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Sums involving the largest prime divisor of an integer

by

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1. Introduction. Let $\gamma(n)$ denote the largest square-free divisor of n and for $n \geqslant 2$, $P(n)$ denote the largest prime divisor of n. In response to a suggestion of P. Erdös, G. J. Rieger (see [5] and also [6], p. 85) proved that as $x \to \infty$

$$\sum_{2 \leqslant n \leqslant x} \frac{1}{n \log \gamma(n)} = \log\log x + C + O\left(\frac{\log\log x}{\log x}\right), \tag{1.1}$$

$$\sum_{2 \leqslant n \leqslant x} \frac{1}{n \log P(n)} = e^{\gamma} \log\log x + O(\log\log\log x) \tag{1.2}$$

where C is a constant and γ denotes the Euler constant. G. J. Rieger [5] also mentions that (1.1) with a weaker remainder term and (1.2) as it stands were also stated independently by P. G. Schmidt. In recent times sums involving the function $P(n)$ have been investigated extensively (see for instance, N. G. De Bruijn [1], J. M. De Koninck and A. Ivić [2] and A. Ivić [4]).

The purpose of this paper is to sharpen the results (1.1) and (1.2) considerably. In fact we prove

THEOREM 1.1. *For each positive integer k, there exist constants $b_0, b_1, \ldots, b_{k-1}$ such that*

$$\sum_{2 \leqslant n \leqslant x} \frac{1}{n \log \gamma(n)} = \log\log x + \sum_{m=0}^{k-1} \frac{b_m}{(\log x)^m} + O\left(\frac{1}{(\log x)^k}\right).$$

THEOREM 1.2. $\displaystyle\sum_{2 \leqslant n \leqslant x} \frac{1}{n \log P(n)} = e^{\gamma} \log\log x + O(1).$

In Section 2, we prove a general result for a wide class of arithmetical functions of which Theorem 1.1 will be a simple consequence. Our attempts to sharpen Theorem 1.2 still further were not successful.

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2. Proof of Theorem 1.1. First we prove a general result. Let r be an integer $\geqslant 2$. A positive integer n is called *r-free* if n is not divisible by the rth power of any prime. A positive integer n is called *r-full* if $p|n$, p a prime implies $p^r|n$. Let Q_r (respy. L_r) denote the set of all r-free (respy. r-full) integers. Let q_r (respy. l_r) denote the characteristic function of Q_r (respy. L_r). Further, let $\zeta(s)$ denote the Riemann zeta function, $\omega(n)$ the number of distinct prime factors of n and ψ_r be the generalized Dedekind ψ-function of order r defined by

$$\psi_r(n) = n\prod_{p|n}\left(1+\frac{1}{p}+\ldots+\frac{1}{p^{r-1}}\right) \tag{2.1}$$

where the product extends over all distinct prime factors of n.

Let S_r denote the class of all multiplicative arithmetical functions f satisfying

$$f(p^m) = p^m \quad \text{for} \quad 1 \leqslant m \leqslant r-1 \tag{2.2}$$

and

$$f(p^m) \geqslant p^{r-1} \quad \text{for} \quad m \geqslant r \tag{2.3}$$

and all primes p.

Let T_r denote the class of all bounded arithmetical functions χ satisfying

$$\chi(mn) = \chi(m) \quad \text{whenever} \quad (m,n) = 1 \text{ and } n \in Q_r. \tag{2.4}$$

Further let λ be a real with $-1/r \leqslant \lambda < 0$ and define $F\colon [\lambda, 0] \to C$ by

$$F(t) = \frac{x^t}{\zeta(r)(t+1)}\sum_{n=1}^{\infty}\frac{\chi(n)\,l_r(n)\big(f(n)n^{-1}\big)^t}{\psi_r(n)} \tag{2.5}$$

for $f \in S_r$ and $\chi \in T_r$ (the convergence of the series on the right will be proved in Lemma 2.3 below).

THEOREM 2.1. *Let $f \in S_r$ and $\chi \in T_r$. Then for each fixed $\varepsilon > 0$ and as $x \to \infty$*

$$\sum_{1 \leqslant n \leqslant x}\chi(n)\big(f(n)\big)^t = F(t)\,x + O_{\varepsilon,r}(x^{t+(1-\lambda)/r+\varepsilon})$$

uniformly for $t \in [\lambda, 0]$.

The proof of Theorem 2.1 is based on the following three lemmas.

LEMMA 2.1 (cf. [7], Lemma 2).

$$\sum_{\substack{m \leqslant x,\\ (m,n)=1}} q_r(m) = \frac{nx}{\zeta(r)\,\psi_r(n)} + O_r\big(\theta(n)\,x^{1/r}\big)$$

uniformly in $x \geqslant 1$ and $n \geqslant 1$. Here $\theta(n) = 2^{\omega(n)}$.

LEMMA 2.2.

$$\sum_{\substack{m \leqslant x\\ (m,n)=1}} q_r(m)\,m^t = \frac{nx^{t+1}}{\zeta(r)\,\psi_r(n)(t+1)} + O_r\big(\theta(n)\,x^{t+1/r}\big)$$

uniformly in $x \geqslant 1$, $n \geqslant 1$ and $t \in [\lambda, 0]$.

Proof. This follows from Lemma 2.1 and the theorem of partial summation.

LEMMA 2.3. *Let $f \in S_r$. Then for each fixed $\varepsilon > 0$ and as $x \to \infty$*

$$\sum_{n \leqslant x} l_r(n)\big(f(n)n^{-1}\big)^t = O_{\varepsilon,r}(x^{(1-\lambda)/r+\varepsilon}), \tag{2.6}$$

$$\sum_{n > x}\frac{l_r(n)\big(f(n)n^{-1}\big)^t}{n} = O_{\varepsilon,r}(x^{-(r-1+\lambda)/r+\varepsilon}) \tag{2.7}$$

and

$$\sum_{n \leqslant x}\frac{l_r(n)\big(f(n)n^{-1}\big)^t}{n^{1/r}} = O_{\varepsilon,r}(x^{-\lambda/r+\varepsilon}) \tag{2.8}$$

uniformly in $t \in [\lambda, 0]$.

Proof. Let $\alpha \geqslant (1-\lambda)/r+\varepsilon$. Since $t \leqslant 0$, we have by (2.3)

$$\begin{aligned} \sum_{m=r}^{\infty}\big(f(p^m)p^{-m}\big)^t p^{-m\alpha} &\leqslant \sum_{m=r}^{\infty}(p^{-m+r-1})^t p^{-m\alpha} \\ &= \frac{p^{(r-1)t}\,p^{-r(t+\alpha)}}{1-p^{-(t+\alpha)}} \leqslant p^{-(r\alpha+\lambda)}(1-2^{-(\alpha+\lambda)})^{-1} \\ &\leqslant p^{-(1+r\varepsilon)}(1-2^{-(\varepsilon+r^{-2})})^{-1} \end{aligned} \tag{2.9}$$

on noting that $r\alpha+\lambda \geqslant 1+r\varepsilon$ and

$$\alpha+t \geqslant \alpha+\lambda \geqslant \frac{1-\lambda}{r}+\lambda+\varepsilon = \frac{1}{r}+\lambda\left(\frac{r-1}{r}\right)+\varepsilon \geqslant \frac{1}{r}-\frac{r-1}{r^2}+\varepsilon = \varepsilon+r^{-2} > 0.$$

Hence the infinite product

$$\prod_p\left\{1+\sum_{m=r}^{\infty}\frac{\big(f(p^m)p^{-m}\big)^t}{p^{m\alpha}}\right\}$$

and consequently the series

$$\sum_{n=1}^{\infty}\frac{l_r(n)\big(f(n)n^{-1}\big)^t}{n^{\alpha}}$$

converge.

Further by (2.9)

$$\sum_{n=1}^{\infty}\frac{l_r(n)\left(f(n)n^{-1}\right)^t}{n^\alpha}=\prod_p\{1+\sum_{m=r}^{\infty}\left(f(p^m)p^{-m}\right)^t p^{-m\alpha}\}\leqslant\prod_p\left\{1+\frac{c_\varepsilon}{p^{1+r\varepsilon}}\right\}.$$

Hence

$$\sum_{n\leqslant x} l_r(n)\left(f(n)n^{-1}\right)^t n^{-\alpha}\ll_{r,\varepsilon}1$$

uniformly for $t\in[\lambda, 0]$ and (2.6), (2.7) and (2.8) follow by the theorem of partial summation.

Now we are in a position to give a proof of Theorem 2.1.

Each positive integer n can be written uniquely as $n=d\delta$, $(d,\delta)=1$ where $d\in Q_r$ and $\delta\in L_r$. Hence by (2.2) and (2.4)

$$\begin{aligned}\sum_{n\leqslant x}\chi(n)\left(f(n)\right)^t &= \sum_{\substack{d\delta\leqslant x\\(d,\delta)=1}}\chi(d\delta)\left(f(d)\right)^t\left(f(\delta)\right)^t q_r(d)\,l_r(\delta)\\
&= \sum_{\substack{d\delta\leqslant x\\(d,\delta)=1}}\chi(d\delta)\,d^t\left(f(\delta)\right)^t q_r(d)\,l_r(\delta)\\
&= \sum_{\delta\leqslant x}\chi(\delta)\,l_r(\delta)\left(f(\delta)\right)^t\sum_{\substack{d<x/\delta\\(d,\delta)=1}}q_r(d)\,d^t\\
&= \sum_{\delta\leqslant x}\chi(\delta)\,l_r(\delta)\left(f(\delta)\right)^t\left\{\frac{\delta(x/\delta)^{t+1}}{\zeta(r)\,\psi_r(\delta)(t+1)}+O_{\lambda,r}\left(\theta(\delta)\left(\frac{x}{\delta}\right)^{1/r+t}\right)\right\}\\
&= \frac{x^{t+1}}{\zeta(r)(t+1)}\sum_{n=1}^{\infty}\frac{\chi(n)\,l_r(n)\left(f(n)n^{-1}\right)^t}{\psi_r(n)}\\
&\quad+O\left(x^{t+1}\sum_{n>x}\frac{l_r(n)\left(f(n)n^{-1}\right)^t}{n}\right)+O_{\lambda,r}\left(x^{t+1/r}\sum_{n\leqslant x}\frac{l_r(n)\left(f(n)n^{-1}\right)^t}{n^{1/r}}\right)\end{aligned}$$

where in the above we used Lemma 2.2 and the fact that χ is bounded and $\psi_r(n)\geqslant n$ for all n. Now Lemma 2.3 completes the proof of Theorem 2.1.

THEOREM 2.2. *Let $f\in S_r$ and $\chi\in T_r$. Then as $x\to\infty$*

$$\sum_{2\leqslant n\leqslant x}\frac{\chi(n)}{\log f(n)}=x\int_{-1/r}^{0}F(t)\,dt+\begin{bmatrix}O(x^{3/4+\varepsilon}) & \text{for} & r=2,\\ O_r(x^{1-1/r}) & \text{for} & r\geqslant 3\end{bmatrix}\tag{2.10}$$

and in particular, for each positive integer k

$$\sum_{2\leqslant n\leqslant x}\frac{\chi(n)}{\log f(n)}=x\sum_{m=1}^{k-1}\frac{(-1)^{m-1}G^{(m-1)}(0)}{(\log x)^m}+O\left(\frac{x}{(\log x)^k}\right)\tag{2.11}$$

where for $t\in[-1/r, 0]$

$$G(t)=x^{-t}F(t).\tag{2.12}$$

Further, for each positive integer k, there exist constants $b_0, b_1, \ldots, b_{k-1}$ such that

$$\sum_{2\leqslant n\leqslant x}\frac{\chi(n)}{n\log f(n)}=G(0)\log\log x+\sum_{m=0}^{k-1}\frac{b_m}{(\log x)^m}+O\left(\frac{x}{(\log x)^k}\right)\tag{2.13}$$

where the order constant depends on parameters other than x.

Proof. The method of proof is similar to the one first used by J. M. De Koninck and J. Galambos [3] to estimate sums of reciprocals of certain additive functions. First, from Theorem 2.1, we get

$$\sum_{2\leqslant n\leqslant x}\chi(n)\int_{-1/r}^{0}\left(f(n)\right)^t dt=x\int_{-1/r}^{0}F(t)\,dt+O\left(x^{(1/r)(1+1/r)+\varepsilon}\int_{-1/r}^{0}x^t\,dt\right),$$

$$\sum_{2\leqslant n\leqslant x}\frac{\chi(n)}{\log f(n)}=x\int_{-1/r}^{0}F(t)\,dt+O\left(x^{(1/r)(1/r+1)+\varepsilon}\right)+\sum_{2\leqslant n\leqslant x}\frac{\chi(n)\left(f(n)\right)^{-1/r}}{\log f(n)}.\tag{2.14}$$

Since χ is bounded and $f(n)\geqslant 2$ for all $n\geqslant 2$, we have

$$\Sigma:=\sum_{2\leqslant n\leqslant x}\frac{\chi(n)\left(f(n)\right)^{-1/r}}{\log f(n)}\ll\sum_{2\leqslant n\leqslant x}\left(f(n)\right)^{-1/r}.$$

Again by Theorem 2.1. (with the function $\chi(n)=1$ for all n, $t=\lambda=-1/r$) and the theorem of partial summation

$$\Sigma\ll\sum_{2\leqslant n\leqslant x}\left(f(n)\right)^{-1/r}\ll x^{1-1/r}+x^{-1/r+(1/r)(1+1/r+\varepsilon)}\ll x^{1-1/r}$$

so that (2.10) follows from (2.14).

Now to prove (2.11), consider

$$\begin{aligned}\int_{-1/r}^{0}F(t)\,dt &= \int_{-1/r}^{0}x^t G(t)\,dt=\frac{x^t G(t)}{\log x}\bigg|_{-1/r}^{0}-\int_{-1/r}^{0}\frac{x^t G'(t)}{\log x}\,dt\\
&= \frac{G(0)}{\log x}-\frac{G(-1/r)}{x^{1/r}\log x}-\int_{-1/r}^{0}\frac{x^t G'(t)}{\log x}\,dt\\
&= \frac{G(0)}{\log x}-\frac{G^{(1)}(0)}{(\log x)^2}+\ldots+\frac{(-1)^{k-1}G^{(k-1)}(0)}{(\log x)^k}\\
&\quad-\frac{1}{x^{1/r}}\left\{\frac{G(-1/r)}{\log x}-\frac{G^{(1)}(-1/r)}{(\log x)^2}+\ldots+\frac{(-1)^{k-1}G^{(k-1)}(-1/r)}{(\log x)^k}\right\}\\
&\quad+(-1)^k\int_{-1/r}^{0}\frac{x^t G^{(k)}(t)}{(\log x)^k}\,dt.\end{aligned}$$

Since

$$\int_{-1/r}^{0} \frac{x^t G^{(k)}(t)}{(\log x)^k} dt \ll (\log x)^{-k},$$

(2.11) follows from (2.10).

Finally (2.13) follows from (2.11) by partial summation. This completes the proof of Theorem 2.2.

Now we deduce Theorem 1.1 from Theorem 2.2. First note that by Euler's infinite product factorization theorem

$$\frac{1}{\zeta(2)} \sum_{n=1}^{\infty} \frac{l_2(n)}{\psi_2(n)} = \frac{1}{\zeta(2)} \prod_p \left\{1 + \frac{1}{p(p+1)} + \frac{1}{p^2(p+1)} + \ldots\right\}$$
$$= \frac{1}{\zeta(2)} \prod_p \left\{1 + \frac{1}{(p+1)(p-1)}\right\} = 1.$$

Hence on taking $r = 2$, $f(n) = \gamma(n)$, the largest square-free divisor of n and $\chi(n) = 1$ for all n in Theorem 2.2, equation (2.13), we obtain Theorem 1.1.

As another application of Theorem 2.2, we have

COROLLARY 2.1. *Let r, s be integers satisfying $2 \leqslant r \leqslant s$ and $\gamma_r(n)$ denote the largest r-free divisor of n. Then, for each positive integer k, there exist constants $d_0, d_1, \ldots, d_{k-1}$ such that as $x \to \infty$*

$$\sum_{2 \leqslant n \leqslant x} \frac{q_s(n)}{n \log \gamma_r(n)} = \frac{1}{\zeta(s)} \log\log x + \sum_{m=0}^{k-1} \frac{d_m}{(\log x)^m} + O\left(\frac{1}{(\log x)^k}\right).$$

Proof. In Theorem 2.2, equation (2.13), we take $f(n) = \gamma_r(n)$, $\chi(n) = q_s(n)$ and note that in this case

$$G(0) = F(0) = \frac{1}{\zeta(r)} \sum_{n=1}^{\infty} \frac{l_r(n) q_s(n)}{\psi_r(n)}$$
$$= \frac{1}{\zeta(r)} \prod_p \left\{1 + \sum_{m=r}^{s-1} \frac{1}{\psi_r(p^m)}\right\}$$
$$= \frac{1}{\zeta(r)} \prod_p \left\{1 + \left(\frac{1-p^{-1}}{1-p^{-r}}\right)\left(\frac{p^{-r}-p^{-s}}{1-p^{-1}}\right)\right\}$$
$$= \frac{1}{\zeta(r)} \prod_p \left\{\frac{1-p^{-s}}{1-p^{-r}}\right\} = \frac{1}{\zeta(s)}.$$

This completes the proof of Corollary 2.1.

3. Proof of Theorem 1.2. For $x \geqslant 1$ and $y \geqslant 2$, let $\psi(x, y)$ denote the number of positive integers $\leqslant x$ all of whose prime factors are $\leqslant y$. Then it is known due to De Bruijn (cf. [1], (1.9)) that there exist positive constants A and B such that for all x and y

$$\psi(x, y) < Ax \exp\{-(B \log x/\log y)\}. \tag{3.1}$$

Now clearly we have

$$\sum_{2 \leqslant n \leqslant x} \frac{1}{n \log P(n)} = \sum_{p \leqslant x} \frac{1}{\log p} \sum_{\substack{n \leqslant x \\ P(n) = p}} \frac{1}{n} = \sum_{p \leqslant x} \frac{1}{\log p} \sum_{\substack{pm \leqslant x \\ P(m) \leqslant p}} \frac{1}{pm} \tag{3.2}$$
$$= \sum_{p \leqslant x} \frac{1}{p \log p} \sum_{\substack{m \leqslant x/p \\ P(m) \leqslant p}} \frac{1}{m}$$
$$= \sum_{p \leqslant x} \frac{1}{p \log p} \sum_{\substack{m \\ P(m) \leqslant p}} \frac{1}{m} - \sum_{p \leqslant x} \frac{1}{p \log p} \sum_{\substack{m > x/p \\ P(m) \leqslant p}} \frac{1}{m}$$
$$= \Sigma_1 - \Sigma_2,$$

say. By Mertens' theorem in the distribution of primes, we have

$$\sum_{\substack{m \\ P(m) \leqslant p}} \frac{1}{m} = \prod_{\substack{q \leqslant p \\ q \text{ prime}}} \left(1 - \frac{1}{q}\right)^{-1} = \left\{\frac{e^{-\gamma}}{\log p} + O\left(\frac{1}{\log^2 p}\right)\right\}^{-1} = e^{\gamma} \log p + O(1) \tag{3.3}$$

so that

$$\Sigma_1 = \sum_{p \leqslant x} \frac{1}{p \log p} (e^{\gamma} \log p + O(1)) = e^{\gamma} \log\log x + O(1). \tag{3.4}$$

The estimation of Σ_2 is done via partial summation and (3.1). In fact

$$\Sigma_2 = \sum_{2 \leqslant p \leqslant x} \frac{1}{p \log p} \left(\frac{-\psi(t, p)}{t}\Big|_{x/p}^{\infty} + \int_{x/p}^{\infty} \frac{\psi(t, p)}{t^2} dt\right) \tag{3.5}$$
$$\ll \sum_{2 \leqslant p \leqslant x} \frac{1}{p \log p} \left(1 + \int_{x/p}^{\infty} \frac{e^{-(B \log t/\log p)}}{t} dt\right)$$
$$\ll 1 + \sum_{p \leqslant x} \frac{1}{p \log p} \int_{x/p}^{\infty} \frac{dt}{t^{1+(B/\log p)}} \ll 1 + \sum_{p \leqslant x} \frac{1}{p} \left(\frac{x}{p}\right)^{-B/\log p}$$
$$\ll 1 + \sum_{p \leqslant x} \frac{1}{p e^{B \log x/\log p}} \ll 1 + \sum_{p \leqslant x} \frac{1}{p} \cdot \frac{\log p}{\log x} \ll 1.$$

Now the theorem follows from (3.2), (3.4) and (3.5).

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Unités de certains sous-anneaux des corps de fonctions algébriques

par

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Introduction. Cette étude est le résultat d'un travail qui s'est développé au sein de l'équipe d'Algèbre et de Théorie des Nombres de Caen à la suite d'une visite du Professeur A. Schinzel en 1981.

Au départ, ce travail était centré sur l'algorithme des fractions continues et les méthodes utilisées s'inspiraient de celles de Schinzel [12] et d'Artin [1].

Une première étude [7] résumait les résultats obtenus, résultats plus généraux pour la partie géométrique, que pour la partie algorithmique. La partie géometrique traitait le cas des extensions $E = k(X, \sqrt[p]{D(X)})$ avec p quelconque > 0 et la partie algorithmique, qui utilisait les fractions continues, était limitée au cas $p = 2$.

Dans une seconde étape [11], Roger Paysant-Le Roux généralisait les résultats de [7] à l'aide d'une notion de meilleure approximation semblable à celle utilisée par E. Dubois et G. Rhin [6] dans le cas des corps de nombres. L'algorithme des meilleures approximations était utilisé pour remplacer celui des fractions continues. Il peut aussi être comparé aux algorithmes de J. H. Davenport [4].

Puis D. L. Mc Quillan, lors d'une visite à Caen, donnait une forme plus générale aux résultats qu'Yves Hellegouarch avait obtenus en utilisant le langage de la théorie des corps de fonctions algébriques [5].

Finalement, une étude axiomatique des notions d'approximation utilisées nous conduisait à réviser notre vocabulaire et à introduire différentes notions: commas, points extrémaux, arêtes qui sont définies de manière simple aussi bien dans les corps de nombres que dans les corps de fonctions pour lesquels on a choisi un élément X non constant [8].

Tel qu'il est composé ici, ce travail est essentiellement algébrique et se réfère aux ouvrages classiques de Deuring [5] et d'Artin [2]. Nous avons donc utilisé les notations de Deuring, qui, c'est regrettable, ne sont pas d'un usage universel. Que le lecteur habitué à d'autres notations veuille bien nous en excuser!