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An integral involving the remainder term in the Piltz divisor problem

by

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1. Introduction. Let $\tau_k(n)$ denote the number of ordered k-tuples $(x_1, x_2, \ldots, x_k)$ of positive integers such that $x_1 x_2 \ldots x_k = n$ and

$$\sum_{n \leqslant x} \tau_k(n) = xP_k(\log x) + \Delta_k(x) \tag{1.1}$$

where $xP_k(\log x)$ is the residue of $\zeta^k(s)x^s/s$ at $s = 1$. Further let

$$P_k(\log x) = a_{k-1}^{(k)}(\log x)^{k-1} + \ldots + a_1^{(k)}(\log x) + a_0^{(k)},$$

$$I_k = \int_1^\infty \frac{\Delta_k(u)}{u^2}\,du,$$

$$\gamma_n = \frac{(-1)^n}{n!} \lim_{M\to\infty} \left[\sum_{1 \leqslant m \leqslant M} \frac{(\log m)^n}{m} - \frac{(\log M)^{n+1}}{n+1}\right]$$

and

$$\beta_n^{(k)} = (-1)^n \left[1 + \sum_{r=1}^{n} (-1)^r \sum_{s=1}^{r} \binom{k}{s} \sum_{\substack{i_1, i_2, \ldots, i_s \geqslant 0 \\ i_1 + i_2 + \ldots + i_s = r-s}} \gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_s}\right].$$

Recently, A. F. Lavrik, M. I. Israilov and Ž. Edgorov [4] proved that for $k \geqslant 1$

$$I_k = a_0^{(k+1)} - \sum_{m=0}^{k-1} m!\,\gamma_m a_m^{(k)} \tag{1.2}$$

and also expressed I_k, $1 \leqslant k \leqslant 5$, explicitly in terms of γ_n's, $0 \leqslant n \leqslant 4$ using Lavrik's [3] representation (in a slightly different notation)

$$a_j^{(k)} = \frac{\beta_{k-1-j}^{(k)}}{j!}, \quad 0 \leqslant j \leqslant k-1. \tag{1.3}$$

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The aim of this note is to give simple proofs of (1.2) and (1.3) and to express I_k explicitly in terms of γ_n's, namely

$$I_k = \beta_k^{(k)}. \tag{1.4}$$

We also prove the following alternate form of (1.4):

$$I_k = \sum_{i=0}^{k} (-1)^i B_{k-i}^{(k)} \tag{1.5}$$

where the numbers $B_n^{(k)}$ are defined recursively by

$$B_0^{(k)} = 1,$$

$$nB_n^{(k)} = \sum_{i=0}^{n-1} \big((i+1)(k+1)-n\big)\gamma_i B_{n-i-1}^{(k)}, \qquad n \geqslant 1,\ k \geqslant 1.$$

2. Proofs of (1.2)–(1.5). By partial summation and (1.1) we have for $\operatorname{Re} s > 1$

$$\sum_{n\leqslant x} \tau_k(n) n^{-s} = \Big(\sum_{n\leqslant x} \tau_k(n)\Big) x^{-s} + s\int_1^x \frac{uP_x(\log u)+\Delta_k(u)}{u^{s+1}}\,du. \tag{2.1}$$

Since

$$\sum_{n=1}^{\infty} \tau_k(n) n^{-s} = \zeta^k(s), \qquad \sum_{n\leqslant x} \tau_k(n) \ll_k x^{1+\varepsilon} \quad \text{for each } \varepsilon > 0$$

and

$$\int_1^{\infty} (\log u)^i u^{-s}\,du = i!\,(s-1)^{-i-1} \quad \text{for} \quad i \in Z^{(0)},$$

we have, on letting $x \to \infty$ in (2.1)

$$\int_1^{\infty} \frac{\Delta_k(u)}{u^{s+1}}\,du = \frac{\zeta^k(s)}{s} - \sum_{i=0}^{k-1} a_i^{(k)} \frac{i!}{(s-1)^{i+1}}. \tag{2.2}$$

By elementary arguments (cf. [5], Chapter 12), we have $\Delta_k(x) \ll x^{1-1/k}$. Hence $\int_1^{\infty} \Delta_k(u) u^{-s-1}\,du$ converges uniformly and absolutely on every compact subset of the half-plane $\operatorname{Re} s > 1-1/k$ and thus defines an analytic function, say $f_k(s)$, there. Thus (2.2) is valid (at least) in the half-plane $\operatorname{Re} s > 1-1/k$ and I_k $(= f_k(1))$ equals the constant term in the Laurent expansion of $\zeta^k(s)/s$ at $s = 1$. To find this, let $\alpha_0 = 1$ and $\alpha_n = \gamma_{n-1}$ for $n \geqslant 1$. It is well known, due to Stieltjes (cf. [1], p. 155), that

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \gamma_n (s-1)^n$$

where $\gamma_0 = \gamma$ is the Euler's constant. Hence for $|s-1| < 1$

$$\begin{aligned} \frac{\zeta^k(s)}{s} &= \Big(1+\sum_{n=0}^{\infty} \gamma_n (s-1)^{n+1}\Big)^k \{1+(s-1)\}^{-1} (s-1)^{-k} \\ &= \Big(\sum_{n=0}^{\infty} \alpha_n (s-1)^n\Big)^k \Big(\sum_{n=0}^{\infty} (-1)^n (s-1)^n\Big)(s-1)^{-k} \end{aligned} \tag{2.3}$$

and consequently

$$\begin{aligned} I_k = f_k(1) &= \sum_{\substack{i+i_1+\ldots+i_k=k\\ i,i_j\geqslant 0}} (-1)^i \alpha_{i_1}\ldots\alpha_{i_k} = \sum_{r=0}^{k} (-1)^{k-r} \sum_{\substack{i_1+\ldots+i_k=r\\ i_j\geqslant 0}} \alpha_{i_1}\ldots\alpha_{i_k} \\ &= (-1)^k + \sum_{r=1}^{k} (-1)^{k-r} \sum_{\substack{1\leqslant s\leqslant r\\ i_1+\ldots+i_s=r,\, i_j\geqslant 1}} \binom{k}{s} \gamma_{i_1-1}\gamma_{i_2-1}\cdots\gamma_{i_s-1} \\ &= \beta_k^{(k)} \end{aligned}$$

which is (1.4).

To prove (1.3), we have by (2.3)

$$\frac{\zeta^k(s)}{s} = \Big(\sum_{n=0}^{\infty} \beta_n^{(k)} (s-1)^n\Big)(s-1)^{-k} \tag{2.4}$$

so that

$$\begin{aligned} P_k(\log x) &= \operatorname*{Res}_{s=1} \frac{\zeta^k(s)\, x^{s-1}}{s} \\ &= \operatorname*{Res}_{s=1} \frac{\sum_{n=0}^{\infty} \beta_n^{(k)} (s-1)^n \sum_{n=0}^{\infty} \frac{(\log x)^n}{n!}(s-1)^n}{(s-1)^k} \\ &= \sum_{j=0}^{k-1} \frac{\beta_{k-1-j}^{(k)} (\log x)^j}{j!}. \end{aligned}$$

Now (1.3) follows in view of $P_k(\log x) = \sum_{j=0}^{k-1} a_j^{(k)} (\log x)^j$.

To prove (1.2), we have by (2.4)

$$\begin{aligned} \sum_{n=0}^{\infty} \beta_n^{(k+1)} (s-1)^n &= \frac{\big((s-1)\zeta(s)\big)^{k+1}}{s} = \frac{\big((s-1)\zeta(s)\big)^k}{s}\big((s-1)\zeta(s)\big) \\ &= \Big(\sum_{n=0}^{\infty} \beta_n^{(k)} (s-1)^n\Big)\Big(\sum_{n=0}^{\infty} \alpha_n (s-1)^n\Big). \end{aligned}$$

Hence

$$\beta_n^{(k+1)} = \sum_{i=0}^{n} \alpha_i \beta_{n-i}^{(k)} = \beta_n^{(k)} + \sum_{i=0}^{n-1} \gamma_i \beta_{n-i-1}^{(k)}$$

and consequently by (1.4) and (1.3)

$$I_k = \beta_k^{(k)} = \beta_k^{(k+1)} - \sum_{i=0}^{k-1} \gamma_i \beta_{k-i-1}^{(k)} = a_0^{(k+1)} - \sum_{i=0}^{k-1} i!\, \gamma_i a_i^{(k)}$$

which is (1.2).

Finally (1.5) follows from (2.3) and Euler's multinominal formula [2] which states that if $b_0 \neq 0$ and s is any real number, then

$$\Big(\sum_{n=0}^{\infty} b_n (z-a)^n\Big)^s = \sum_{n=0}^{\infty} B_n^{(s)} (z-a)^n$$

where

$$B_0^{(s)} = b_0^s \quad \text{and} \quad B_n^{(s)} = \frac{1}{nb_0} \sum_{i=1}^{n} \big(i(s+1)-n\big) b_i B_{n-i}^{(s)} \quad \text{for} \quad n \geqslant 1.$$

Remark. We note that the numbers $B_n^{(k)}$ and $\beta_n^{(k)}$ are related by

$$\beta_n^{(k)} = (-1)^n \sum_{i=0}^{n} (-1)^i B_i^{(k)}.$$

and that $B_n^{(k)}$'s satisfy the recurrence formula

$$B_n^{(k)} = \sum_{i=0}^{n} \alpha_i B_{n-i}^{(k-1)} = B_n^{(k-1)} + \sum_{i=0}^{n-1} \gamma_i B_{n-i-1}^{(k-1)}.$$

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On sum-free sequences

by

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A sequence A: $a_1 < a_2 < a_3 \ldots$ of positive integers is said to be *sum-free* if no member of A is the sum of two or more other members of A. P. Erdős [1] proved a number of results concerning sum-free sequences. One of these is that for any such sequence

$$\sum (1/a_i) < 103.$$

This leads one to define ϱ by

$$\varrho = \sup_A \{\sum_{a\in A} 1/a\}$$

where the supremum is taken over all sum-free sequences A. The powers of 2 form a sum-free sequence so that $2 \leqslant \varrho < 103$. Levine and O'Sullivan [2] considerably improved on Erdős' upper bound by showing that $\varrho < 3.97$ and they constructed an example which shows $\varrho > 2.0351$.

The object of this note is to exhibit an example of a sum-free sequence which establishes $\varrho > 2.0648$. The construction is fairly elaborate. The relatively modest improvement over the result of Levine and O'Sullivan can perhaps be considered as evidence supporting their conjecture that ϱ is much closer to 2 than to 4. The construction is given in the following theorem.

Theorem. *Let A be a (finite) sum-free set. Let $s = \sum_{a\in A} a$ and let t be an integer exceeding s. Define integers l, m, n, r and p as follows:*

$$l = \binom{t-s+2}{2}, \quad m = \binom{t-s+1}{2},$$

$$n = \left[\frac{l-1+s}{t}\right], \quad r = l-nt-1,$$

$$p = \binom{l+1}{2} - \binom{r+1}{2} + n.$$