Necessary condition for the existence of an incongruent covering system with odd moduli II

by

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1. Explanation of results. For \( a, m \in \mathbb{Z}, m \geq 2 \) denote by \( a(m) \) the residue class \( a(m) = \{a + km: k \in \mathbb{Z}\} \). We refer to \( m \) as the modulus of this residue class. Let \( \Delta = \{a(m)_i: 1 \leq i \leq l\} \) be a covering system, i.e. a system of residue classes which cover \( \mathbb{Z} \). We say \( \Delta \) is incongruent if the moduli \( m_i \) are all distinct. An old conjecture of Erdős-Selfridge (see [3], (1.9)) asserts that if \( \Delta \) is incongruent then some modulus \( m_i \) must be even. In [1] we showed that if the moduli \( m_i \) are all odd then a necessary condition for \( \Delta \) to be incongruent is

\[
\begin{equation}
 f(x) \geq 2
\end{equation}
\]

where \( f \) is the \( n \)-variate polynomial

\[
\begin{equation}
 f(x) = \prod_{i=1}^{n} (1 + x_i) - \sum_{i=1}^{n} x_i, \quad x \in \mathbb{R}^n;
\end{equation}
\]

\( \bar{x} \) is the point with coordinates

\[
\begin{equation}
 \bar{x}_i = \frac{p_i^{\nu_i} - 1}{(p_i - 2) p_i^{\nu_i} + 1}, \quad 1 \leq i \leq n;
\end{equation}
\]

and \( N = \text{l.c.m.}(m_1, \ldots, m_l) \) has the prime factorization

\[
\begin{equation}
 N = \prod_{i=1}^{n} p_i^{\nu_i}.
\end{equation}
\]

It is clear that in the domain \( x_1, \ldots, x_n > 0 \), \( f(x) \) is increasing in each of the variables \( x_1, \ldots, x_n \). Since

\[
\begin{equation}
 \bar{x}_i < \frac{1}{p_i - 2}, \quad 1 \leq i \leq n,
\end{equation}
\]
we also arrived at the necessary condition

\[ f \left( \frac{1}{p_1-2}, \ldots, \frac{1}{p_n-2} \right) = \prod_{i=1}^{n} \frac{p_1 - 1}{p_i - 2} - \sum_{i=2}^{n} \frac{1}{p_i - 2} > 2, \]

independent of the exponents \( s_i \). From this condition followed at once that \( n \) must be at least five. Observe that for

\[ n = 5; \quad p_1 = 3, \quad p_2 = 5, \quad p_3 = 7, \quad p_4 = 11, \quad p_5 = 13 \]

the left-hand side of (6) equals \( 2 + \frac{1}{4} \Delta \). Nevertheless, Churchhouse [2] has conjectured that this particular case (7) is also impossible. Actually our condition (1) gives some partial information. For example if \( n \) is to be five, then necessarily \( p_1 = 3, \; s_1 \geq 3 \).

We present now a new necessary condition, from which will follow in particular that if \( \Delta \) is incongruent and the moduli \( m_i \) are all odd, then \( n \) must be at least six. This then rules out (7), establishing Churchhouse's conjecture.

Theorem. If the moduli are all odd then a necessary condition for \( \Delta \) to be incongruent is

\[ g(\overline{w}, \overline{z}) \geq 2 \]

where \( g \) is the \((n+1)\)-variate polynomial

\[ g(w, z) = (1 + w) \prod_{i=2}^{n} (1 + z_i) - w(1 + w - z_1) \sum_{i=2}^{n} z_i - z_1 z_2 z_3 z_4 z_5 (z_1^{-1} + 2z_2^{-1} + 3z_3^{-1} + 3z_4^{-1} + 3z_5^{-1}), \quad w \in \mathbb{R}, \; z \in \mathbb{R}^n; \]

and

\[ \bar{w} = \frac{p_1^{n+1} - 1}{(p_1 - 2)p_1^{n+1} + 1}, \quad \bar{z}_1 = \frac{p_1^{n+1} - 1}{(p_1 - 2)p_1^{n+1} + 1}, \]

\[ \bar{z}_i = \frac{p_1^{n+1} - 1}{(p_1 - 2)p_1^{n+1} + 2}, \quad 2 \leq i \leq n. \]

To see how we arrive at the conclusion \( n \geq 6 \) observe that in the domain

\[ w, z_1, \ldots, z_n > 0; \quad w \geq 3z_1; \quad z_2, z_3 < 1; \quad z_4, z_5 < 1/3 \]

g(\overline{w}, \overline{z}) \text{ is increasing in each of the variables } w, z_1, \ldots, z_n. \text{ Since}

\[ \bar{w} < \frac{1}{p_1 - 2}, \quad \bar{z}_1 < \frac{1}{p_1(p_1 - 2)}, \]

\[ \bar{z}_i < \frac{1}{p_i - 3}, \quad 2 \leq i \leq n, \]

we arrive at the necessary condition

\[ g \left( \frac{1}{p_1 - 2}, \frac{1}{p_1(p_1 - 2)}, \frac{1}{p_2 - 3}, \ldots, \frac{1}{p_n - 3} \right) \]

\[ = \frac{p_1 - 1}{p_1 - 2} \prod_{i=2}^{n} \frac{p_i - 2}{p_i - 3} - \sum_{i=2}^{n} \frac{1}{p_i - 2} \]

\[ = \frac{p_1 - 1}{p_1 - 2} \prod_{i=2}^{n} \frac{p_i - 2}{p_i - 3} - \frac{p_2 + 2p_3 + 3p_4 + 3p_5 - 27}{p_1(p_1 - 2)(p_2 - 3)(p_3 - 4)(p_4 - 5)(p_5 - 6)} \geq 2. \]

From (15) now follows that \( n \geq 6 \). (Simply check case (7) -- the worst case for \( n = 5 \).)

2. A geometric approach. We shall use the following

Lemma. Let \( G \) be a forest (i.e. a finite undirected graph with no cycles) with vertex set \( V = V(G) \) and edge set \( E = E(G) \). Let \( \{S_i \; i \in V\} \) be a family of sets. Then

\[ |\bigcup_{i \in V} S_i| \leq |\sum_{i \in V} S_i| - \sum_{\{i,j\} \in E} |S_i \cap S_j|. \]

Proof. It suffices to prove (1) when \( G \) is a tree. We use induction on \( |V| \). When \( |V| = 1 \) (1) is clear, so we assume now that (1) holds for \( |V| = k \).

Let \( v \in V \) be an endpoint of \( G \), and let \( u \in V \) be adjacent to \( v \). Set \( G' = G - v \). Then \( G' \) is also a tree and, by the induction assumption,

\[ |\bigcup_{i \in V'} S_i| \leq \sum_{i \in V'} |S_i| - \sum_{\{i,j\} \in E'} |S_i \cap S_j|. \]

Here \( V' = V(G') = V \setminus \{v\} \) and \( E' = E(G') = E \setminus \{\{u, v\}\} \). Since

\[ |\bigcup_{i \in V'} S_i| = |\bigcup_{i \in V} S_i| + |S_u \cup S_v| - \left| \bigcup_{i \in V} S_i \right| \cap (S_u \cup S_v) | \leq |\bigcup_{i \in V} S_i| + |S_u| - |S_u \cap S_v|; \]

(1) follows now from (2). \( \Box \)

A product set, \( \mathcal{R} \), in \( \mathbb{Z}^n \) is any finite nonempty set of the form

\[ \mathcal{R} = R_1 \times \cdots \times R_n \]

where \( R_1, \ldots, R_n \subset \mathbb{Z} \). The set \( R_i \) is referred to as the \( i \)-th projection of \( \mathcal{R} \), denoted

\[ R_i = \pi_i(\mathcal{R}), \quad 1 \leq i \leq n. \]
For \( h \in \mathbb{N}^* \) the set
\[
\mathcal{P} = \{ c \in \mathbb{Z}^*: 0 \leq c_i < h_i; 1 \leq i \leq n \}
\]
is called the \((n; h)\)-parallelepiped. Let \( p_1, \ldots, p_n \) be distinct primes. We define \( \Phi(n; p_1, \ldots, p_n) \) to be the family of those product sets in \( \mathbb{Z}^* \) of the form \( a_1 p_1^{\alpha_1} \cdots a_n p_n^{\alpha_n} + \mathcal{P} \), where \( a_1, \ldots, a_n, \alpha_1, \ldots, \alpha_n \) are any non-negative integers and \( \mathcal{P} \) is the \((n; (p_1^{\alpha_1}, \ldots, p_n^{\alpha_n}))\)-parallelepiped.

**Proposition.** Let \( p_1, \ldots, p_n \) be distinct odd primes and let \( \mathcal{P} \) be the \((n; (p_1^{\alpha_1}, \ldots, p_n^{\alpha_n}))\)-parallelepiped. Let \( \Gamma \subseteq \Phi(n; p_1, \ldots, p_n) \) be a family of proper subsets of \( \mathcal{P} \) which cover \( \mathcal{P} \). If \( g(w, \overline{z}) < 2 \), where \( g, w, \overline{z} \) are given by (1.9)-(1.11) then \( \Gamma \) contains two sets of the same cardinality.

**Proof.** Assume, on the contrary, that the sets in \( \Gamma \) have distinct cardinalities. Set \( N = |\mathcal{P}| \). Modify \( \Gamma \) to \( \Gamma^* \) as follows. Enlarge each \( \mathcal{A} \in \Gamma \) with
\[
|\mathcal{A}| = \frac{N}{p_i \cdots p_n^{s_i-1}}, \quad 2 \leq i \leq n, 0 \leq t < s_i,
\]
to a larger product set \( \mathcal{A}^* \) by enlarging \( \pi_i(\mathcal{A}) \) to \( \pi_i(\mathcal{A}^*) \). In other words if \( \mathcal{A} \in \Gamma \) satisfies (7), then replace \( \mathcal{A} \) with \( \mathcal{A}^* \), where \( \pi_i(\mathcal{A}^*) = \pi_i(\mathcal{A}) \) and \( \pi_i(\mathcal{A}^*) = 2 \), \( 2 \leq i \leq n \). Since we assumed that the sets in \( \Gamma \) have distinct cardinalities, it follows that \( \Gamma^* \) can contain at most two sets of cardinality \( N/p_i^{s_i-1} \), \( 2 \leq i \leq n \), no sets of cardinality \( N/p_i^{t_s-1} \), \( 2 \leq i \leq n \), \( 0 \leq t < s_i \) and at most one set of any other cardinality.

Each set \( \mathcal{A} \in \Gamma^* \) can be partitioned into "building blocks" \( \mathcal{A} = \bigcup_{\mathcal{A}_i} \mathcal{A}_i \) where each \( \mathcal{A}_i \in \Phi(n; p_1, \ldots, p_n) \) has cardinality \( |\mathcal{A}_i| = \prod_{i=1}^{n} p_i^{\alpha_i} \) and
\[
I = \text{ind}(\mathcal{A}) = \{ 1 \leq i \leq n: \pi_i(\mathcal{A}) = \pi_i(\mathcal{A}_i) \}.
\]
We now modify \( \Gamma^* \) to a new family \( \Gamma^{**} \) by replacing each \( \mathcal{A} \in \Gamma^* \) with all of its building blocks \( \mathcal{A}_i \). The sets in \( \Gamma^{**} \) all have cardinalities of the form \( \prod_{i=1}^{n} p_i^{\alpha_i} \) for some \( I = \{1, \ldots, n\}, I \neq \emptyset \). Furthermore, at most \( \alpha(I) \) sets in \( \Gamma^{**} \) have this cardinality, where
\[
\alpha(I) = \begin{cases} 2 \prod_{i=1}^{n} p_i^{i-1}, & I = \{i\}, \quad i \neq 1, \\ p_i^{i-1} p_i^{s_i-1}, & I = \{1, i\}, \quad i \neq 1, \\ \prod_{i=1}^{n} p_i^{i-1}, & \text{otherwise}, \end{cases}
\]

We are going to forget about \( I \) now. Instead we will assume that \( \Gamma^{**} \subseteq \Phi(n; p_1, \ldots, p_n) \) is any family of subsets of \( \mathcal{P} \) containing precisely \( \alpha(I) \) distinct (and therefore disjoint) sets of cardinality \( \prod_{i=1}^{n} p_i^{\alpha_i} \), for each \( I \neq \emptyset \). Our conclusion will be that \( \Gamma^{**} \) cannot cover \( \mathcal{P} \). This is obviously more than what the Proposition states.

We introduce some notation. Let
\[
A = \{ \mathcal{A} \in \Gamma^{**}: |\text{ind}(\mathcal{A})| = 1 \}.
\]
For \( I = \{1, \ldots, n\}, |I| \geq 2 \) set
\[
\mathcal{A}(I) = \bigcup \{ R \in \Gamma^{**}: \text{ind}(\mathcal{A}) = I \}.
\]
The basic observation about \( A \) is that for \( \mathcal{A} \in A \) that set \( \mathcal{P} \setminus \mathcal{A} \) is also a product set. Thus
\[
\mathcal{P} = \mathcal{P} \setminus \bigcup_{\mathcal{A} \in A} \mathcal{A} = \bigcap_{\mathcal{A} \in A} \mathcal{P} \setminus \mathcal{A}
\]
is also a product set. For any \( i, 1 \leq i \leq n \)
\[
|\pi_i(\mathcal{P})| = |\pi_i(\mathcal{P} \setminus \mathcal{A})| = \prod_{\mathcal{A} \in A} \pi_i(\mathcal{A}) = y_i
\]
where
\[
y_i = p_i^{n_i} - \alpha(I_i).
\]
Thus for \( \mathcal{A} \in \Gamma^{**} \), \( \text{ind}(\mathcal{A}) = I, |I| \geq 2 \), \( \mathcal{A} \cap \mathcal{P} \neq \emptyset \)
\[
|\pi_i(\mathcal{A} \cap \mathcal{P})| = |\pi_i(\mathcal{A}) \cap \pi_i(\mathcal{P})| = \begin{cases} 1, & i \in I, \\ y_i, & i \notin I, \end{cases}
\]
so that
\[
|\mathcal{A} \cap \mathcal{P}| = |\mathcal{A}| \prod_{i \notin I} y_i^{-1}.
\]
From this we obtain the bounds
\[
|\mathcal{A}(I) \cap \mathcal{P}| \leq |\mathcal{P}| \alpha(I) \prod_{i \notin I} y_i^{-1} = \begin{cases} |\mathcal{A}| \prod_{i \notin I} \bar{y}_i, & |I| \neq 1 \text{ or } |I| = 2, \\ |\mathcal{A}| \prod_{i \notin I} \bar{z}_i, & |I| \geq 3. \end{cases}
\]
Thus
\[
\sum_{|I| \geq 2} |\mathcal{A}(I) \cap \mathcal{P}| \leq |\mathcal{P}| \sum_{|I| \geq 2} [g_4(w, z) - 1]
\]
where
\[
g_4(w, z) = (1 + w) \prod_{i=2}^{n} (1 + z_i) - w(1 + w - z_i) \prod_{i=2}^{n} z_i
\]
To prove the Proposition we show that the sets in $\mathcal{F}^* \setminus \mathcal{F}$ cannot cover $\mathcal{F}$; more precisely,

$$|\bigcup_{|I| \geq 2} (\mathcal{A}(I) \cap \mathcal{F})| < |\mathcal{F}|.$$  

(20)

We make an assumption now which is worst possible regarding (20).

ASSUMPTION. Each set $\mathcal{A} \in \mathcal{F}^*$, $|\text{ind} (\mathcal{A})| = 2$, intersects $\mathcal{F}$.

It follows from this assumption that if $|I_1| = |I_2| = 2$, $I_1 \cap I_2 = \emptyset$, then

$$|\mathcal{A}(I_1) \cap \mathcal{A}(I_2) \cap \mathcal{F}| = |\mathcal{F}||\prod_{i \in I_1 \cup I_2} z_i|.$$  

(21)

Let $\mathcal{F}$ denote the family of subsets $I \subset \{1, \ldots, n\}$, $|I| = 2$. According to the Lemma

$$|\bigcup_{|I| = 2} (\mathcal{A}(I) \cap \mathcal{F})| \leq \sum_{|I| = 2} |\mathcal{A}(I) \cap \mathcal{F}| - \sum_{g_1, g_2 \in E(G)} |\mathcal{A}(g_1) \cap \mathcal{A}(g_2) \cap \mathcal{F}|$$

for any forest $G$ with $V(G) = \mathcal{F}$. Thus

$$|\bigcup_{|I| \geq 3} (\mathcal{A}(I) \cap \mathcal{F})| \leq \sum_{|I| \geq 3} |\mathcal{A}(I) \cap \mathcal{F}| + |\bigcup_{|I| = 2} (\mathcal{A}(I) \cap \mathcal{F})|$$

$$\leq \sum_{|I| \geq 2} |\mathcal{A}(I) \cap \mathcal{F}| - \sum_{g_1, g_2 \in E(G)} |\mathcal{A}(g_1) \cap \mathcal{A}(g_2) \cap \mathcal{F}|.$$  

(22)

In view of (18), (21) it suffices now, to establish (20), to exhibit a forest $G$, with $V(G) = \mathcal{F}$, satisfying

$$[I_1, I_2] \in E(G) \Rightarrow I_1 \cap I_2 = \emptyset,$$

such that

$$\sum_{I_1, I_2 \in E(G)} \prod_{i \in I_1 \cup I_2} z_i = z_1 (3z_2 z_3 z_4 + 3z_2 z_3 z_5 + 2z_2 z_4 z_5 + z_3 z_4 z_5).$$  

(24)

such a graph appears in Figure 1 (all vertices not in the figure are isolated). $lacksquare$

![Figure 1](image_url)