This is considered to be an analogue of the expansion of an Euler product into an infinite product of "basic Euler products (L-functions)" used in the method of Estermann [1] (cf. [3]).

Remark 3. Our result can be generalized to some extent by similar method, but, for example, we have no result on the analytic nature of $\zeta(s, X)$ when X is the set of all Mersenne primes.

References

[1] T. Estermann, *On certain functions represented by Dirichlet series*, Proc. London Math. Soc. 27 (1928), pp. 435–448.
[2] A. Fujii, *On the zeros of Dirichlet L-functions (V)*, Acta Arith. 28 (1976), pp. 395–403.
[3] N. Kurokawa, *On the meromorphy of Euler products*, Proc. Japan Acad. 54 A (1978), pp. 163–166.
[4] H. L. Montgomery, *Zeros of L-functions*, Invent. Math. 8 (1969), pp. 346–354.
[5] E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Clarendon Press, Oxford 1951.

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On well distribution modulo 1 and systems of numeration

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1. Introduction. A sequence $(\omega(n))$, $n = 0, 1, \ldots$, of real numbers is said to be *well distributed modulo* 1 (w.d. mod 1) if for all real numbers a, b with $0 \leqslant a < b \leqslant 1$ we have

$$\lim_{N\to\infty} \left(N^{-1} \operatorname{card} \{n \geqslant 0\colon\ k \leqslant n \leqslant k+N-1 \ \text{and}\ a \leqslant \{\omega(n)\} < b\}\right) = b-a$$

uniformly in $k = 0, 1, \ldots$, $\{x\}$ denoting the fractional part of x. We shall say that a sequence $(\alpha(n))$, $n = 0, 1, \ldots$, of nonnegative integers has the *substitution property with respect to w.d.* mod 1 (swd-property) if for every w.d. mod 1 sequence ω the sequence $\omega\circ\alpha$ is also w.d. mod 1.

In recent years Coquet [1], [2] has constructed certain sequences having the swd-property. In particular he showed that if $q \geqslant 2$ is an integer and $\sigma(n)$ denotes the sum of digits of n in the q-adic expansion ($n = 0, 1, \ldots$), then the sequence σ has the swd-property. The aim of this note is to generalize this result and also to give some new classes of sequences with the swd-property.

Our basic tool will be the Weyl criterion for w.d. mod 1 ([5], p. 41). Therefore, the following notation will be convenient.

Let $h \neq 0$ be an arbitrary integer and put $e(t) = e^{2\pi iht}$, $t \in R$. For any sequence ω of real numbers and for any integers $u \geqslant 0$, $v \geqslant u$, and $M \geqslant 1$, denote

$$S(\omega; u, v) = \sum_{n=u}^{v} e(\omega(n))$$

and

$$\delta(\omega; M) = \sup_{N\geqslant M} \sup_{k\geqslant 0} |N^{-1} S(\omega; k, k+N-1)|.$$

Clearly S and δ depend also on h, but we have no need to point out this dependence explicitly.

2. Two criteria for the swd-property. Let α be a sequence of nonnegative integers. If for all sufficiently large n we have $\alpha(n) = n+K$ with some constant K, then clearly α has the swd-property. Therefore, we may suppose that this is not the case.

Define a sequence $(m_n) = (m_n(\alpha))$, $n = 0, 1, \ldots$, in the following way. Put $m_0 = 0$. If for an integer $n \geqslant 0$ the number m_n has been already defined, then denote $\mu_n = \alpha(m_n+1) - \alpha(m_n)$ and put

$$(1) \qquad m_{n+1} = \begin{cases} 1+\min\{u > m_n \colon \alpha(u+1)-\alpha(u) \neq \mu_n\} & \text{if} \quad |\mu_n| = 1, \\ 1+m_n & \text{otherwise.} \end{cases}$$

Clearly, for $n = 0, 1, \ldots$, we have $\alpha(m_n+j) = \alpha(m_n)+j\mu_n$, $j = 0, 1, \ldots, m_{n+1}-m_n-1$. It follows that the sequence α has been partitioned into infinitely many segments, each of them being an increasing or decreasing finite sequence of consecutive integers.

THEOREM 1. *If*

$$(2) \qquad \lim_{n\to\infty} (m_{n+1}(\alpha) - m_n(\alpha)) = \infty,$$

then α has the swd-property.

Proof. Let ω be any w.d. mod 1 sequence of real numbers. Using the Weyl criterion, we get

$$(3) \qquad \delta(\omega; M) \to 0 \text{ monotonically as } M \to \infty.$$

Moreover, (1) implies that for all integers $n \geqslant 0$, u, and v with $m_n \leqslant u \leqslant v < m_{n+1}$, we have

$$(4) \qquad |S(\omega\circ\alpha; u, v)| = \left|S\big(\omega; \min(\alpha(u), \alpha(v)), \max(\alpha(u), \alpha(v))\big)\right|$$
$$\leqslant (v-u+1)\,\delta(\omega; v-u+1).$$

Let $\varepsilon > 0$ be arbitrary. By (3), there exists $N_1 \geqslant 1$ such that

$$(5) \qquad \delta(\omega; [N^{1/2}]) \leqslant \varepsilon \quad \text{for all } N \geqslant N_1.$$

Next, by (2), we may choose an integer $n_1 \geqslant 1$ such that

$$(6) \qquad m_{n+1} - m_n \geqslant N_1 \quad \text{for all } n \geqslant n_1.$$

Now let an integer $k \geqslant 0$ be arbitrary and let an integer N satisfy

$$(7) \qquad N \geqslant \max(N_1, \varepsilon^{-1} m_{n_1}, \varepsilon^{-2}).$$

From (1) it follows that there exist integers r and s such that $m_r \leqslant k < m_{r+1}$ and $m_s \leqslant k+N-1 < m_{s+1}$. By (3) and (4), we have

$$(8) \qquad |S(\omega\circ\alpha; k, k+N-1)| \leqslant N\varepsilon \quad \text{if} \quad r = s.$$

Now suppose that $r < s$ and put $t = \max(n_1, r+1)$. Then we may write

$$(9) \qquad S(\omega\circ\alpha; k, k+N-1) = S(\omega\circ\alpha; k, m_{r+1}-1) + S(\omega\circ\alpha; m_{r+1}, m_t-1)$$
$$+ \sum_{n=t}^{s-1} S(\omega\circ\alpha; m_n, m_{n+1}-1) + S(\omega\circ\alpha; m_s, k+N-1),$$

where empty sums are understood to be zeros. Since $m_t - m_{r+1} \leqslant m_{n_1}$, (7) implies

$$|S(\omega\circ\alpha; m_{r+1}, m_t-1)| \leqslant N\varepsilon.$$

Moreover, by (4), (5), and (6), we have

$$|S(\omega\circ\alpha; m_n, m_{n+1}-1)| \leqslant (m_{n+1}-m_n)\varepsilon \quad \text{for all } n \geqslant t.$$

Next we observe that for every integer $T \geqslant 1$ we have

$$T\delta(\omega; T) \leqslant N\varepsilon + T\varepsilon.$$

In fact, if $T < N^{1/2}$, then (7) implies $T\delta(\omega; T) \leqslant T \leqslant N\varepsilon$, and if $T \geqslant N^{1/2}$, then (3), (5), and (7) yield $T\delta(\omega; T) \leqslant T\varepsilon$. Now, by (9), we have

$$(10) \qquad |S(\omega\circ\alpha; k, k+N-1)|$$
$$\leqslant N\varepsilon + (m_{r+1}-k)\varepsilon + N\varepsilon + \sum_{n=t}^{s-1} (m_{n+1}-m_n)\varepsilon + N\varepsilon + (k+N-m_s)\varepsilon \leqslant 4N\varepsilon.$$

Combining (8) and (10) and using the Weyl criterion, we obtain that the sequence $\omega\circ\alpha$ is w.d. mod 1. This completes the proof.

COROLLARY 1. *Let α be a sequence of nonnegative integers satisfying condition* (2). *Let η be a sequence of irrational numbers such that the set $E = \{\eta(n) \colon n = 0, 1, \ldots\}$ is finite. Then the sequence η_α defined by*

$$\eta_\alpha(m_n(\alpha)+j) = (m_n(\alpha)+j)\,\eta(n),$$

$j = 0, 1, \ldots, m_{n+1}(\alpha) - m_n(\alpha) - 1$, $n = 0, 1, \ldots$, *is w.d.* mod 1.

Proof. Let an integer $h \neq 0$ be arbitrary and put $C_h = \min_{x\in E} |\sin \pi hx|$. Clearly, we have $C_h > 0$. Now, for every integer $M \geqslant 1$, we have

$$\sup_{l\geqslant 0} \sup_{N\geqslant M} \sup_{k\geqslant 0} \left|\frac{1}{N} \sum_{n=k}^{k+N-1} e^{2\pi i hn\eta(l)}\right| \leqslant \sup_{l\geqslant 0} \frac{1}{M|\sin \pi h\eta(l)|} = \frac{1}{C_h M}.$$

Next we proceed as in the proof of Theorem 1 with $\delta(\omega; M)$ replaced by $(C_h M)^{-1}$.

Let us note that in the proof of Theorem 1 we have made no use of the type of monotonicity of particular segments of the sequence α. This fact suggests the subsequent theorem the proof of which is similar to that of Theorem 1.

Let $0 = p_0 < p_1 < \ldots$ be a sequence of integers and let an integer $K \geqslant 2$ be arbitrary. Given a sequence α of nonnegative integers let α^* be any sequence that can be obtained by the following procedure. For $n = 0, 1, \ldots$, define

$$(\alpha^*(p_n), \alpha^*(p_n+1), \ldots, \alpha^*(p_{n+1}-1)) = \begin{cases} S_1 & \text{if } p_{n+1}-p_n \leqslant K, \\ S_2 & \text{otherwise}, \end{cases}$$

where S_1 is a permutation of $(\alpha(p_n), \alpha(p_n+1), \ldots, \alpha(p_{n+1}-1))$ and S_2 is either $(\alpha(p_n), \alpha(p_n+1), \ldots, \alpha(p_{n+1}-1))$ or $(\alpha(p_{n+1}-1), \alpha(p_{n+1}-2), \ldots, \alpha(p_n))$.

THEOREM 2. *The sequence α has the swd-property if and only if α^* has.*

To give our second criterion for the swd-property, we shall need the following notation.

Let α be a sequence of nonnegative integers. For any integers a, $u \geqslant 0$, and $v \geqslant u$, denote

(11) $$C(\alpha; u, v, a) = \operatorname{card}\{n \geqslant 0\colon u \leqslant n \leqslant v \text{ and } \alpha(n) = a\}.$$

Now let $u \geqslant 0$ and $v \geqslant u$ be fixed and put $m(u, v) = \max\limits_{u \leqslant n \leqslant v} \alpha(n)$. If $a < 0$ or $a > m(u, v)$, then clearly $C(\alpha; u, v, a) = 0$. Thus there exist integers $t(u, v) \geqslant 1$ and $-1 = a_1 < a_2 < \ldots < a_{2t(u,v)+1} = m(u, v)+1$ such that for $i = 1, 2, \ldots, t(u, v)$ we have

$$C(a_{2i\pm 1}) < C(a_{2i}),$$

(12) $$C(a) \leqslant C(a+1), \quad a = a_{2i-1}, a_{2i-1}+1, \ldots, a_{2i}-1,$$

and

(13) $$C(a) \geqslant C(a+1), \quad a = a_{2i}, a_{2i}+1, \ldots, a_{2i+1}-1,$$

where $C(a) = C(\alpha; u, v, a)$, $a \in Z$. Denote

(14) $$V(\alpha; u, v) = \sum_{i=1}^{t(u,v)} C(\alpha; u, v, a_{2i}).$$

With the above notation we have the following

LEMMA 1. *Let ω be a sequence of real numbers. For any integers $u \geqslant 0$, $v \geqslant u$, $l \geqslant 0$, and $M \geqslant 1$, we have*

(15) $$|S(\omega \circ (\alpha+l); u, v)| \leqslant (v-u+1)\,\delta(\omega; M) + 2MV(\alpha; u, v),$$

where $(\alpha+l)(n) = \alpha(n)+l$, $n = 0, 1, \ldots$

Proof. Using (11), we may write

(16) $$S(\omega \circ (\alpha+l); u, v) = \sum_{a=0}^{m(u,v)} C(a)\, e(\omega(a+l)).$$

Let an integer i, $1 \leqslant i \leqslant t(u, v)$, be fixed. Denote

$$a' = \max(a_{2i-1}-1, a_{2i}-M), \quad C'(a_{2i-1}-1) = 0 \quad \text{and} \quad C'(a) = C(a)$$

for $a = a_{2i-1}, a_{2i-1}+1, \ldots, a_{2i}-1$. Using Abel summation formula and (12), we get

(17) $$\Big|\sum_{a=a_{2i-1}}^{a_{2i}-1} C(a)\, e(\omega(a+l))\Big|$$
$$\leqslant \sum_{a=a_{2i-1}}^{a_{2i}-1} (C'(a)-C'(a-1))\,|S(\omega; a+l, a_{2i}+l-1)|$$
$$\leqslant \delta(\omega; M) \sum_{a=a_{2i-1}}^{a'} (C'(a)-C'(a-1))(a_{2i}-a) + \sum_{a=a'+1}^{a_{2i}-1} (C'(a)-C'(a-1))(a_{2i}-a)$$
$$\leqslant \delta(\omega; M) \sum_{a=a_{2i-1}}^{a_{2i}-1} C(a) + MC(a_{2i}).$$

In a similar way, using (13), we obtain

(18) $$\Big|\sum_{a=a_{2i}}^{a_{2i+1}-1} C(a)\, e(\omega(a+l))\Big| \leqslant \delta(\omega; M) \sum_{a=a_{2i}}^{a_{2i+1}-1} C(a) + MC(a_{2i}).$$

Now we observe that (11) implies $\sum\limits_{a \in Z} C(a) = v-u+1$. Thus, (16), (17), (18), and (14) yield (15).

THEOREM 3. *If*

(19) $$\lim_{N\to\infty} \big(N^{-1} \sup_{k \geqslant 0} V(\alpha; k, k+N-1)\big) = 0,$$

then α has the swd-property.

Proof. Let ω be a w.d. mod 1 sequence of real numbers and denote

$$L_N = N^{-1} \sup_{k \geqslant 0} |S(\omega \circ \alpha; k, k+N-1)|, \quad N = 1, 2, \ldots$$

By Lemma 1 and (19), we have

$$\varlimsup_{N\to\infty} L_N \leqslant \delta(\omega; M), \quad M = 1, 2, \ldots$$

An application of the Weyl criterion completes the proof.

It is clear that criteria (2) and (19) are only sufficient ones for α to have the swd-property. It can be shown that they agree whenever

$$(m_{n+1}(\alpha) - m_n(\alpha))/n \to \infty \quad \text{as} \quad n \to \infty.$$

EXAMPLE 1. Let $1 = \Psi_1 < \Psi_2 < \ldots < \Psi_m$ be a finite sequence of integers. Given an integer $n \geqslant 0$, denote $d_m = n$ and, for $i = m, m-1, \ldots, 1$, put

$c_i(n) = [d_i/\Psi_i]$ and $d_{i-1} = d_i - c_i(n)\Psi_i$. Clearly, we have $n = c_1(n)\Psi_1 + \ldots + c_m(n)\Psi_m$. For $n = 0, 1, \ldots$, denote

$$\sigma(n) = \sum_{i=1}^{m} c_i(n).$$

Let integers $k \geqslant 0$ and $N \geqslant 1$ be arbitrary and denote

$$C(a) = C(\sigma; k, k+N-1, a), \quad a \in Z.$$

For every integer a, we have

$$C(a) \leqslant \sum_{i=0}^{\infty} C(\sigma; i\Psi_m, (i+1)\Psi_m - 1, a) \tag{20}$$

$$= \sum_{i=0}^{\infty} C(\sigma; 0, \Psi_m - 1, a-i) \leqslant \Psi_m.$$

Now put $H = \max_{0 \leqslant n \leqslant \Psi_m} \sigma(n)$ and suppose that $N > 2H\Psi_m$. There exist integers r and s such that

$$r\Psi_m \leqslant k < (r+1)\Psi_m \quad \text{and} \quad s\Psi_m \leqslant k+N-1 < (s+1)\Psi_m.$$

We have $C(a) = 0$ if $a < r$ or $a > s+H$ and $C(a) = \Psi_m$ if $r+H < a \leqslant s$. Thus, using (20), we obtain $V(\sigma; k, k+N-1) \leqslant (H+1)\Psi_m$. By Theorem 3, σ has the swd-property.

We end this section with an application of Lemma 1 to sequences of the form $(\alpha(n)\theta)$, θ being an irrational number.

Let α be a nondecreasing sequence of nonnegative integers such that $\alpha(n) \to \infty$ as $n \to \infty$. For $i = 0, 1, \ldots$ denote

$$l_i = \operatorname{card}\{n \geqslant 0: \alpha(n) = i\} \quad \text{and} \quad L_i = l_0 + l_1 + \ldots + l_i.$$

Using Lemma 1, we may prove the following

COROLLARY 2. *If*

$$\lim_{m\to\infty} \left(L_m^{-1} V(\alpha; 0, L_m - 1)\right) = 0, \tag{21}$$

then for every irrational θ *the sequence* $(\alpha(n)\theta)$, $n = 0, 1, \ldots$, *is uniformly distributed* mod 1.

If the sequence (l_n) is nondecreasing, then $V(\alpha; 0, L_m - 1) = l_m$ and so in this case condition (21) has the form given by Dress [3].

3. Sequences connected with mixed radix systems of numeration. Let (φ_n), $n = 1, 2, \ldots$, be a sequence of integers such that $\varphi_1 = 1$ and $\varphi_n \geqslant 2$ for all $n \geqslant 2$. For $n = 1, 2, \ldots$, denote

$$\Phi_n = \prod_{i=1}^{n} \varphi_i. \tag{22}$$

Then every integer $n \geqslant 0$ has the following unique representation in the system Φ (see Fraenkel [4]):

$$n = \sum_{i=1}^{\infty} c_i(n)\Phi_i, \tag{23}$$

where $c_i(n)$ are integers satisfying $0 \leqslant c_i(n) < \varphi_{i+1}$, $i = 1, 2, \ldots$

Define sequences σ_Φ and γ_Φ by

$$\sigma_\Phi(n) = \sum_{i=1}^{\infty} c_i(n) \tag{24}$$

and

$$\gamma_\Phi(n) = \operatorname{card}\{i \geqslant 1: c_i(n) \neq 0\}. \tag{25}$$

LEMMA 2. *Let* α *be one of the sequences* σ_Φ *and* γ_Φ. *If*

$$\varliminf_{m\to\infty} (\Phi_m^{-1} V(\alpha; 0, \Phi_m - 1)) = 0,$$

then α *has the swd-property.*

Proof. Let ω be any w.d. mod 1 sequence of real numbers and let integers $m \geqslant 2$ and $M \geqslant 1$ be arbitrary.

Given any integers $k \geqslant 0$ and $N \geqslant 3\Phi_m$, there exist integers r and s such that $r\Phi_m \leqslant k < (r+1)\Phi_m$ and $s\Phi_m \leqslant k+N-1 < (s+1)\Phi_m$. Clearly, we have $N \geqslant (s-r-1)\Phi_m > 0$.

By (24) and (25), for all integers $t \geqslant 0$ and v, $0 \leqslant v < \Phi_m$, we have

$$\alpha(t\Phi_m + v) = \alpha(t\Phi_m) + \alpha(v).$$

Thus, Lemma 1 yields

$$\begin{aligned} (26) \quad & |S(\omega\circ\alpha; k, k+N-1)| \\ & \leqslant |S(\omega\circ\alpha; k, (r+1)\Phi_m - 1)| + \sum_{n=r+1}^{s-1} |S(\omega\circ\alpha; n\Phi_m, (n+1)\Phi_m - 1)| \\ & \quad + |S(\omega\circ\alpha; s\Phi_m, k+N-1)| \\ & \leqslant 2\Phi_m + \sum_{n=r+1}^{s-1} |S(\omega\circ(\alpha + \alpha(n\Phi_m)); 0, \Phi_m - 1)| \\ & \leqslant 2\Phi_m + N\delta(\omega; M) + 2MN\Phi_m^{-1} V(\alpha; 0, \Phi_m - 1). \end{aligned}$$

Now, by the hypothesis, we may choose a sequence (m_i) such that

$$\Phi_{m_i}^{-1} V(\alpha; 0, \Phi_{m_i} - 1) \to 0 \quad \text{as} \quad i \to \infty.$$

To complete the proof, it suffices to divide both sides of (26) by N, and then let $N \to \infty$, $m \to \infty$ through the values $m_1, m_2, \ldots$, and $M \to \infty$. By the Weyl criterion $\omega\circ\alpha$ is w.d. mod 1 and so α has the swd-property.

In order to use Lemma 2, we shall need the following lemma from probability theory.

LEMMA 3. *Let (X_n), $n=1, 2, \ldots$, be a sequence of uniformly bounded independent random variables. For $n = 1, 2, \ldots$, denote $Y_n = X_1+X_2+\ldots+X_n$ and let s_n^2 be the variance of Y_n. If $s_n \to \infty$ as $n\to\infty$, then for every sequence (y_n) of real numbers we have $P(Y_n = y_n) \to 0$ as $n\to\infty$, P denoting the probability measure.*

Proof. The sequence (X_n) satisfies Lindeberg condition, so that the lemma follows immediately from the central limit theorem.

THEOREM 4. *The sequence σ_Φ has the swd-property.*

Proof. Let an integer $m\geqslant 2$ be arbitrary and denote

$$T_m = \sum_{j=2}^{m} (\varphi_j - 1). \tag{27}$$

Denoting $C(m, a) = C(\sigma_\Phi; 0, \Phi_m - 1, a)$, $a\in Z$, and using (23) and (24), we get

$$C(m, a) = 0 \quad \text{if} \quad a<0 \text{ or } a>T_m, \tag{28}$$

$$C(m, a) = C(m, T_m - a) \quad \text{for all } a\in Z, \tag{29}$$

and

$$C(m, a-1) \leqslant C(m, a) \quad \text{if} \quad a\leqslant [T_m/2], \tag{30}$$

the above inequality being proved by induction on m (cf. Coquet [1]). Now, (28), (29), and (30) imply

$$V(\sigma_\Phi; 0, \Phi_m - 1) = C(m, [T_m/2]). \tag{31}$$

We shall complete the proof by considering two cases. First suppose that $\overline{\lim_{m\to\infty}}\, \varphi_m = \infty$. As in Example 1 it can be shown that $C(m, a)\leqslant \Phi_{m-1}$ for all $a\in Z$. Thus, using (31), we get $\Phi_m^{-1} V(\sigma_\Phi; 0, \Phi_m - 1) \leqslant \varphi_m^{-1}$. By Lemma 2, σ_Φ has the swd-property.

Next suppose that the sequence (φ_m) is bounded. Let (X_m), $m = 2, 3, \ldots$, be a sequence of independent random variables with the distribution $P(X_m = i) = \varphi_m^{-1}$, $i=0, 1, \ldots, \varphi_m - 1$. Denoting $Y_m = X_2+\ldots+X_m$, we obtain that the variance of Y_m is equal to $12^{-1}\sum_{n=2}^{m}(\varphi_n^2-1)$. Since $P(Y_m = a) = \Phi_m^{-1} C(m, a)$, $a\in Z$, Lemma 3 yields $\Phi_m^{-1} C(m, [T_m/2]) \to 0$ as $m\to\infty$. Thus, by (31) and Lemma 2, σ_Φ has the swd-property.

COROLLARY 3. *The sequence $(\sigma_\Phi(n)\theta)$, $n=0, 1, \ldots$, is w.d.* mod 1 *(u.d.* mod 1*) if and only if θ is irrational.*

The q-adic version of Theorem 4 was proved by Coquet [1]. For a weaker version of Corollary 3 we refer to [6].

THEOREM 5. *The sequence γ_Φ has the swd-property if and only if the series $\sum_{n=1}^{\infty}\varphi_n^{-1}$ is divergent.*

Proof. Let an integer $m\geqslant 2$ be arbitrary. Denoting

$$C(m, a) = C(\gamma_\Phi; 0, \Phi_m - 1, a), \quad a\in Z,$$

and using (23) and (25), we get

$$C(m, a) = 0 \quad \text{if} \quad a<0 \text{ or } a\geqslant m \tag{32}$$

and

$$C(m+1, a) = C(m, a) + (\varphi_{m+1}-1)C(m, a-1) \quad \text{for all } a\in Z. \tag{33}$$

Next, using induction on m, we shall construct a sequence (T_m) such that

$$C(m, a-1)\leqslant C(m, a) \quad \text{if} \quad a\leqslant T_m \tag{34}$$

and

$$C(m, a)\geqslant C(m, a+1) \quad \text{if} \quad a\geqslant T_m. \tag{35}$$

Since $C(2, 0) = 1$ and $C(2, 1) = \varphi_2 - 1$, we may put $T_2 = 1$. Now suppose that for an integer $m\geqslant 2$ the number T_m has been already defined. If $a\leqslant T_m$, then (33), (34), and (35) imply

$$\begin{aligned} C(m+1, a-1) &= C(m, a-1) + (\varphi_{m+1}-1)C(m, a-2) \\ &\leqslant C(m, a) + (\varphi_{m+1}-1)C(m, a-1) = C(m+1, a). \end{aligned}$$

In the same way (33) and (35) yield $C(m+1, a)\geqslant C(m+1, a+1)$ whenever $a\geqslant T_m+1$. Putting

$$T_{m+1} = \begin{cases} T_m & \text{if} \quad C(m+1, T_m) > C(m+1, T_m+1), \\ T_m + 1 & \text{otherwise}, \end{cases}$$

we obtain that (34) and (35) hold with $m+1$ substitued for m.

To prove the sufficiency part of the theorem suppose that the series $\sum_{n=1}^{\infty}\varphi_n^{-1}$ is divergent. Let (X_m), $m = 2, 3, \ldots$, be a sequence of independent random variables with the distribution $P(X_m = 0) = \varphi_m^{-1}$ and $P(X_m = 1) = 1-\varphi_m^{-1}$. Denoting $Y_m = X_2+\ldots+X_m$, it can be shown that the variance s_m^2 of Y_m is $s_m^2 = \sum_{n=2}^{m}\varphi_n^{-1}(1-\varphi_n^{-1})$. Since $\varphi_n\geqslant 2$ for all $n\geqslant 2$, we have $s_m^2 \geqslant 2^{-1}\sum_{n=2}^{m}\varphi_n^{-1}$, so that $s_m\to\infty$ as $m\to\infty$. We observe that $P(Y_m = a) = \Phi_m^{-1} C(m, a)$, $a\in Z$. Thus, (34) and (35) imply $\Phi_m^{-1} V(\gamma_\Phi; 0, \Phi_m - 1) = P(Y_m = T_m)$. By Lemmas 3 and 2, γ_Φ has the swd-property.

To prove the necessity of the condition suppose that $\sum_{n=1}^{\infty} \varphi_n^{-1} < \infty$. Let ω be any sequence of real numbers. For $m = 2, 3, \ldots$, denote

$$F_m = \operatorname{card}\{n < \Phi_m:\ \omega(\gamma_\Phi(n)) = \omega(m-1)\}.$$

Since

$$C(\gamma_\Phi;\ 0, \Phi_m - 1, m-1) = \prod_{n=2}^{m} (\varphi_n - 1),$$

we have

$$\varliminf_{m\to\infty} (F_m/\Phi_m) \geqslant \prod_{n=2}^{\infty} (1 - \varphi_n^{-1}) > 0.$$

It follows that the sequence $\omega \circ \gamma_\Phi$ can not be w.d. mod 1, and so γ_Φ has not swd-property. This completes the proof of the theorem.

COROLLARY 4. *If $\sum_{n=1}^{\infty} \varphi_n^{-1} < \infty$, then for no sequence ω of reals the sequence $\omega \circ \gamma_\Phi$ is u.d.* mod 1.

COROLLARY 5. *The sequence $(\gamma_\Phi(n)\theta)$, $n = 0, 1, \ldots$, is w.d.* mod 1 (*u.d.* mod 1) *if and only if θ is irrational and the series $\sum_{n=1}^{\infty} \varphi_n^{-1}$ is divergent.*

We remark that if α and β are sequences having the swd-property, then the sequence $\alpha \circ \beta$ has it also. Using this fact and Theorems 1, 2, 3, 4, and 5, we may construct other sequences with the swd-property.

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References

[1] J. Coquet, *Sur certaines suites uniformément équiréparties modulo* 1, Acta Arith. 36 (1980), pp. 157–162.

[2] — *Sur certaines suites uniformément équiréparties modulo un* (II), Bull. Soc. R. Sci. Liège 48 (1979), pp. 426–431.

[3] F. Dress, *Sur l'équirépartition de certaines suites* $(x\lambda_n)$, Acta Arith. 14 (1968), pp. 169–175.

[4] A. S. Fraenkel, *Systems of numeration*, Amer. Math. Monthly 92 (1985), pp. 105–114.

[5] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, Wiley, New York 1974.

[6] M. Mendès-France, *Nombres normaux. Applications aux fonctions pseudoaléatoires*, J. Analyse Math. 20 (1967), pp. 1–56.

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Zeros of p-adic L-functions

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Introduction. In this paper, we study the zeros of the Kubota–Leopoldt p-adic L-functions. These functions are known to be related to certain formal power series over $\boldsymbol{Z}_p$ defined by Iwasawa, and it is this relationship which will be exploited to analyze the occurrence of zeros for the L-functions. The zeros of these power series are better understood (via the Main Conjecture), and are more readily computed. The assumption that a zero of the Iwasawa series $f_\chi(T)$ for a Dirichlet character χ will yield a corresponding zero of the L-function $L_p(s, \chi)$ is in general false, however. For instance, in [8], Wagstaff has computed zeros of the Iwasawa series for various fields and has used the relationship with $L_p(s, \chi)$ to compute values for s which in many cases are not within the domain of convergence of the L-function, so clearly cannot be zeros of it. The relationship between zeros of the series $f_\chi(T)$ and those of $L_p(s, \chi)$ will be given. We then will consider the more general case of p-adic L-functions defined over the fields $\boldsymbol{Q}_m$ constituting the layers of the basic $\boldsymbol{Z}_p$-extension of $\boldsymbol{Q}$, and the relationship of their zeros to those of the series $f_\chi(T)$.

The results presented here are based upon material from the first-named author's Ph. D. dissertation. The authors would like to thank Warren Sinnott for many useful conversations and suggestions.

1. p-adic L-functions. Let p be an odd prime. Let $\boldsymbol{Z}_p$, $\boldsymbol{Q}_p$ and $\boldsymbol{C}_p$ denote the p-adic integers, the p-adic rationals, and the completion of the algebraic closure of $\boldsymbol{Q}_p$, respectively. We use the normalization $|p| = p^{-1}$ for the p-adic absolute value. Let ω be the p-adic Teichmüller character, defined as follows: for $a \in \boldsymbol{Z}_p^\times$, let $\omega(a)$ be the unique $(p-1)$st root of unity in $\boldsymbol{Z}_p$ satisfying $\omega(a) \equiv a \pmod p$. [Then ω is a Dirichlet character on $\boldsymbol{Z}$ of order $p-1$ and conductor p.] Let $\langle a \rangle = \omega(a)^{-1} a$.

Let d be a positive integer prime to p. Assume $d \not\equiv 2 \pmod 4$. Let $q_n = p^{n+1} d$, $K_n = \boldsymbol{Q}(\zeta_{q_n})$, and $K_\infty = \bigcup_{n\geqslant 0} \boldsymbol{Q}(\zeta_{q_n})$, where ζ_{q_n} is a q_n-th root of unity. Then $\mathrm{Gal}(K_\infty/\boldsymbol{Q}) \cong G \times \Gamma$, where $G = \mathrm{Gal}(K_0/\boldsymbol{Q})$ and $\Gamma = \mathrm{Gal}(K_\infty/K_0) \cong 1 + q_0 \boldsymbol{Z}_p = (1+q_0)^{\boldsymbol{Z}_p}$. Let $\varkappa$ be the isomorphism between Γ and $1 + q_0 \boldsymbol{Z}_p$. [So $1 + q_0$ is the image, under $\varkappa$, of a topological generator for