To prove the necessity of the condition suppose that $\sum_{n=1}^{\infty} \varphi_n^{-1} < \infty$. Let ω be any sequence of real numbers. For $m = 2, 3, \ldots$, denote

$$F_m = \operatorname{card}\{n < \Phi_m:\ \omega(\gamma_\Phi(n)) = \omega(m-1)\}.$$

Since

$$C(\gamma_\Phi;\ 0,\ \Phi_m - 1,\ m-1) = \prod_{n=2}^{m} (\varphi_n - 1),$$

we have

$$\varliminf_{m\to\infty} (F_m/\Phi_m) \geqslant \prod_{n=2}^{\infty} (1-\varphi_n^{-1}) > 0.$$

It follows that the sequence $\omega \circ \gamma_\Phi$ can not be w.d. mod 1, and so γ_Φ has not swd-property. This completes the proof of the theorem.

Corollary 4. *If $\sum_{n=1}^{\infty} \varphi_n^{-1} < \infty$, then for no sequence ω of reals the sequence $\omega \circ \gamma_\Phi$ is u.d.* mod 1.

Corollary 5. *The sequence $(\gamma_\Phi(n)\theta)$, $n = 0, 1, \ldots$, is w.d.* mod 1 (*u.d.* mod 1) *if and only if θ is irrational and the series $\sum_{n=1}^{\infty} \varphi_n^{-1}$ is divergent.*

We remark that if α and β are sequences having the swd-property, then the sequence $\alpha \circ \beta$ has it also. Using this fact and Theorems 1, 2, 3, 4, and 5, we may construct other sequences with the swd-property.

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Zeros of p-adic L-functions

by

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Introduction. In this paper, we study the zeros of the Kubota–Leopoldt p-adic L-functions. These functions are known to be related to certain formal power series over $\mathbf{Z}_p$ defined by Iwasawa, and it is this relationship which will be exploited to analyze the occurrence of zeros for the L-functions. The zeros of these power series are better understood (via the Main Conjecture), and are more readily computed. The assumption that a zero of the Iwasawa series $f_\chi(T)$ for a Dirichlet character χ will yield a corresponding zero of the L-function $L_p(s, \chi)$ is in general false, however. For instance, in [8], Wagstaff has computed zeros of the Iwasawa series for various fields and has used the relationship with $L_p(s, \chi)$ to compute values for s which in many cases are not within the domain of convergence of the L-function, so clearly cannot be zeros of it. The relationship between zeros of the series $f_\chi(T)$ and those of $L_p(s, \chi)$ will be given. We then will consider the more general case of p-adic L-functions defined over the fields $\mathbf{Q}_m$ constituting the layers of the basic $\mathbf{Z}_p$-extension of $\mathbf{Q}$, and the relationship of their zeros to those of the series $f_\chi(T)$.

The results presented here are based upon material from the first-named author's Ph. D. dissertation. The authors would like to thank Warren Sinnott for many useful conversations and suggestions.

1. p-adic L-functions. Let p be an odd prime. Let $\mathbf{Z}_p$, $\mathbf{Q}_p$ and $\mathbf{C}_p$ denote the p-adic integers, the p-adic rationals, and the completion of the algebraic closure of $\mathbf{Q}_p$, respectively. We use the normalization $|p| = p^{-1}$ for the p-adic absolute value. Let ω be the p-adic Teichmüller character, defined as follows: for $a \in \mathbf{Z}_p^\times$, let $\omega(a)$ be the unique $(p-1)$st root of unity in $\mathbf{Z}_p$ satisfying $\omega(a) \equiv a \pmod p$. [Then ω is a Dirichlet character on $\mathbf{Z}$ of order $p-1$ and conductor p.] Let $\langle a\rangle = \omega(a)^{-1} a$.

Let d be a positive integer prime to p. Assume $d \not\equiv 2 \pmod 4$. Let $q_n = p^{n+1} d$, $K_n = \mathbf{Q}(\zeta_{q_n})$, and $K_\infty = \bigcup_{n\geqslant 0} \mathbf{Q}(\zeta_{q_n})$, where ζ_{q_n} is a q_n-th root of unity. Then $\operatorname{Gal}(K_\infty/\mathbf{Q}) \cong G \times \Gamma$, where $G = \operatorname{Gal}(K_0/\mathbf{Q})$ and $\Gamma = \operatorname{Gal}(K_\infty/K_0) \cong 1+q_0\mathbf{Z}_p = (1+q_0)^{\mathbf{Z}_p}$. Let $\varkappa$ be the isomorphism between Γ and $1+q_0\mathbf{Z}_p$. [So $1+q_0$ is the image, under $\varkappa$, of a topological generator for

Γ.] The elements of Γ fixing K_n are just the elements of $\Gamma^{p^n} \cong 1+q_n \boldsymbol{Z}_p$. So we have $\mathrm{Gal}(K_n/K_0) = \Gamma/\Gamma^{p^n} = \Gamma_n$, and $\mathrm{Gal}(K_n/\boldsymbol{Q}) \cong G \times \Gamma_n$.

Let φ be a Dirichlet character of conductor dp^j, for some $j \geqslant 0$. Then φ may be regarded as a character of $\mathrm{Gal}(K_n/\boldsymbol{Q})$, so that it may be written uniquely as $\varphi = \chi\psi$, with $\chi \in G^{\wedge}$ and $\psi \in \Gamma^{\wedge}$. [$A^{\wedge}$ denotes the character group of A.] χ is called a *character of the first kind*, and ψ is called a *character of the second kind*. Note that characters of the first kind are associated with $K_0 = \boldsymbol{Q}(\zeta_{pd})$ and hence have conductors dividing pd, while characters of the second kind are associated with the subfield of $\boldsymbol{Q}(\zeta_{p^{n+1}})$ of degree p^n over $\boldsymbol{Q}$, and hence are either trivial or have p-power order and conductor of the form p^j with $j \geqslant 2$.

Let φ be any primitive Dirichlet character. Then the p-adic L-function $L_p(s, \varphi)$ is the unique continuous p-adic function $\boldsymbol{Z}_p \to \boldsymbol{C}_p$ such that:

$$L_p(1-n, \varphi) = -\big(1-\varphi\omega^{-n}(p)\,p^{n-1}\big)\big(B_{n,\varphi\omega^{-n}}/n\big)$$

for every positive integer n. [$B_{n,\chi}$ denotes the generalized Bernoulli number.] Note that if $p-1$ divides n, then $L_p(1-n, \varphi)$ agrees with the classical Dirichlet L-function except for the Euler factor at p, which does not appear. Also, if φ is odd (i.e. $\varphi(-1) = -1$), then $L_p(s, \varphi)$ is identically zero.

Let χ be a non-trivial even Dirichlet character of the first kind associated to K, a real cyclic extension of $\boldsymbol{Q}$. Let $q = \mathrm{l.c.m.}(f, p)$, where f is the conductor of the character χ, and let B_j be the jth Bernoulli number.

Then the p-adic L-function, $L_p(s, \chi)$, may be written (according to a formula of Washington, [10]):

$$L_p(s, \chi) = q^{-1}(s-1)^{-1} \sum_{\substack{a=1 \\ (a,p)=1}}^{q} \chi(a)\,\langle a\rangle^{1-s} \sum_{j=1}^{\infty} \binom{1-s}{j} (q/a)^j B_j.$$

Now Iwasawa, [3], has defined a formal power series $f_\chi(T) \in \boldsymbol{Z}_p[\chi][[T]]$ connected to $L_p(s, \chi)$ via:

$$f_\chi\big(\varkappa(\gamma_0)^s - 1\big) = L_p(s, \chi)$$

where $\varkappa$ is defined by $\gamma_0(\zeta_{p^n}) = \zeta_{p^n}^{\varkappa(\gamma_0)}$ for every n, where ζ_{p^n} is a p^n-th root of unity, and where γ_0 is a topological generator for $\mathrm{Gal}(K_\infty/K)$. We may take $\varkappa(\gamma_0) = 1+q$, as above. We note that $L_p(s, \chi)$ is then defined and analytic for $|s| < p^{1-1/(p-1)}$. Morita, [6], has shown that $L_p(s, \chi)$ cannot be continued to a single-valued analytic function on any larger s-disk. Now the change of variable $T = \varkappa(\gamma_0)^s - 1$ relates zeros of $f_\chi(T)$ with zeros of $L_p(s, \chi)$. Under this relation, the s-disk, $\mathscr{D}_0 = \{s\colon |s| < p^{1-1/(p-1)}\}$, maps into the open unit T-disk. [A partial inverse is given by $s = \log(1+T)/\log(1+q)$, where "log" denotes the p-adic logarithm (see below).] We note that if $s_0 \in \mathscr{D}_0$ is a zero of $L_p(s, \chi)$, then we have a corresponding $\alpha_0 = (1+q)^{s_0} - 1$ in the open unit T-disk which is a zero of $f_\chi(T)$. As we will show, however, the converse is not necessarily true.

The power series $f_\chi(T)$ is known to satisfy the hypothesis of the Weierstrass Preparation Theorem, [9]. Thus it may be written as a product:

$$f_\chi(T) = \pi^\mu D_\chi(T)\,U(T)$$

where $U(T)$ is a unit power series, $D_\chi(T)$ is a distinguished polynomial, π is a prime element above p in the ring $\boldsymbol{Z}_p[\chi]$, and μ is a non-negative integer depending on K and χ. We consider f_χ as a function on the open unit T-disk. Then we note that the zeros of f_χ are exactly the zeros of the distinguished polynomial D_χ. In particular, f_χ has only finitely many zeros. Also, it is known that for abelian extensions of $\boldsymbol{Q}$, $\mu = 0$, so that in our situation,

$$f_\chi(T) = D_\chi(T)\,U(T).$$

2. Zeros of $L_p(s, \chi)$. We first state some general results concerning p-adic logarithms and exponentials. Define:

$$\exp(X) = \sum_{n=0}^{\infty} X^n/n!,$$

$$\log(1+X) = \sum_{n=1}^{\infty} (-1)^{n+1} X^n/n.$$

Then:

1. $\exp(X)$ has radius of convergence $p^{-1/(p-1)}$.

2. $\log(1+X)$ may be extended to all of $\boldsymbol{C}_p^\times$, such that $\log p = 0$, and $\log(xy) = \log x + \log y$ for all $x, y \in \boldsymbol{C}_p^\times$.

3. If $|x| < p^{-1/(p-1)}$, then $|\log(1+x)| = |x|$; if $|x| \leqslant p^{-1/(p-1)}$, then $|\log(1+x)| \leqslant |x|$.

(For proofs, see Washington, [9].)

In fact, let $D = \{x\colon |x| < p^{-1/(p-1)}\}$. So D is an additive group, and $1+D$ is a multiplicative group. Then it is well known, [5], that $\exp\colon D \to 1+D$, and $\log\colon 1+D \to D$ are inverse isomorphisms.

LEMMA 1. *Let $|\alpha| \leqslant 1$, $\alpha \in \boldsymbol{C}_p$. Then for all positive $n \in \boldsymbol{Z}$, $\alpha^{p^n} \in D$ if and only if $(1+\alpha)^{p^n} - 1 \in D$.*

Proof. We know:

$$(1+\alpha)^{p^n} - 1 = p^n\alpha + \ldots + \binom{p^n}{r}\alpha^r + \ldots + \alpha^{p^n}.$$

Each of the terms on the right, except α^{p^n} is of absolute value at most $|p\alpha| \leqslant |p| < p^{-1/(p-1)}$. If $|\alpha^{p^n}| < p^{-1/(p-1)}$ as well, then $|(1+\alpha)^{p^n} - 1| < p^{-1/(p-1)}$.

Conversely, if $(1+\alpha)^{p^n}-1 \in D$, then α^{p^n} clearly must also be in D. Hence we obtain the desired result. ∎

LEMMA 2. *Let q be as in Section 1. Then $(1+q)^s$ converges for $|s| < p^{1-1/(p-1)}$, and for such s we have*

$$|(1+q)^s-1| < p^{-1/(p-1)}.$$

Proof. For proof that $(1+q)^s = 1+\ldots+\binom{s}{n}q^n+\ldots$ converges on $|s| < p^{1-1/(p-1)}$, see [9], Proposition 5.8. Note that $v_p(n!) = [n/p]+[n/p^2]+\ldots$, so if $p^m \leqslant n \leqslant p^{m+1}$, we have

$$v_p(n!) \leqslant n/p+n/p^2+\ldots+n/p^m = (p^{m-1}+\ldots+1)\,n/p^m$$
$$= ((p^m-1)/(p-1))\,n/p^m \leqslant (n-1)/(p-1).$$

Hence

$$\left|\binom{s}{n}\right| < p^{(n-1)/(p-1)}(p^{1-1/(p-1)})^n = p^{n-1/(p-1)}.$$

So $\left|\binom{s}{n}q^n\right| < p^{-1/(p-1)}$ which gives

$$|(1+q)^s-1| \leqslant \max_{n\geqslant 1}\left\{\left|\binom{s}{n}q^n\right|\right\} < p^{-1/(p-1)}$$

as desired. ∎

Now let $|x| < 1$, $x \in C_p$, and suppose $\log(1+x) = 0$. Then we may choose an integer n such that $x^{p^n} \in D$. By Lemma 1, $(1+x)^{p^n} \in 1+D$. But $\log(1+x)^{p^n} = p^n \log(1+x) = 0$, so $(1+x)^{p^n} = 1$. [Since log is an isomorphism on $1+D$, it will have trivial kernel there.] So $1+x$ is a p^n-th root of unity.

LEMMA 3. *Let $|\alpha| < 1$, $\alpha \in C_p$, and $s_\alpha = \log(1+\alpha)/\log(1+q)$. If $|s_\alpha p^m| < p^{1-1/(p-1)}$, then $(1+q)^{s_\alpha p^m} = \varrho^{-1}(1+\alpha)^{p^m}$ for some p-power root of unity ϱ, which is uniquely determined by $|(1+\alpha)^{p^m}-\varrho| < p^{-1/(p-1)}$.*

Proof. Note that $\log(1+\alpha)^{p^m} = \log(1+q)^{s_\alpha p^m}$, and that Lemma 2 gives $(1+q)^{s_\alpha p^m} \in 1+D$. Thus $\log[(1+\alpha)^{p^m}/(1+q)^{s_\alpha p^m}] = 0$, and $(1+\alpha)^{p^m}/(1+q)^{s_\alpha p^m}$ may be written as $1+x$ for $|x| < 1$. Thus $(1+\alpha)^{p^m}/(1+q)^{s_\alpha p^m} = \varrho$ for some p-power root of unity ϱ, by the remark above.

Now, since $(1+q)^{s_\alpha p^m}-1 \in D$, we have $\varrho^{-1}(1+\alpha)^{p^m}-1 \in D$. So

$$|(1+\alpha)^{p^m}-\varrho| < p^{-1/(p-1)}.$$

Suppose ϱ_1, ϱ_2 are p-power roots of unity, and

$$|(1+\alpha)^{p^m}-\varrho_j| < p^{-1/(p-1)} \quad \text{for} \quad j = 1, 2.$$

Then

$$|\varrho_2-\varrho_1| = |((1+\alpha)^{p^m}-\varrho_1)-((1+\alpha)^{p^m}-\varrho_2)|$$
$$\leqslant \max_{j=1,2}\{|(1+\alpha)^{p^m}-\varrho_j|\} < p^{-1/(p-1)}.$$

So

$$|\varrho_1^{-1}\varrho_2-1| < p^{-1/(p-1)}.$$

Now $\varrho_1^{-1}\varrho_2$ is a p-power root of unity, say of order p^n. Thus

$$|\varrho_1^{-1}\varrho_2-1| = p^{-p/p^n(p-1)} < p^{-1/(p-1)}.$$

So $p^{n-1}(p-1) < p-1$, and hence $n = 0$. Hence $\varrho_1 = \varrho_2$. ∎

Now let ψ be a Dirichlet character of the second kind for the prime p. Such characters are in bijective correspondence with the p-power roots of unity via:

$$\psi \text{ corresponds to } \zeta \text{ if and only if } \psi(\gamma_0)^{-1} = \zeta.$$

Write ψ_ζ for the character of the second kind associated to ζ. Then:

$$L_p(s, \chi\psi_\zeta) = f_\chi(\zeta\varkappa(\gamma_0)^s-1),$$

where χ is a character of the first kind, as in Section 1.

Because of this relationship, it is natural to consider $\{L_p(s, \chi\psi_\zeta)\}$ for ζ ranging through the p-power roots of unity. We study the question of when a zero of $f_\chi(T)$ corresponds to a zero of any one of these related L-series.

THEOREM 4. *Let α be a zero of $f_\chi(T)$, and let $s_\alpha = \log(1+\alpha)/\log(1+q)$. If $|s_\alpha| < p^{1-1/(p-1)}$, then $L_p(s_\alpha, \chi\psi_\varrho) = 0$ for some p-power root of unity ϱ.*

Proof. We know $|\alpha| < 1$. Since $|s_\alpha| < p^{1-1/(p-1)}$, we take $m = 0$ in Lemma 3 to get $(1+q)^{s_\alpha} = \varrho^{-1}(1+\alpha)$ for some p-power root of unity ϱ. So:

$$L_p(s_\alpha, \chi\psi_\varrho) = f_\chi(\varrho(1+q)^{s_\alpha}-1) = f_\chi(\varrho\varrho^{-1}(1+\alpha)-1) = f_\chi(\alpha) = 0. \ ∎$$

In fact, Lemma 3 gives us a characterization of the p-power root of unity ϱ, i.e. ϱ is the unique root of unity which satisfies $|1+\alpha-\varrho| < p^{-1/(p-1)}$.

THEOREM 5. *Let α be a zero of $f_\chi(T)$, and let $s_\alpha = \log(1+\alpha)/\log(1+q)$. If s_α is a zero of at least one of $\{L_p(s, \chi\psi_\zeta)\}$ then there is a (necessarily unique) p-power root of unity ϱ such that*

$$|\alpha+1-\varrho| < p^{-1/(p-1)}.$$

Conversely, if $|\alpha+1-\varrho| < p^{-1/(p-1)}$ for some p-power root of unity ϱ, then $L_p(s_\alpha, \chi\psi_\varrho) = 0$.

Proof. Say $L_p(s_\alpha, \chi\psi_\zeta) = 0$ for some p-power root of unity ζ. Then, since $L_p(s_\alpha, \chi\psi_\zeta)$ is defined, we must have $|s_\alpha| < p^{1-1/(p-1)}$. Lemma 3 gives

$(1+q)^{s_\alpha} = \varrho^{-1}(1+\alpha)$ for some p-power root of unity ϱ. And Lemma 2 gives $(1+q)^{s_\alpha}-1 \in D$. So we have $\varrho^{-1}(1+\alpha)-1 \in D$, i.e. $|1+\alpha-\varrho| < p^{-1/(p-1)}$.

Conversely, say there is a p-power root of unity ϱ, such that $|\alpha+1-\varrho| < p^{-1/(p-1)}$. Then $\varrho^{-1}(1+\alpha) \in 1+D$, so that $\log(\varrho^{-1}(1+\alpha)) \in D$, i.e. $\log(1+\alpha) \in D$. So $|s_\alpha| < p^{1-1/(p-1)}$ and Lemma 3 gives $(1+q)^{s_\alpha} = \varrho^{-1}(1+\alpha)$. As in Theorem 4, we obtain $L_p(s_\alpha, \chi\psi_\varrho) = 0$. ∎

Note that the uniqueness of ϱ does not imply that $L_p(s, \chi\psi_\varrho)$ is the only member of $\{L_p(s, \chi\psi_\zeta)\}$ having s_α as a zero. For example, if both α and $(1+\alpha)\zeta-1$ are zeros of f_χ, then s_α could be a zero of both $L_p(s, \chi)$ and $L_p(s, \chi\psi_\zeta)$. However, we do have the following:

COROLLARY 6. *Suppose α is a zero of f_χ and ϱ is a p-power root of unity such that $|\alpha+1-\varrho| < p^{-1/(p-1)}$. Let ζ be any p-power root of unity, and $s_\alpha = \log(1+\alpha)/\log(1+q)$. Then:*

$$L_p(s_\alpha, \chi\psi_\zeta) = 0 \quad \text{if and only if} \quad f_\chi(\zeta\varrho^{-1}(1+\alpha)-1) = 0.$$

Proof. By Lemma 3, $(1+q)^{s_\alpha} = \varrho^{-1}(1+\alpha)$, so

$$L_p(s_\alpha, \chi\psi_\zeta) = f_\chi(\zeta(1+q)^{s_\alpha}-1) = f_\chi(\zeta\varrho^{-1}(1+\alpha)-1). \ ∎$$

3. L-functions over the totally real fields, Q_m. Suppose now that α is a zero of f_χ, but that s_α is outside the domain of the L-function, i.e. $|s_\alpha| \geq p^{1-1/(p-1)}$. This seems to occur quite frequently. Using Washington's formula, Wagstaff, [8], has computed approximations for the zeros of $f_\chi(T)$ for χ a real quadratic character and $p = 3, 5$. For example, for χ of conductor 181 and $p = 3$ we find D_χ to be an irreducible (over Q_3) quadratic which has zeros $\alpha \equiv 2055 \pm 647(3)^{1/2} (\text{mod } 3^7)$. He determines $s_\alpha \equiv 7 \pm 1048(3)^{-1/2} (\text{mod } 3^7)$, each of which has absolute value $3^{1/2} = p^{1-1/(p-1)}$. (For further such examples, see Kobayashi, [4].) In this section we provide an interpretation for these values of s_α which do not fall inside the domain of $L_p(s, \chi)$.

Define $\mathcal{S}_n = \{s: |s| < p^{n+1-1/(p-1)}\}$ for each non-negative $n \in Z$. Let ζ_0 be a primitive p^{m+1}-th root of unity, and let Q_m be the unique subfield of $Q(\zeta_0)$ which is cyclic of degree p^m over Q (i.e. the mth field in the tower constituting the basic Z_p-extension of Q). Then, for $s_\alpha \in \mathcal{S}_m$, we have $L_p(s_\alpha, \chi, Q_m)$, the p-adic L-function defined over Q_m, with χ now considered as a character of $\mathrm{Gal}(K_m/Q_m)$. For a detailed definition of these functions, see Deligne–Ribet, [2], and Ribet, [7].

We note that, for $s \in \mathcal{S}_0$, $L_p(s, \chi, Q_m) = \prod_\zeta L_p(s, \chi\psi_\zeta)$, where ζ runs through all p-power roots of unity of order dividing p^m. Moreover, $L_p(s, \chi, Q_m)$ is Iwasawa analytic on $\mathcal{S}_m$.

Corresponding to $L_p(s, \chi\psi_\zeta)$ is the Iwasawa series $f_\chi(\zeta(1+T)-1)$. So $\prod_\zeta L_p(s, \chi\psi_\zeta)$ corresponds to $\prod_\zeta f_\chi(\zeta(1+T)-1)$, i.e.

$$\prod_\zeta L_p(s, \chi\psi_\zeta) = \prod_\zeta f_\chi(\zeta\varkappa(\gamma_0)^s-1) \quad \text{for} \quad s \in \mathcal{S}_0.$$

Coleman, [1], has shown that $\prod_\zeta f_\chi(\zeta(1+T)-1)$ can be written as a power series $g((1+T)^{p^m}-1)$.

Let $h_\chi(T)$ be the Iwasawa series associated to $L_p(s, \chi, Q_m)$, i.e. $L_p(s, \chi, Q_m) = h_\chi(\varkappa(\gamma_m)^s-1)$, where γ_m is a topological generator for $\mathrm{Gal}(K_\infty/K_m)$. We may take $\gamma_m = \gamma_0^{p^m}$ and $\varkappa(\gamma_0) = 1+q$, so that

$$L_p(s, \chi, Q_m) = h_\chi((1+q)^{sp^m}-1), \quad \text{for} \quad s \in \mathcal{S}_m.$$

Now for $s \in \mathcal{S}_0$, we have

$$h_\chi((1+q)^{sp^m}-1) = L_p(s, \chi, Q_m) = \prod_\zeta L_p(s, \chi\psi_\zeta)$$
$$= \prod_\zeta f_\chi(\zeta(1+q)^s-1) = g((1+q)^{sp^m}-1).$$

If $g(T)-h_\chi(T)$ is not identically zero, then the Weierstrass Preparation Theorem implies that it has finitely many zeros is the open unit T-disk. But, for all $T \in \{(1+q)^{sp^m}-1:\ s \in \mathcal{S}_0\} = S$, this function is zero and S is an infinite set. So $g(T)-h_\chi(T) = 0$ for all T in the open unit T-disk, i.e.

$$g((1+q)^{sp^m}-1) = h_\chi((1+q)^{sp^m}-1) \quad \text{for all } s \in \mathcal{S}_m.$$

We consider characters of the second kind. For $s \in \mathcal{S}_m$, and ξ a p-power root of unity, we know (Ribet, [7]), that

$$L_p(s, \chi\psi_\xi, Q_m) = h_\chi(\xi^{p^m}(1+q)^{sp^m}-1).$$

Analogues of the results of the previous section are now possible.

THEOREM 7. *Let α be a zero of $f_\chi(T)$, and $s_\alpha = \log(1+\alpha)/\log(1+q)$. Say $m = \min\{n: s_\alpha \in \mathcal{S}_n\}$. Then s_α is a zero of one of $\{L_p(s, \chi\psi_\xi, Q_m)\}$, where ξ runs through all p-power roots of unity.*

Proof. We know $|\alpha| < 1$. Since $|s_\alpha p^m| < p^{1-1/(p-1)}$, we use Lemma 3 to get $(1+q)^{s_\alpha p^m} = \varrho^{-1}(1+\alpha)^{p^m}$ for some p-power root of unity ϱ. Let ξ be any p^m-th root of ϱ. Then:

$$L_p(s_\alpha, \chi\psi_\xi, Q_m) = h_\chi(\varrho(1+q)^{s_\alpha p^m}-1) = h_\chi(\varrho\varrho^{-1}(1+\alpha)^{p^m}-1)$$
$$= h_\chi((1+\alpha)^{p^m}-1) = 0. \ ∎$$

THEOREM 8. *Suppose $f_\chi(\alpha) = 0$, and let $s_\alpha = \log(1+\alpha)/\log(1+q)$. If s_α is a* *zero of $L_p(s, \chi\psi_\xi, Q_m)$ for some p-power root of unity ξ, then there is a (necessarily unique) p-power root of unity ϱ such that*

$$|(1+\alpha)^{p^m}-\varrho| < p^{-1/(p-1)}.$$

Conversely, if $|(1+\alpha)^{p^m}-\varrho| < p^{-1/(p-1)}$ for some p-power root of unity ϱ, then

$$L_p(s_\alpha, \chi\psi_\xi, Q_m) = 0$$

for ξ any p^m-th root of ϱ.

Proof. Say $L_p(s_\alpha, \chi\psi_\xi, Q_m) = 0$ for some p-power root of unity ξ. Then, since $L_p(s_\alpha, \chi\psi_\zeta, Q_m)$ is defined, we must have $|s_\alpha| < p^{m+1-1/(p-1)}$, i.e. $|s_\alpha p^m| < p^{1-1/(p-1)}$. Lemma 3 gives $(1+q)^{s_\alpha p^m} = \varrho^{-1}(1+\alpha)^{p^m}$ for some p-power root of unity ϱ. And Lemma 2 gives $(1+q)^{s_\alpha p^m}-1 \in D$. So we have $\varrho^{-1}(1+\alpha)^{p^m} - 1 \in D$, i.e.

$$|(1+\alpha)^{p^m}-\varrho| < p^{-1/(p-1)}.$$

Conversely, say there is a p-power root of unity ϱ, such that

$$|(1+\alpha)^{p^m}-\varrho| < p^{-1/(p-1)}.$$

Then $\varrho^{-1}(1+\alpha)^{p^m} \in 1+D$, so that $\log(\varrho^{-1}(1+\alpha)^{p^m}) \in D$, i.e. $\log(1+\alpha)^{p^m} \in D$. So $|s_\alpha p^m| < p^{1-1/(p-1)}$ and Lemma 3 gives

$$(1+q)^{s_\alpha p^m} = \varrho^{-1}(1+\alpha)^{p^m}.$$

As in Theorem 7, we obtain $L_p(s_\alpha, \chi\psi_\xi, Q_m) = 0$, where ξ is any of the p^m-th roots of ϱ. ∎

Furthermore, we have a result analogous to Corollary 6.

COROLLARY 9. *Say α is a zero of f_χ [so that $(1+\alpha)^{p^m}-1$ is a zero of h_χ], and ϱ is a p-power root of unity such that $|(1+\alpha)^{p^m}-\varrho| < p^{-1/(p-1)}$. Let ζ be any p-power root of unity, and $s_\alpha = \log(1+\alpha)/\log(1+q)$. Then:*

$$L_p(s_\alpha, \chi\psi_\zeta, Q_m) = 0$$

if and only if

$$h_\chi(\zeta^{p^m}\varrho^{-1}(1+\alpha)^{p^m}-1) = 0. \ ∎$$

At this point, we note a result of Morita, [6], which provides a "multi-valued analytic continuation" of $L_p(s, \chi)$ to any of the disks $\mathscr{D}_m$, defined as follows:

There exists a polynomial $C(X) = X^{p^m}+a_1(s)X^{p^m-1}+\ldots+a_{p^m}(s)$, which is irreducible over the quotient field of the ring of all Krasner analytic functions that converge on $\mathscr{D}_m$, with $a_j(s)$ Krasner analytic on $\mathscr{D}_m$, and any root of $C(X) = 0$ for $s \in \mathscr{D}_0$ has the form $L_p(s, \chi\psi_\zeta)$ for some p-power root of unity ζ of order dividing p^m.

So s_0 is a zero of the multi-valued analytic continuation if and only if one of the zeros of $C_{s_0}(X) = X^{p^m}+a_1(s_0)X^{p^m-1}+\ldots+a_{p^m}(s_0)$ is $X = 0$, which occurs precisely when $a_{p^m}(s_0) = 0$. But, on $\mathscr{D}_0$, $a_{p^m}(s) = \prod_\zeta L_p(s, \chi\psi_\zeta)$ $= L_p(s, \chi, Q_m)$, and since $a_{p^m}(s)$ and $L_p(s, \chi, Q_m)$ are both Krasner analytic on $\mathscr{D}_m$, we must have $a_{p^m}(s) = L_p(s, \chi, Q_m)$ on $\mathscr{D}_m$. So s_0 is a zero of the multi-valued analytic continuation if and only if $L_p(s_0, \chi, Q_m) = 0$.

Theorem 7 gives $L_p(s_\alpha, \chi\psi_\xi, Q_m) = 0$ for some ξ, a p-power root of unity. Say the order of ξ is $p^n \geq p^m$. For $s \in \mathscr{D}_0$, $L_p(s, \chi\psi_\xi, Q_m)$ divides $L_p(s, \chi, Q_n)$, but each is analytic on $\mathscr{D}_m$. Hence $L_p(s_\alpha, \chi, Q_n) = 0$, and this gives s_α as a zero of the multi-valued analytic continuation of $L_p(s, \chi)$ to $\mathscr{D}_n$.

Returning to our example from Wagstaff, we see that $L_p(s_\alpha, \chi, Q_1)$ is defined for both values of α. So, for each α, $L_p(s_\alpha, \chi\psi_\xi, Q_1) = 0$ for some p-power root of unity ξ. Using Theorem 8, we seek a p-power root of unity ϱ such that $|(1+\alpha)^p-\varrho| < p^{-1/(p-1)}$. $\varrho = 1$ is seen to satisfy this for either α. Hence $L_p(s_\alpha, \chi, Q_1) = 0$ for both values s_α. Alternatively, by the above, we may say that the multi-valued analytic continuation of $L_p(s, \chi)$, to the disk $\mathscr{D}_1$, has a zero at s_α for each α.

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