ACTA ARITHMETICA
XLVIII (1987)

Necessary condition for the existence of an incongruent covering system with odd moduli II

by

MARC A. BERGER, ALEXANDER FELZENBAUM and AVIEZRI S. FRAENKEL
(Rehovot)

1. Explanation of results. For $a, m \in \mathbf{Z}$, $m \geqslant 2$ denote by $a(m)$ the *residue class* $a(m) = \{a+km: k \in \mathbf{Z}\}$. We refer to m as the *modulus* of this residue class. Let $\Delta = \{a_i(m_i): 1 \leqslant i \leqslant l\}$ be a *covering system*, i.e. a system of residue classes which cover $\mathbf{Z}$. We say Δ is *incongruent* if the moduli m_i are all distinct. An old conjecture of Erdős–Selfridge (see [3], (1.9)) asserts that if Δ is incongruent then some modulus m_k must be even. In [1] we showed that if the moduli m_i are all odd then a necessary condition for Δ to be incongruent is

$$f(\bar{x}) \geqslant 2 \tag{1}$$

where f is the n-variate polynomial

$$f(x) = \prod_{i=1}^{n} (1+x_i) - \sum_{i=1}^{n} x_i, \qquad x \in \mathbf{R}^n; \tag{2}$$

$\bar{x}$ is the point with coordinates

$$\bar{x}_i = \frac{p_i^{s_i}-1}{(p_i-2)\,p_i^{s_i}+1}, \qquad 1 \leqslant i \leqslant n; \tag{3}$$

and $N = \text{l.c.m.}(m_1, \ldots, m_l)$ has the prime factorization

$$N = \prod_{i=1}^{n} p_i^{s_i}. \tag{4}$$

It is clear that in the domain $x_1, \ldots, x_n > 0$, $f(x)$ is increasing in each of the variables $x_1, \ldots, x_n$. Since

$$\bar{x}_i < \frac{1}{p_i-2}, \qquad 1 \leqslant i \leqslant n, \tag{5}$$

we also arrived at the necessary condition

$$f\left(\frac{1}{p_1-2}, \ldots, \frac{1}{p_n-2}\right) = \prod_{i=1}^{n} \frac{p_i-1}{p_i-2} - \sum_{i=2}^{n} \frac{1}{p_i-2} > 2, \tag{6}$$

independent of the exponents s_i. From this condition followed at once that n must be at least five. Observe that for

$$n = 5; \quad p_1 = 3, \quad p_2 = 5, \quad p_3 = 7, \quad p_4 = 11, \quad p_5 = 13 \tag{7}$$

the left-hand side of (6) equals $2+\frac{71}{495}$. Nevertheless, Churchhouse [2] has conjectured that this particular case (7) is also impossible. Actually our condition (1) gives some partial information. For example if n is to be five, then necessarily $p_1 = 3$, $s_1 \geqslant 3$.

We present now a new necessary condition, from which will follow in particular that if Δ is incongruent and the moduli m_i are all odd, then n must be at least six. This then rules out (7), establishing Churchhouse's conjecture.

THEOREM. *If the moduli are all odd then a necessary condition for Δ to be incongruent is*

$$g(\bar{w}, \bar{z}) \geqslant 2 \tag{8}$$

where g is the $(n+1)$-variate polynomial

$$g(w, z) = (1+w)\prod_{i=2}^{n}(1+z_i) - w - (1+w-z_1)\sum_{i=2}^{n} z_i \tag{9}$$
$$- z_1 z_2 z_3 z_4 z_5 (z_2^{-1}+2z_3^{-1}+3z_4^{-1}+3z_5^{-1}), \qquad w \in \boldsymbol{R},\ z \in \boldsymbol{R}^n;$$

and

$$\bar{w} = \frac{p_1^{s_1}-1}{(p_1-2)\,p_1^{s_1}+1}, \qquad \bar{z}_1 = \frac{p_1^{s_1-1}-1}{(p_1-2)\,p_1^{s_1}+1}, \tag{10}$$

$$\bar{z}_i = \frac{p_i^{s_i}-1}{(p_i-3)\,p_i^{s_i}+2}, \qquad 2 \leqslant i \leqslant n. \tag{11}$$

To see how we arrive at the conclusion $n \geqslant 6$ observe that in the domain

$$w, z_1, \ldots, z_n > 0; \quad w \geqslant 3z_1; \quad z_2, z_3 < 1; \quad z_4, z_5 < 1/3 \tag{12}$$

$g(w, z)$ is increasing in each of the variables $w, z_1, \ldots, z_n$. Since

$$\bar{w} < \frac{1}{p_1-2}, \qquad \bar{z}_1 < \frac{1}{p_1(p_1-2)}, \tag{13}$$

$$\bar{z}_i < \frac{1}{p_i-3}, \qquad 2 \leqslant i \leqslant n, \tag{14}$$

we arrive at the necessary condition

$$g\left(\frac{1}{p_1-2}, \frac{1}{p_1(p_1-2)}, \frac{1}{p_2-3}, \ldots, \frac{1}{p_n-3}\right) \tag{15}$$
$$= \frac{p_1-1}{p_1-2}\prod_{i=2}^{n}\frac{p_i-2}{p_i-3} - \frac{1}{p_1-2}$$
$$- \frac{p_1^2-p_1-1}{p_1(p_1-2)}\sum_{i=2}^{n}\frac{1}{p_i-3}$$
$$- \frac{p_2+2p_3+3p_4+3p_5-27}{p_1(p_1-2)(p_2-3)(p_3-3)(p_4-3)(p_5-3)} > 2.$$

From (15) now follows that $n \geqslant 6$. (Simply check case (7) – the worst case for $n = 5$.)

2. A geometric approach. We shall use the following

LEMMA. *Let G be a forest (i.e. a finite undirected graph with no cycles) with vertex set $V = V(G)$ and edge set $E = E(G)$. Let $\{S_i\colon i \in V\}$ be a family of sets. Then*

$$\Big|\bigcup_{i\in V} S_i\Big| \leqslant \sum_{i\in V}|S_i| - \sum_{\{i,j\}\in E}|S_i \cap S_j|. \tag{1}$$

Proof. It suffices to prove (1) when G is a tree. We use induction on $|V|$. When $|V| = 1$ (1) is clear, so we assume now that (1) holds for $|V| = k$. Let $v \in V$ be an endpoint of G, and let $u \in V$ be adjacent to v. Set $G' = G - v$. Then G' is also a tree and, by the induction assumption,

$$\Big|\bigcup_{i\in V'} S_i\Big| \leqslant \sum_{i\in V'}|S_i| - \sum_{\{i,j\}\in E'}|S_i \cap S_j|. \tag{2}$$

Here $V' = V(G') = V\setminus\{v\}$ and $E' = E(G') = E\setminus\{\{u, v\}\}$. Since

$$\Big|\bigcup_{i\in V} S_i\Big| = \Big|\bigcup_{i\in V'} S_i\Big| + |S_u \cup S_v| - \Big|\Big(\bigcup_{i\in V'} S_i\Big) \cap (S_u \cup S_v)\Big| \tag{3}$$
$$\leqslant \Big|\bigcup_{i\in V'} S_i\Big| + |S_v| - |S_u \cap S_v|$$

(1) follows now from (2). ∎

A *product set*, $\mathscr{R}$, in $\boldsymbol{Z}^n$ is any finite nonempty set of the form

$$\mathscr{R} = R_1 \times \ldots \times R_n \tag{4}$$

where $R_1, \ldots, R_n \subset \boldsymbol{Z}$. The set R_i is referred to as the *i-th projection of $\mathscr{R}$*, denoted

$$R_i = \pi_i(\mathscr{R}), \qquad 1 \leqslant i \leqslant n. \tag{5}$$

For $b \in N^n$ the set

(6) $$\mathscr{P} = \{c \in Z^n:\ 0 \leqslant c_i < b_i;\ 1 \leqslant i \leqslant n\}$$

is called the $(n;\, b)$-*parallelepiped*. Let $p_1, \ldots, p_n$ be distinct primes. We define $\Phi(n;\, p_1, \ldots, p_n)$ to be the family of those product sets in Z^n of the form $(a_1 p_1^{t_1}, \ldots, a_n p_n^{t_n}) + \mathscr{P}$, where $a_1, \ldots, a_n$, $t_1, \ldots, t_n$ are any non-negative integers and $\mathscr{P}$ is the $(n;\, (p_1^{t_1}, \ldots, p_n^{t_n}))$-parallelepiped.

PROPOSITION. *Let $p_1, \ldots, p_n$ be distinct odd primes and let $\mathscr{P}$ be the $(n;\, (p_1^{s_1}, \ldots, p_n^{s_n}))$-parallelepiped. Let $\Gamma \subset \Phi(n;\, p_1, \ldots, p_n)$ be a family of proper subsets of $\mathscr{P}$ which cover $\mathscr{P}$. If $g(\bar{w}, \bar{z}) < 2$, where g, $\bar{w}$, $\bar{z}$ are given by* (1.9)–(1.11) *then Γ contains two sets of the same cardinality.*

Proof. Assume, on the contrary, that the sets in Γ have distinct cardinalities. Set $N = |\mathscr{P}|$. Modify Γ to Γ^* as follows. Enlarge each $\mathscr{R} \in \Gamma$ with

(7) $$|\mathscr{R}| = \frac{N}{p_1 p_i^{s_i - t}}, \quad 2 \leqslant i \leqslant n,\ 0 \leqslant t < s_i,$$

to a larger product set $\mathscr{R}^*$ by enlarging $\pi_1(\mathscr{R})$ to $\pi_1(\mathscr{P})$. In other words if $\mathscr{R} \in \Gamma$ satisfies (7), then replace $\mathscr{R}$ with $\mathscr{R}^*$, where $\pi_1(\mathscr{R}^*) = \pi_1(\mathscr{P})$ and $\pi_i(\mathscr{R}^*) = \pi_i(\mathscr{R})$, $2 \leqslant i \leqslant n$. Since we assumed that the sets in Γ have distinct cardinalities, it follows that Γ^* can contain at most two sets of cardinality $N/p_i^{s_i - t}$, $2 \leqslant i \leqslant n$, $0 \leqslant t < s_i$, no sets of cardinality $N/p_1 p_i^{s_i - t}$, $2 \leqslant i \leqslant n$, $0 \leqslant t < s_i$ and at most one set of any other cardinality.

Each set $\mathscr{R} \in \Gamma^*$ can be partitioned into "building blocks" $\mathscr{R} = \bigcup_l \mathscr{R}_l$ where each $\mathscr{R}_l \in \Phi(n;\, p_1, \ldots, p_n)$ has cardinality $|\mathscr{R}_l| = \prod_{i \notin I} p_i^{s_i}$ and

(8) $$I = ind(\mathscr{R}) = \{1 \leqslant i \leqslant n:\ \pi_i(\mathscr{R}) \neq \pi_i(\mathscr{P})\}.$$

We now modify Γ^* to a new family Γ^{**} by replacing each $\mathscr{R} \in \Gamma^*$ with all of its building blocks $\mathscr{R}_l$. The sets in Γ^{**} all have cardinalities of the form $\prod_{i \notin I} p_i^{s_i}$ for some $I \subset \{1, \ldots, n\}$, $I \neq \emptyset$. Furthermore, at most $\alpha(I)$ sets in Γ^{**} have this cardinality, where

(9) $$\alpha(I) = \begin{cases} 2\dfrac{p_i^{s_i} - 1}{p_i - 1}, & I = \{i\}, \quad i \neq 1, \\[2mm] \dfrac{p_1^{s_1 - 1} - 1}{p_1 - 1} \cdot \dfrac{p_i^{s_i} - 1}{p_i - 1}, & I = \{1, i\}, \quad i \neq 1, \\[2mm] \prod\limits_{i \in I} \dfrac{p_i^{s_i} - 1}{p_i - 1}, & \text{otherwise.} \end{cases}$$

We are going to forget about Γ now. Instead we will assume that $\Gamma^{**} \subset \Phi(n;\, p_1, \ldots, p_n)$ is any family of subsets of $\mathscr{P}$ containing precisely $\alpha(I)$ distinct (and therefore disjoint) sets of cardinality $\prod_{i \notin I} p_i^{s_i}$, for each $I \neq \emptyset$. Our conclusion will be that Γ^{**} cannot cover $\mathscr{P}$. This is obviously more than what the Proposition states.

We introduce some notation. Let

(10) $$\Lambda = \{\mathscr{R} \in \Gamma^{**}:\ |ind(\mathscr{R})| = 1\}.$$

For $I \subset \{1, \ldots, n\}$, $|I| \geqslant 2$ set

(11) $$\mathscr{R}(I) = \bigcup (R \in \Gamma^{**}:\ ind(\mathscr{R}) = I).$$

The basic observation about Λ is that for $\mathscr{R} \in \Lambda$ that set $\mathscr{P} \setminus \mathscr{R}$ is also a product set. Thus

(12) $$\mathscr{S} = \mathscr{P} \setminus \bigcup_{\mathscr{R} \in \Lambda} \mathscr{R} = \bigcap_{\mathscr{R} \in \Lambda} (\mathscr{P} \setminus \mathscr{R})$$

is also a product set. For any i, $1 \leqslant i \leqslant n$

(13) $$|\pi_i(\mathscr{S})| = \Big|\pi_i\Big(\bigcap_{\mathscr{R} \in \Lambda} (\mathscr{P} \setminus \mathscr{R})\Big)\Big| = \Big|\bigcap_{\mathscr{R} \in \Lambda} \pi_i(\mathscr{P} \setminus \mathscr{R})\Big| = y_i$$

where

(14) $$y_i = p_i^{s_i} - \alpha(\{i\}).$$

Thus for $\mathscr{R} \in \Gamma^{**}$, $ind(\mathscr{R}) = I$, $|I| \geqslant 2$, $\mathscr{R} \cap \mathscr{S} \neq \emptyset$

(15) $$|\pi_i(\mathscr{R} \cap \mathscr{S})| = |\pi_i(\mathscr{R}) \cap \pi_i(S)| = \begin{cases} 1, & i \in I, \\ y_i, & i \notin I, \end{cases}$$

so that

(16) $$|\mathscr{R} \cap \mathscr{S}| = |\mathscr{S}| \prod_{i \in I} y_i^{-1}.$$

From this we obtain the bounds

(17) $$|\mathscr{R}(I) \cap \mathscr{S}| \leqslant |\mathscr{S}|\, \alpha(I) \prod_{i \in I} y_i^{-1} = \begin{cases} |\mathscr{S}| \prod\limits_{i \in I} \bar{z}_i, & 1 \notin I \ \text{or} \ |I| = 2, \\ |\mathscr{S}|\, \bar{w} \prod\limits_{\substack{i \in I \\ i \neq 1}} \bar{z}_i, & 1 \in I \ \text{and} \ |I| \geqslant 3. \end{cases}$$

Thus

(18) $$\sum_{|I| \geqslant 2} |\mathscr{R}(I) \cap \mathscr{S}| \leqslant |\mathscr{S}|\, [g_1(w, z) - 1]$$

where

(19) $$g_1(w, z) = (1 + w) \prod_{i=2}^{n} (1 + z_i) - w - (1 + w - z_1) \sum_{i=2}^{n} z_i.$$

To prove the Proposition we show that the sets in $\Gamma^{**}\setminus\Delta$ cannot cover $\mathscr{S}$; more precisely,

$$\left|\bigcup_{|I|\geq 2}(\mathscr{R}(I)\cap\mathscr{S})\right| < |\mathscr{S}|. \tag{20}$$

We make an assumption now which is worst possible regarding (20).

Assumption. Each set $\mathscr{R}\in\Gamma^{**}$, $|ind(\mathscr{R})|=2$, intersects $\mathscr{S}$.

It follows from this assumption that if $|I_1|=|I_2|=2$, $I_1\cap I_2=\emptyset$, then

$$|\mathscr{R}(I_1)\cap\mathscr{R}(I_2)\cap\mathscr{S}| = |\mathscr{S}|\prod_{i\in I_1\cup I_2} z_i. \tag{21}$$

Let $\mathscr{J}$ denote the family of subsets $I\subset\{1,\ldots,n\}$, $|I|=2$. According to the Lemma

$$\left|\bigcup_{|I|=2}(\mathscr{R}(I)\cap\mathscr{S})\right| \leq \sum_{|I|=2}|\mathscr{R}(I)\cap\mathscr{S}| - \sum_{\{I_1,I_2\}\in E(G)}|\mathscr{R}(I_1)\cap\mathscr{R}(I_2)\cap\mathscr{S}| \tag{22}$$

for any forest G with $V(G)=\mathscr{J}$. Thus

$$\begin{aligned}\left|\bigcup_{|I|\geq 2}(\mathscr{R}(I)\cap\mathscr{S})\right| &\leq \sum_{|I|\geq 3}|\mathscr{R}(I)\cap\mathscr{S}| + \left|\bigcup_{|I|=2}(\mathscr{R}(I)\cap\mathscr{S})\right| \\ &\leq \sum_{|I|\geq 2}|\mathscr{R}(I)\cap\mathscr{S}| - \sum_{\{I_1,I_2\}\in E(G)}|\mathscr{R}(I_1)\cap\mathscr{R}(I_2)\cap\mathscr{S}|.\end{aligned} \tag{23}$$

In view of (18), (21) it suffices now, to establish (20), to exhibit a forest G, with $V(G)=\mathscr{J}$, satisfying

$$\{I_1, I_2\}\in E(G)\Rightarrow I_1\cap I_2=\emptyset, \tag{24}$$

such that

$$\sum_{\{I_1,I_2\}\in E(G)}\ \prod_{i\in I_1\cup I_2} z_i = z_1(3z_2z_3z_4+3z_2z_3z_5+2z_2z_4z_5+z_3z_4z_5). \tag{25}$$

Such a graph appears in Figure 1 (all vertices not in the figure are isolated). ∎

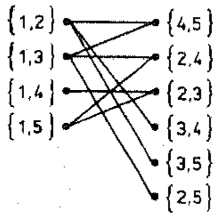

Fig. 1

Corollary. *Let H be a finite cyclic group of odd order with prime factorization*

$$|H| = \prod_{i=1}^{n} p_i^{s_i}.$$

Let Δ be a family of cosets which cover H. If $H\notin\Delta$ and $g(\bar{w},\bar{z})<2$ then Δ contains two cosets of the same order.

The proof is exactly as in [1]. Since a covering system of residue classes is equivalent to a cover of cosets for a cyclic group, our Theorem follows.

References

[1] M. A. Berger, A. Felzenbaum and A. S. Fraenkel, *Necessary condition for the existence of an incongruent covering system with odd moduli*, Acta Arith. 45 (1986), pp. 375–379.

[2] R. F. Churchhouse, *Covering sets and systems of congruences*, in *Computers in Mathematical Research* (eds. R. F. Churchhouse and J.-C. Herz,), North Holland, Amsterdam 1968, pp. 20–36.

[3] Š. Porubský, *Results and Problems on Covering Systems of Residue Classes*, Mitteilungen aus dem Math. Sem. Giessen, Heft 150, Universität Giessen, 1981.

FACULTY OF MATHEMATICS
THE WEIZMANN INSTITUTE OF SCIENCE
Rehovot 76100, Israel

Received on 28. 6. 1985 (1525)