pp.
 H. Steinhaus, *Selected papers*, 1985, 899 pp.

MONOGRAFIE MATEMATYCZNE

43. J. Szarski, *Differential inequalities*, 2nd ed., 1967, 256 pp.
 50. K. Borsuk, *Multidimensional analytic geometry*, 1969, 443 pp.
 51. R. Sikorski, *Advanced calculus, Functions of several variables*, 1969, 460 pp.
 58. C. Bessaga and A. Pełczyński, *Selected topics in infinite-dimensional topology*, 1975, 353 pp.
 59. K. Borsuk, *Theory of shape*, 1975, 379 pp.
 62. W. Narkiewicz, *Classical problems in number theory*, 1986, 363 pp.

BANACH CENTER PUBLICATIONS

- Vol. 1. *Mathematical control theory*, 1976, 166 pp.
 Vol. 5. *Probability theory*, 1979, 289 pp.
 Vol. 6. *Mathematical statistics*, 1980, 377 pp.
 Vol. 7. *Discrete mathematics*, 1982, 224 pp.
 Vol. 8. *Spectral theory*, 1982, 603 pp.
 Vol. 9. *Universal algebra and applications*, 1982, 454 pp.
 Vol. 10. *Partial differential equations*, 1983, 422 pp.
 Vol. 11. *Complex analysis*, 1983, 362 pp.
 Vol. 12. *Differential geometry*, 1984, 288 pp.
 Vol. 13. *Computational mathematics*, 1984, 792 pp.
 Vol. 14. *Mathematical control theory*, 1985, 643 pp.
 Vol. 15. *Mathematical models and methods in mechanics*, 1985, 725 pp.
 Vol. 16. *Sequential methods in statistics*, 1985, 554 pp.
 Vol. 17. *Elementary and analytic theory of numbers*, 1985, 498 pp.
 Vol. 18. *Geometric and algebraic topology*, in the press.
 Vol. 19. *Partial differential equations*, in the press.
 Vol. 20. *Singularities*, in the press.
 Vol. 21. *Mathematical problems in computation theory*, in the press.