An integral involving the remainder term in the Piltz divisor problem
by

R. SITARAMACHANDRANARAO* (Toledo, Ohio)

1. Introduction. Let \( \tau_k(n) \) denote the number of ordered \( k \)-tuples \((x_1, x_2, \ldots, x_k)\) of positive integers such that \( x_1 x_2 \cdots x_k = n \)
and
\[
\sum_{n \leq x} \tau_k(n) = x P_k(\log x) + A_k(x)
\]
where \( x P_k(\log x) \) is the residue of \( \zeta^k(s) x^s / s \) at \( s = 1 \). Further let
\[
P_k(\log x) = d_k^{(k-1)}(\log x)^{k-1} + \cdots + d_k^{(0)}(\log x) + d_k^{(0)},
\]

\[
I_k = \int \frac{A_k(u)}{u^k} \, du,
\]

\[
\gamma_n = \lim_{M \to \infty} \left[ \frac{1}{n!} \sum_{m \leq M} \frac{\log m)^n}{m} \frac{\log M)^{n+1}}{n+1} \right]
\]
and

\[
\beta_n^{(k)} = (-1)^n \left[ 1 + \sum_{r=1}^n (-1)^r \sum_{s=1}^r \binom{k}{s} \sum_{i_1 + i_2 + \cdots + i_r = r} \gamma_{i_1} \gamma_{i_2} \cdots \gamma_{i_r} \right].
\]

Recently, A. F. Lavrik, M. I. Israilov and Z. Edgorov [4] proved that for \( k \geq 1 \)
\[
I_k = d_k^{(k-1)} + \sum_{m=0}^{k-1} m! \gamma_m d_k^{(m)}
\]
and also expressed \( I_k, 1 \leq k \leq 5 \), explicitly in terms of \( \gamma_m, 0 \leq m \leq 4 \) using

\[
d_k^{(j)} = \frac{\beta_n^{(k-1)-j}}{j!}, \quad 0 \leq j \leq k-1.
\]

* On leave from Andhra University, Waltair, India.
The aim of this note is to give simple proofs of (1.2) and (1.3) and to express $I_k$ explicitly in terms of $\gamma_k$, namely

\begin{equation}
I_k = \beta_k^{(a)}.
\end{equation}

We also prove the following alternate form of (1.4):

\begin{equation}
I_k = \sum_{i=0}^k (-1)^i B_{k-i}^{(a)},
\end{equation}

where the numbers $B_{k}^{(a)}$ are defined recursively by

\begin{align*}
B_0^{(a)} &= 1, \\
nB_{k}^{(a)} &= \sum_{i=0}^{n-1} \frac{(i+1)(k+1) - n)}{i!} B_{k-i}^{(a)}, \quad n \geq 1, \quad k \geq 1.
\end{align*}

2. Proofs of (1.2)-(1.5). By partial summation and (1.1) we have for $\Re s > 1$

\begin{equation}
\sum_{n \leq x} \tau_k(n) n^{-s} = \left( \sum_{n \leq x} \tau_k(n) \right) x^{-s} + \int_1^x \frac{uP_x(\log u) + A_k(u)}{u^{s+1}} du.
\end{equation}

Since

\begin{equation}
\sum_{n=1}^\infty \tau_k(n) n^{-s} = \zeta^k(s), \quad \sum_{n \leq x} \tau_k(n) \ll_k x^{1+\varepsilon} \quad \text{for each } \varepsilon > 0
\end{equation}

and

\begin{align*}
\int_1^\infty (\log u)^i u^{-s} du &= i! (s-1)^{-i-1} \quad \text{for } i \in \mathbb{Z}_{\geq 0},
\end{align*}

we have, on letting $x \to \infty$ in (2.1)

\begin{equation}
\int_1^\infty \frac{A_k(u)}{u^{s+1}} du = \frac{\zeta^k(s)}{s} = \sum_{i=0}^k \frac{k!}{(s-1)^{i+1}}.
\end{equation}

By elementary arguments (cf. [5], Chapter 12), we have $A_k(x) \ll x^{1-1/k}$. Hence $\int \frac{A_k(u)}{u^{s+1}} du$ converges uniformly and absolutely on every compact subset of the half-plane $\Re s > 1-1/k$ and thus defines an analytic function, say $f_k(s)$, there. Thus (2.2) is valid (at least) in the half-plane $\Re s > 1-1/k$ and $I_k (= f_k(1))$ equals the constant term in the Laurent expansion of $\zeta^k(s)/s$ at $s = 1$. To find this, let $\alpha_0 = 1$ and $\alpha_n = \gamma_{n-1}$ for $n \geq 1$. It is well known, due to Stieiljes (cf. [1], p. 155), that

\begin{equation}
\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^\infty \gamma_n (s-1)^n
\end{equation}

where $\gamma_0 = \gamma$ is the Euler's constant. Hence for $|s-1| < 1$

\begin{equation}
\zeta^k(s) = \left( 1 + \sum_{n=0}^\infty \gamma_n (s-1)^n \right)^k \left( 1 + (s-1)^{-1} \right)^{-k}
\end{equation}

\begin{equation}
\zeta^k(s) = \left( \sum_{n=0}^\infty \alpha_n (s-1)^n \right)^k \left( \sum_{n=0}^\infty (s-1)^n \right)^{-k}
\end{equation}

and consequently

\begin{align*}
I_k &= f_k(1) = \sum_{l_1 + \ldots + l_k = k} (-1)^{l_1} \ldots \alpha_{l_k} = \sum_{r=0}^k \sum_{l_1 + \ldots + l_k = r} \alpha_{l_1} \ldots \alpha_{l_k} \\
&= (-1)^k \sum_{r=1}^k \sum_{l_1 + \ldots + l_k = r} \gamma_{l_1-1} \gamma_{l_2-1} \ldots \gamma_{l_k-1}
\end{align*}

\begin{equation}
= \beta_k^{(a)}
\end{equation}

which is (1.4).

To prove (1.3), we have by (2.3)

\begin{equation}
\frac{\zeta^k(s)}{s} = \left( \sum_{n=0}^\infty \alpha_n (s-1)^n \right) (s-1)^{-k}
\end{equation}

so that

\begin{align*}
P_k(x) &= \text{Res}_{s=1} \frac{\zeta^k(s) x^{-s}}{s} \\
&= \sum_{n=0}^\infty \alpha_n (s-1)^n \sum_{n=0}^\infty \frac{\left( \log x \right)^n}{n!} (s-1)^n \\
&= \sum_{n=0}^\infty \frac{\beta_n^{(a)} (s-1)^n \sum_{n=0}^\infty \left( \log x \right)^n}{n!}
\end{align*}

\begin{equation}
= \frac{k-1}{k} \beta_k^{(a)}
\end{equation}

Now (1.3) follows in view of $P_k(x) = \sum_{j=0}^{k-1} \beta_j^{(a)} (\log x)^j$.

To prove (1.2), we have by (2.4)

\begin{equation}
\sum_{n=0}^\infty \beta_n^{(k+1)} (s-1)^n = \frac{(s-1) \zeta(s)^{k+1}}{s} = \frac{(s-1)^k \zeta(s)}{s}
\end{equation}

\begin{equation}
= \frac{\sum_{n=0}^\infty \beta_n^{(a)} (s-1)^n \sum_{n=0}^\infty \alpha_n (s-1)^n}{n!}
\end{equation}

Hence

\begin{equation}
\beta_n^{(k+1)} = \sum_{i=0}^n \alpha_i \beta_{n-i}^{(a)} = \beta_n^{(a)} + \sum_{i=0}^{n-1} \gamma_i \beta_{n-i-1}^{(a)}
\end{equation}
and consequently by (1.4) and (1.3)

\[ I_k = \beta_k^{(k)} = \beta_k^{(k+1)} - \sum_{l=0}^{k-1} \frac{1}{k} b_{k-l-1} = a_k^{(k-1)} - \sum_{l=0}^{k-1} \frac{1}{l!} \gamma_l a_l^{(k)} \]

which is (1.2).

Finally (1.5) follows from (2.3) and Euler's multinomial formula [2] which states that if \( b_0 \neq 0 \) and \( s \) is any real number, then

\[ \left( \sum_{n=0}^{\infty} b_n (z-a)^n \right)^s = \sum_{n=0}^{\infty} B_n^{(s)} (z-a)^n \]

where

\[ B_n^{(s)} = b_n^{(s)} \quad \text{and} \quad B_n^{(s)} = \frac{1}{n b_0} \sum_{i=1}^{n} \left( \begin{array}{c} z+1 \end{array} n \right) b_i B_i^{(s)} \quad \text{for} \quad n \geq 1. \]

Remark. We note that the numbers \( B_n^{(s)} \) and \( \beta_n^{(k)} \) are related by

\[ \beta_n^{(k)} = (-1)^n \sum_{i=0}^{n} (-1)^i B_i^{(k)} \]

and that \( B_n^{(s)} \)s satisfy the recurrence formula

\[ B_n^{(k)} = \sum_{i=0}^{n} \alpha_i B_i^{(k-1)} = B_n^{(k-1)} + \sum_{i=0}^{n-1} \gamma_i B_i^{(k-1)} \]

References


On sum-free sequences

by

H. L. Abbott (Edmonton, Canada)

A sequence \( A = a_1 < a_2 < a_3 \ldots \) of positive integers is said to be sum-free if no member of \( A \) is the sum of two or more other members of \( A \). P. Erdős [1] proved a number of results concerning sum-free sequences. One of these is that for any such sequence

\[ \sum \left( \frac{1}{a_i} \right) < 103. \]

This leads one to define \( \varphi \) by

\[ \varphi = \sup_{A} \sum_{a \in A} \left( \frac{1}{a} \right) \]

where the supremum is taken over all sum-free sequences \( A \). The powers of 2 form a sum-free sequence so that \( 2 < \varphi < 103 \). Levine and O'Sullivan [2] considerably improved on Erdős' upper bound by showing that \( \varphi < 3.97 \) and they constructed an example which shows \( \varphi > 2.0351 \).

The object of this note is to exhibit an example of a sum-free sequence which establishes \( \varphi > 2.0648 \). The construction is fairly elaborate. The relatively modest improvement over the result of Levine and O'Sullivan can perhaps be considered as evidence supporting their conjecture that \( \varphi \) is much closer to 2 than to 4. The construction is given in the following theorem.

**Theorem.** Let \( A \) be a (finite) sum-free set. Let \( s = \sum_{a \in A} a \) and let \( t \) be an integer exceeding \( s \). Define integers \( l, m, n, r \) and \( p \) as follows:

\[ l = \left( \frac{l+1}{2} \right), \quad m = \left( \frac{l-s+1}{2} \right), \quad n = \left[ \frac{l-1+s}{t} \right], \quad r = l-n-t-1, \quad p = \left( \frac{l+1}{2} \right) - \left( \frac{r+1}{2} \right) + n. \]