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# On certain Euler products

by

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We denote by $\boldsymbol{P}$ the set of all rational primes. Let $X$ be a subset of $\boldsymbol{P}$. We define the zeta function $\zeta(s, X)$ of $X$ by

$$\zeta(s, X) = \prod_{p \in X} (1-p^{-s})^{-1},$$

where $s$ is a variable in $\boldsymbol{C}$, the complex numbers; obviously this product converges absolutely in $\operatorname{Re} s > 1$, and $\zeta(s, \boldsymbol{P})$ is equal to the Riemann zeta function $\zeta(s)$. Since

$$\zeta(s, X) = \zeta(s, \boldsymbol{P})\zeta(s, \boldsymbol{P}-X)^{-1} = \zeta(s)\prod_{p \notin X}(1-p^{-s}),$$

we see that if $X$ or $\boldsymbol{P}-X$ is a finite set then $\zeta(s, X)$ is a meromorphic function on $\boldsymbol{C}$. It seems that the analytic nature of $\zeta(s, X)$ is not so clear when both $X$ and $\boldsymbol{P}-X$ are infinite sets. In this paper we prove the following

THEOREM. *Let $\chi$ be a Dirichlet character of order* 2. *Let*

$$X_+ = \{p \in \boldsymbol{P};\ \chi(p) = 1\} \quad \textit{and} \quad X_- = \{p \in \boldsymbol{P};\ \chi(p) = -1\}.$$

*Then $\zeta(s, X_+)$ and $\zeta(s, X_-)$ are continued as analytic functions (with singularities) in* $\operatorname{Re} s > 0$ *with natural boundaries* $\operatorname{Re} s = 0$.

EXAMPLE. Let $m = 3$, 4 and 6, then the infinite product $\prod_{p \equiv 1(m)} (1-p^{-s})^{-1}$ is analytic in $\operatorname{Re} s > 0$ with the natural boundary $\operatorname{Re} s = 0$.

Proof of theorem. We use a modification of the method of Estermann [1] (cf. [3]). Let $X_0 = \{p \in \boldsymbol{P};\ \chi(p) = 0\}$, which is a finite set. We put: $a(s) = \zeta(s, X_-)$, $b(s) = \zeta(s, \boldsymbol{P}-X_0)$, and $c(s) = L(s, \chi)$, where $L(s, \chi)$ denotes the Dirichlet $L$-function. From $\boldsymbol{P} = X_+ \cup X_0 \cup X_-$ (a disjoint union) we see that $b(s) = \zeta(s, X_+)\zeta(s, X_-)$ so $\zeta(s, X_+) = b(s)/a(s)$. Since $b(s)$ $(= \zeta(s)/\zeta(s, X_0))$ is meromorphic on $\boldsymbol{C}$, to prove theorem, it is sufficient to show that $a(s)$ is analytic in $\operatorname{Re} s > 0$ with the natural boundary $\operatorname{Re} s = 0$.

By definition

$$\frac{a(s)^2}{a(2s)} = \prod_{p\in X_-} \frac{1+p^{-s}}{1-p^{-s}} = \frac{b(s)}{c(s)},$$

hence we have via iteration

$$(*) \qquad \frac{a(s)^{2^{n+1}}}{a(2^{n+1}s)} = \prod_{k=0}^{n} \left(\frac{b(2^k s)}{c(2^k s)}\right)^{2^{n-k}}$$

for each integer $n \geqslant 0$. Note that $a(2^{n+1}s)$ is non-zero holomorphic in $\operatorname{Re} s > 2^{-n-1}$ and that the right hand side of $(*)$ is meromorphic on $\boldsymbol{C}$. Hence $a(s)^{2^{n+1}}$ is meromorphic in $\operatorname{Re} s > 2^{-n-1}$. Thus, by letting $n \to \infty$, we see that $a(s)$ is analytic (with singularities) in $\operatorname{Re} s > 0$.

We now prove that the line $\operatorname{Re} s = 0$ is the natural boundary of $a(s)$. It is sufficient to show that $a(s)$ has at least one singularity (actually, infinitely many singularities) in the region

$$D(t, \varepsilon) = \{s \in \boldsymbol{C};\ 0 < \operatorname{Re} s < \varepsilon,\ t < \operatorname{Im} s \leqslant t+\varepsilon\}$$

for each real number $t$ and $0 < \varepsilon < 1$. We treat the case $t > 0$, since the case $t \leqslant 0$ is exactly similar. For a positive integer $n$ satisfying $5 \cdot 2^{-n-2} < \varepsilon$, we put

$$D_n(t, \varepsilon) = \{s \in \boldsymbol{C};\ 3 \cdot 2^{-n-2} < \operatorname{Re} s < 5 \cdot 2^{-n-2},\ t < \operatorname{Im} s \leqslant t+\varepsilon\} \subset D(t, \varepsilon).$$

To simplify the notation, for a function $f(s)$ meromorphic in $D_n(t, \varepsilon)$ we denote by $P(n; f(s))$ (resp. $Z(n; f(s))$) the number of poles (resp. zeros) with multiplicities of $f(s)$ in $D_n(t, \varepsilon)$. Note that $a(s)^{2^{n+1}}$ is meromorphic in $D_n(t, \varepsilon)$ since $3 \cdot 2^{-n-2} > 2^{-n-1}$. We put:

$$P(n) = P(n; a(s)^{2^{n+1}}),$$
$$P_k(n) = 2^{n-k} P(n; b(2^k s)/c(2^k s)),$$
$$Z_k(n) = 2^{n-k} Z(n; b(2^k s)/c(2^k s))$$

for $k = 0, \ldots, n$. Then, by $(*)$, we see that

$$P(n) \geqslant P_{n-1}(n) - (Z_0(n) + \ldots + Z_{n-2}(n) + Z_n(n)).$$

We prove that there are positive constants $C_1$ and $C_2$ such that

$$(**) \qquad P_{n-1}(n) \geqslant C_1 \cdot n 2^n \quad \text{for sufficiently large } n,$$

and

$$(***) \qquad Z_0(n) + \ldots + Z_{n-2}(n) + Z_n(n) \leqslant C_2 \cdot 2^n \quad \text{for all } n.$$

Then we have the desired result: $P(n) \to \infty$ as $n \to \infty$.

First we show $(**)$. For each meromorphic function $f(s)$ in $\operatorname{Re} s > 0$ we



denote by $N(\sigma_1, \sigma_2; T; f(s))$ the number of zeros of $f(s)$ with multiplicities in the region $\{s \in \boldsymbol{C};\ \sigma_1 < \operatorname{Re} s < \sigma_2,\ 0 < \operatorname{Im} s \leqslant T\}$ for $0 < \sigma_1 < \sigma_2 < 1$ and $T > 0$. Let $f(s) = b(s)$ and $c(s)$. Then it is known that (see Titchmarsh [5], Chap. 9; Montgomery [4] and Fujii [2]):

$$N(\sigma_1, \sigma_2; T; f(s)) = \frac{T}{2\pi} \log T + O(T) \quad \text{as} \quad T \to \infty$$

if $\sigma_1 < 1/2 < \sigma_2$. Hence $(**)$ follows from a result of Fujii [2] (Theorem 1' and § 4) saying that positive proportion of zeros of $\zeta(s)$ and $L(s, \chi)$ are non-coincident, by noting that: $s \in D_n(t, \varepsilon)$ if and only if $3/8 < \operatorname{Re}(2^{n-1}s) < 5/8$ and $2^{n-1}t < \operatorname{Im}(2^{n-1}s) \leqslant 2^{n-1}(t+\varepsilon)$.

Next we show $(***)$. Note that if $s \in D_n(t, \varepsilon)$ then $\operatorname{Re}(2^k s) < 5/16$ for $k \leqslant n-2$ and $\operatorname{Re}(2^n s) > 3/4$. Hence, for $k = 0, \ldots, n-2$ and $n$, we have:

$$Z_k(n) \leqslant 2^{n-k} Z(n; \zeta(2^k s)) \leqslant 2^{n-k} N(0, 5/16; 2^k(t+\varepsilon); \zeta(s))$$
$$= 2^{n-k} N(11/16, 1; 2^k(t+\varepsilon); \zeta(s)).$$

By Titchmarsh ([5], Theorem 9.17) we see that for each $1/2 < \sigma < 1$ there are positive constants $0 < c(1) < 1$ and $c(2)$ such that

$$N(\sigma, 1; T; \zeta(s)) \leqslant c(2)(T+1)^{1-c(1)}$$

for all $T > 0$. Hence there are positive constants $0 < c(3) < 1$ and $c(4)$ such that

$$Z_k(n) \leqslant c(4) 2^{n-k} 2^{k(1-c(3))} = c(4) 2^{n-c(3)k}$$

for all $n$ and for $k = 0, \ldots, n-2$ and $n$. Hence we have

$$Z_0(n) + \ldots + Z_{n-2}(n) + Z_n(n) \leqslant c(4) 2^n \sum_{k=0}^{\infty} 2^{-c(3)k}$$
$$= c(4) 2^n (1 - 2^{-c(3)})^{-1} = C_2 \cdot 2^n$$

for all $n$. Thus the required estimations $(**)$ and $(***)$ hold, and theorem is proved. ■

Remark 1. We may look at zeros of $a(s)^{2^{n+1}}$ instead of poles by using

$$Z(n; a(s)^{2^{n+1}}) \geqslant Z_{n-1}(n) - (P_0(n) + \ldots + P_{n-2}(n) + P_n(n));$$

here we use an estimation of the form

$$N(\sigma, 1; T; L(s, \chi)) \leqslant c(2)(T+1)^{1-c(1)}$$

which is contained in Montgomery [4].

Remark 2. From the equation $(*)$ we have formally the following:

$$a(s) = \prod_{n=0}^{\infty} b(2^n s)^{2^{-n}} c(2^n s)^{-2^{-n}}.$$

This is considered to be an analogue of the expansion of an Euler product into an infinite product of "basic Euler products (L-functions)" used in the method of Estermann [1] (cf. [3]).

Remark 3. Our result can be generalized to some extent by similar method, but, for example, we have no result on the analytic nature of $\zeta(s, X)$ when $X$ is the set of all Mersenne primes.

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# On well distribution modulo 1 and systems of numeration

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**1. Introduction.** A sequence $(\omega(n))$, $n = 0, 1, \ldots$, of real numbers is said to be *well distributed modulo* 1 (w.d. mod 1) if for all real numbers $a$, $b$ with $0 \leqslant a < b \leqslant 1$ we have

$$\lim_{N\to\infty} \left(N^{-1} \operatorname{card}\{n \geqslant 0\colon\ k \leqslant n \leqslant k+N-1 \text{ and } a \leqslant \{\omega(n)\} < b\}\right) = b-a$$

uniformly in $k = 0, 1, \ldots$, $\{x\}$ denoting the fractional part of $x$. We shall say that a sequence $(\alpha(n))$, $n = 0, 1, \ldots$, of nonnegative integers has the *substitution property with respect to w.d.* mod 1 (swd-property) if for every w.d. mod 1 sequence $\omega$ the sequence $\omega\circ\alpha$ is also w.d. mod 1.

In recent years Coquet [1], [2] has constructed certain sequences having the swd-property. In particular he showed that if $q \geqslant 2$ is an integer and $\sigma(n)$ denotes the sum of digits of $n$ in the $q$-adic expansion ($n = 0, 1, \ldots$), then the sequence $\sigma$ has the swd-property. The aim of this note is to generalize this result and also to give some new classes of sequences with the swd-property.

Our basic tool will be the Weyl criterion for w.d. mod 1 ([5], p. 41). Therefore, the following notation will be convenient.

Let $h \neq 0$ be an arbitrary integer and put $e(t) = e^{2\pi i h t}$, $t \in R$. For any sequence $\omega$ of real numbers and for any integers $u \geqslant 0$, $v \geqslant u$, and $M \geqslant 1$, denote

$$S(\omega;\, u, v) = \sum_{n=u}^{v} e(\omega(n))$$

and

$$\delta(\omega;\, M) = \sup_{N \geqslant M}\sup_{k \geqslant 0} |N^{-1} S(\omega;\, k, k+N-1)|.$$

Clearly $S$ and $\delta$ depend also on $h$, but we have no need to point out this dependence explicitly.

**2. Two criteria for the swd-property.** Let $\alpha$ be a sequence of nonnegative integers. If for all sufficiently large $n$ we have $\alpha(n) = n+K$ with some