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The greatest prime factor of the integers in an interval

by

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1. Introduction. Let $P(x)$ denote the greatest prime factor of

$$\prod_{x<n\leqslant x+x^{1/2}} n.$$

It is conjectured that $P(x) > x$ for large x. As an approximation to this conjecture it was shown by Ramachandra [17] that $P(x) > x^{15/26}$, and later [18] that $P(x) > x^{5/8}$, for large x. The same problem was considered by Jutila [15], but the strongest result in the literature is

$$P(x) > x^{0.66}$$

(Graham [7]). The object of the present paper is to show that, for large x,

$$P(x) > x^{7/10}. \tag{1}$$

Our proof is far longer than that of Graham.

Let $\varepsilon > 0$. The greatest prime factor of the integers in the slightly longer interval

$$(x, x+x^{1/2+\varepsilon}]$$

was shown by Jutila [14] to be larger than $x^{2/3-\varepsilon}$. His result was improved in the series of papers [1], [2] by Balog and [3], [4] (Balog, Harman and Pintz) culminating in the lower bound $x^{0.82}$. The stronger result is due to the availability of Dirichlet polynomial methods, which do not apply to $P(x)$.

The main tools we use to improve Graham's result are

(i) a more elaborate treatment of 'Type I' sums

$$\sum_h \sum_m a_m \sum_n e\left(\frac{h\zeta}{mn}\right)$$

than Graham's [6], [7], though the same basic principles are used;

(ii) estimates for 'Type II' sums

$$\sum_h \sum_m a_m \sum_n b_n e\left(\frac{h\zeta}{mn}\right)$$

adapted from work of Iwaniec and Laborde [13] on P_2's in short intervals, and Heath-Brown's work [10] on prime numbers of the form $[n^c]$;

(iii) Heath-Brown's decomposition [9] of sums of the form $\sum_n \Lambda(n) f(n)$;

(iv) Iwaniec's form of the error term in the linear sieve [12]; and

(v) a small but significant 'deduction' owing to almost primes counted by the upper bound sieve, as in [4], [11] for example.

We begin like Graham [7]. We shall write

$$y = x^{1/2}, \quad L = \log x$$

and

$$N(d) = \sum_{\substack{x < n \leqslant x+y \\ d \mid n}} 1.$$

Then

$$yL + O(y) = \sum_{x < n \leqslant x+y} \log n = \sum_{x < n \leqslant x+y} \sum_{d \mid n} \Lambda(d) = \sum_d \Lambda(d) N(d), \tag{2}$$

where Λ is von Mangoldt's function.

Let ε be sufficiently small. Constants implied by '$\ll$' and 'O' notations will depend at most on ε. With

$$\varphi = 13/22 - \varepsilon,$$

we have

$$\sum_d \Lambda(d) N(d) = \sum_{d \leqslant x^\varphi} \Lambda(d) N(d) + \sum_{\substack{p^j > x^\varphi \\ j \geqslant 2}} N(p^j) \log p + \sum_{x^\varphi < p \leqslant P(x)} N(p) \log p. \tag{3}$$

Here and later, $p, p_1, p_2, \ldots$ denote prime variables.

Since $N(d) \leqslant 1$ for $d > y$,

$$\sum_{\substack{p^j \leqslant x^\varphi \\ j \geqslant 2}} N(p^j) \log p \leqslant \sum_{\substack{p^j \leqslant x+y \\ j \geqslant 2}} \log p \ll y. \tag{4}$$

In Section 3 we shall prove the following asymptotic formula:

PROPOSITION 1. *We have*

$$\sum_{d \leqslant x^\varphi} \Lambda(d) N(d) = y \sum_{d \leqslant x^\varphi} \Lambda(d) d^{-1} + O(y) = \varphi y L + O(y). \tag{5}$$

In Section 6 we shall prove the following upper bound:

PROPOSITION 2. *We have*

$$\sum_{x^\varphi < p \leqslant x^{7/10}} N(p) \log p < (1 - \varphi - \varepsilon) yL \tag{6}$$

for large x.

It is a simple matter to deduce (1) from these two propositions. In fact, combining (2)–(6), we see that

$$\sum_{x^{7/10} < p \leqslant P(x)} N(p) \log p > yL(1-\varphi) + O(y) - (1 - \varphi - \varepsilon) yL > \tfrac{1}{2} \varepsilon y L. \tag{7}$$

In particular, the sum on the left in (7) is non-empty and (1) follows.

Throughout the remainder of the paper we use the following notations:

$$e(\theta) = e^{2\pi i \theta},$$

$$\psi(\theta) = \theta - [\theta] - 1/2,$$

$$|E| = \text{Lebesgue measure of the Borel set } E.$$

The dependence of constants, such as $C_1(\gamma)$, will be shown explicitly, so that C_2 denotes a positive numerical constant. It is convenient to put $\eta = \varepsilon^2$. We assume that x is larger than a suitable function of ε.

2. Type I sums. We first state some lemmas on transformations of exponential sums. We then derive our two basic results on Type I sums, Lemmas 4 and 5.

LEMMA 1. *Let I be a subinterval of $(Y, 2Y]$ and let Z be a positive integer. Then for any complex numbers w_n $(n \in I)$, we have*

$$\Big|\sum_{n \in I} w_n\Big|^2 \leqslant (1 + YZ^{-1}) \sum_{|k| \leqslant Z} (1 - |k| Z^{-1}) \sum_{n, n+k \in I} w_n \bar{w}_{n+k}.$$

Proof. See [10], Lemma 5.

LEMMA 2. *Let $0 \leqslant Z < Z_1 \leqslant 2Z$. Let $f(z/Z)$ be holomorphic on an open convex set $\mathscr{R}$ containing the real line segment $[1, Z_1/Z]$ and suppose $|f''(z)| \leqslant M^{-1}$ on $Z\mathscr{R}$. Let $f(x)$ be real when $x \in Z\mathscr{R}$ is real and let $f''(x) \leqslant -cM^{-1}$ with $c > 0$. Let $f'(Z_1) = \alpha$, $f'(Z) = \beta$ and define y_ν for each integer ν in the range $\alpha < \nu \leqslant \beta$ by $f'(y_\nu) = \nu$. Then*

$$\Big|\sum_{Z < n \leqslant Z_1} e(f(n)) - e(-1/8) \sum_{\alpha < \nu \leqslant \beta} |f''(y_\nu)|^{-1/2} e\big(f(y_\nu) - \nu y_\nu\big)\Big| < C\,(M^{1/2} + \log(2 + M^{-1} Z)),$$

where C depends only on c and $\mathscr{R}$.

Proof. See [10], Lemma 6.

We now consider Srinivasan's analogue [19] of Lemma 2 for real functions f on $\mathscr{B}$, where $\mathscr{B} = [M, 2M] \times [N, 2N]$ is a rectangle. This is the simplest result from the literature that serves our present purpose. In Lemma 3 we suppose that f has continuous partial derivatives of all orders in $\mathscr{B}$ and we write $D_{i,j} f$ for the common value of

$$\frac{\partial^{i+j} f}{\partial x_1^i \, \partial x_2^j}, \quad \frac{\partial^{i+j} f}{\partial x_2^j \, \partial x_1^i}, \quad \ldots$$

Let $\mathscr{D}$ be a subset of $\mathscr{B}$ and suppose that lines parallel to either coordinate axis meet $\mathscr{D}$ in $O(1)$ segments, and that the same is true for $\mathscr{D}\cap\mathscr{E}$ where $\mathscr{E}$ is a subset of $\mathscr{B}$ defined by any inequality

$$D_{r,s}f \geqslant c \quad \text{or} \quad D_{r,s}f \leqslant c \quad \text{with } r+s=1.$$

We suppose that the transformation

$$(y_1, y_2) = (D_1 f, D_2 f)$$

is one-to-one on $\mathscr{D}$, with image set Δ. We further suppose that lines parallel to either coordinate axis meet the curve

$$D_{j,k} f = 0$$

in $O(1)$ points whenever $j+k=2$. (These conditions are easily seen to be fulfilled when we apply Srinivasan's result in the proof of Lemma 5.) We write

$$H(f) = H(f;x) = D_{2,0}f\ D_{0,2}f-(D_{1,1}f)^2.$$

LEMMA 3. *With the above notation, suppose that throughout $\mathscr{D}$ we have, for some $B>0$,*

$$|H(f)| \gg B^2 M^{-2} N^{-2},$$

$$D_{r,s} f \ll BM^{-r}N^{-s} \quad \textit{for} \quad 2\leqslant r+s\leqslant 3,$$

$$BM^{-2} \ll |D_{2,0}f| \ll BM^{-2}, \quad BN^{-2} \ll |D_{0,2}f| \ll BN^{-2}.$$

Let $\boldsymbol{x}(\boldsymbol{v}) = \boldsymbol{x}(v_1, v_2)$ denote the unique solution in $\mathscr{D}$ of

$$\big(D_1 f(\boldsymbol{x}), D_2 f(\boldsymbol{x})\big) = (v_1, v_2)$$

for $\boldsymbol{v}\in\Delta$, and write $H_*(\boldsymbol{v}) = |H(f;\boldsymbol{x}(\boldsymbol{v}))|$. Let w denote the number of changes of sign in the sequence

$$1,\ D_{2,0}f,\ H(f).$$

Then

$$\sum_{(m,n)\in\mathscr{D}} e\big(f(m,n)\big) - i^{1-w}\sum_{\boldsymbol{v}\in\Delta} H_*(\boldsymbol{v})^{-1/2} e\big(f(\boldsymbol{x}(\boldsymbol{v}))-v_1x_1(\boldsymbol{v})-v_2x_2(\boldsymbol{v})\big)$$
$$\ll B^{11/12} + (B^{1/2}+MB^{-1/2})\big(NB^{-1/2}+\log(BN^{-1}+2)\big)$$
$$+(B^{1/2}+NB^{-1/2})\big(MB^{-1/2}+\log(BM^{-1}+2)\big).$$

Proof. This is a special case of Theorem 2 of [19] (take $r=B^{1/2}$, $R=B^{-1/2}$).

In the next lemma and later on, we use *exponent pairs* $(\varkappa,\lambda)$; for a full discussion see Phillips [16].

LEMMA 4. *Let $(\varkappa,\lambda)$ be an exponent pair. Let*

$$M\geqslant 1,\quad N\geqslant 1,\quad M_1\leqslant 2M,\quad N_1\leqslant 2N,\quad x\leqslant\zeta\leqslant 2x,$$

(8) $$x^{1/2}\leqslant v\leqslant x^{7/10},\quad v\leqslant v_1\leqslant ev,$$

(9) $$1\leqslant H\leqslant vx^{-1/2+4\eta},\quad H\leqslant H_1\leqslant 2H.$$

Suppose that

(10) $$M\leqslant x^{-26\eta}\min\big(v^{1/2}x^{26\eta},\ xv^{-1},\ (x^{(1+\varkappa)/2}v^{-\varkappa})^{1/(1+\lambda)}\big).$$

Let

$$S=\sum_{H<h\leqslant H_1}\ \sum_{M<m\leqslant M_1}\Big|\sum_{\substack{N<n\leqslant N_1\\ v<mn\leqslant v_1}} e\Big(\frac{h\zeta}{mn}\Big)\Big|.$$

Then

$$S\ll vx^{-2\eta}.$$

Proof. In this proof and later on we use, without comment, the inequality $v\ll MN\ll v$ for nonempty S. Suppose first that

(11) $$HM^{2/3}\leqslant vx^{-1/3-2\eta}.$$

Then we use the estimate

$$\sum_{\max(N,vm^{-1})<n\leqslant\min(N_1,v_1m^{-1})} e\Big(\frac{h\zeta}{mn}\Big) \ll (HxM^{-1})^{1/2}N^{-1/2}+(HxM^{-1})^{-1/2}N^{3/2},$$

which follows at once from [21], Theorem 5.9. Thus

$$S\ll HM\{(Hx)^{1/2}v^{-1/2}+(Hx)^{-1/2}v^{1/2}N\}$$
$$\ll (M^{2/3}H)^{3/2}x^{1/2}v^{-1/2}+H^{1/2}x^{-1/2}v^{3/2}$$
$$\ll vx^{-2\eta}+v^2x^{-3/4+2\eta}\ll vx^{-2\eta}$$

in view of (8), (9) and (11).

Now suppose that (11) fails. Because of (9), we must have

(12) $$M>x^{1/4-9\eta}.$$

Then, from Lemma 2, with $y_\nu=(h\zeta)^{1/2}(m\nu)^{-1/2}$, we have

$$\sum_{\substack{N<n\leqslant N_1\\ v<mn\leqslant v_1}} e\Big(\frac{-h\zeta}{mn}\Big)$$
$$= e(-1/8)\sum_{a_m<\nu\leqslant b_m}\{2(h\zeta)^{-1/2}m^{1/2}\nu^{3/2}\}^{-1/2}e(-2h^{1/2}\zeta^{1/2}m^{-1/2}\nu^{1/2})$$
$$+O\big((hx)^{-1/2}M^{1/2}N^{3/2}+L\big).$$

Here

$$HxM^{-1}N^{-2} \ll a_m,\ b_m \ll HxM^{-1}N^{-2}.$$

Thus an easy partial summation argument yields (with $c_m \in (a_m, b_m]$)

(13)
$$S \ll (Hx)^{-1/2} M^{1/2} N^{3/2} \sum_{H<h\leqslant H_1} \sum_{M<m\leqslant M_1} \Big|\sum_{a_m<\nu\leqslant c_m} e(2h^{1/2}\zeta^{1/2} m^{-1/2}\nu^{1/2})\Big| + H^{1/2}x^{-1/2}(MN)^{3/2} + HML.$$

Moreover,

(14) $$H^{1/2}x^{-1/2}(MN)^{3/2} + HML \ll v^2 x^{-3/4+2\eta} + v^{3/2} x^{-1/2+4\eta} \ll vx^{-2\eta}$$

from (8), (9) and (10).

Combining (13) and (14), and Cauchy's inequality together with Lemma 1, we find that

$$S^2 \ll v^2 x^{-4\eta} + x^{-1} M^2 N^3 \sum_{H<h\leqslant H_1} \sum_{M<m\leqslant M_1} (1 + HxM^{-1}N^{-2}Z^{-1})$$
$$\times \sum_{|k|\leqslant Z} \left(1 - \frac{|k|}{Z}\right) \sum_{a_m<\nu,\nu+k\leqslant c_m} e\left(-2h^{1/2}\zeta^{1/2}m^{-1/2}\{(\nu+k)^{1/2} - \nu^{1/2}\}\right),$$

where the natural number Z is at our disposal. We put

(15) $$Z = [\min(M^2 x^{-1/2+18\eta},\ HxM^{-1}N^{-2})].$$

Note that

$$HxM^{-1}N^{-2}$$
$$\geqslant (vx^{-1/3-2\eta}M^{-2/3})xM^{-1}N^{-2} \gg vx^{2/3-2\eta}v^{-2}M^{1/3} > x^{3/4-5\eta}v^{-1} > x^{\eta}$$

using (12) and the falsity of (11). Thus $Z \geqslant 1$, and for some h, ν, $H \leqslant h \leqslant H_1$, $HxM^{-1}N^{-2} \ll \nu \ll HxM^{-1}N^{-2}$, we have

(16) $$S^2 \ll v^2 x^{-4\eta} + HMNZ^{-1}H(HxM^{-1}N^{-2}) \times \sum_{|k|\leqslant Z} \Big| \sum_{M<m\leqslant M_1} e\left(-2h^{1/2}\zeta^{1/2}m^{-1/2}\{(\nu+k)^{1/2} - \nu^{1/2}\}\right)\Big|$$
$$\ll v^2 x^{-4\eta} + H^3 xMN^{-1}Z^{-1} + H^3 xN^{-1}Z^{-1} \sum_{1\leqslant |k|\leqslant Z} |S_k|.$$

Here the sum

$$S_k = \sum_{M<m\leqslant M_1} e\left(-2h^{1/2}\zeta^{1/2}m^{-1/2}\{(\nu+k)^{1/2} - \nu^{1/2}\}\right)$$

may be estimated by the method of exponent pairs, since

$$D \ll \frac{d}{dm}\left(-2h^{1/2}\zeta^{1/2}m^{-1/2}\{(\nu+k)^{1/2} - \nu^{1/2}\}\right) \ll D,$$

where

$$D = H^{1/2}x^{1/2}M^{-3/2}|k|(HxM^{-1}N^{-2})^{-1/2} = |k|\,NM^{-1} \gg 1.$$

(For the last inequality we simply observe that

$$M < v^{1/2} \quad \text{so} \quad N \gg v^{1/2}.)$$

Thus

(17) $$S_k \ll D^{\varkappa} M^{\lambda} \ll |k|^{\varkappa} N^{\varkappa} M^{\lambda-\varkappa}.$$

Combining (15), (16), (17) and (9) we find that

$$S^2 \ll v^2 x^{-4\eta} + H^3 xMN^{-1}(M^{-2}x^{1/2-18\eta} + H^{-1}x^{-1}MN^2) + H^3 xN^{\varkappa-1}M^{\lambda-\varkappa}(M^2 x^{-1/2+18\eta})^{\varkappa}$$
$$\ll v^2 x^{-4\eta} + (vx^{-1/2+4\eta})^3 v^{-1} x^{3/2-18\eta} + (vx^{-1/2+4\eta})^2 vM + (vx^{-1/2+4\eta})^3 x^{1-\varkappa/2+9\eta} v^{\varkappa-1} M^{\lambda+1}.$$

On inserting the bounds (8) and (10) we get

$$S^2 \ll v^2 x^{-4\eta} + v^3 x^{-1+8\eta} x^{-26\eta}(xv^{-1}) + v^{2+\varkappa} x^{-(1+\varkappa)/2+22\eta}(x^{(1+\varkappa)/2-26\eta} v^{-\varkappa})$$
$$\ll v^2 x^{-4\eta}.$$

This completes the proof of Lemma 4.

Lemma 5. *Make all the hypotheses of Lemma* 4, *except that* (10) *is replaced by*

(18) $$x^{1/2} \leqslant v \leqslant x^{\varphi}, \qquad N \gg x^{1/4-6\eta}.$$

Then

$$S \ll vx^{-2\eta}.$$

Proof. In the first place, we may assume that

(19) $$N \ll v^{1/2},$$

otherwise the result follows from Lemma 4 with $(\varkappa, \lambda) = (1/2, 1/2)$.

It suffices to show for a fixed h, $H < h \leqslant H_1$, that

(20) $$T = \sum_{M<m\leqslant M_1} \Bigg| \sum_{\substack{N<n\leqslant N_1 \\ v<mn\leqslant v_1}} e\left(\frac{h\zeta}{mn}\right) \Bigg| \ll x^{1/2-6\eta}.$$

We begin with an application of Lemma 1 to the inner sum, in conjunction with Cauchy's inequality:

(21)
$$T^2 \leqslant M(1+NZ^{-1}) \sum_{M<m\leqslant M_1} \sum_{|k|\leqslant Z} \left(1 - \frac{|k|}{Z}\right) \sum_{n;(24)} e\left(h\zeta m^{-1}\{(n+k)^{-1} - n^{-1}\}\right).$$

Here the natural number Z is at our disposal; we choose

(22) $$Z = [v^2 x^{-1+12\eta}]$$

so that

(23) $$1 \leqslant Z \ll Nx^{-\eta}$$

as an easy consequence of (18). The sum $\sum_{n;(24)}$ runs over integers n satisfying

(24) $$\max(N, vm^{-1}) < n,\ n+k \leqslant \min(N_1, v_1 m^{-1}).$$

We apply Lemma 3 to the sum

$$T_k = \sum_{M < m \leqslant M_1} \sum_{n;(24)} e\left(h\zeta m^{-1}\{(n+k)^{-1} - n^{-1}\}\right).$$

It is not difficult to verify that (writing $\mathscr{D}$ for the domain of summation of T_k) the function

$$f(m, n) = f_k(m, n) = h\zeta m^{-1}\{(n+k)^{-1} - n^{-1}\}$$

satisfies the conditions of Lemma 3 with

$$B = B_k = HxM^{-1}|k|\,N^{-2} \gg xv^{-1}N^{-1} \gg xv^{-3/2} > x^{\eta}.$$

Moreover, it is easy to see that

$$B_k^{11/12} + (B_k^{1/2} + MB_k^{-1/2})\left(NB_k^{-1/2} + \log(B_k N^{-1}+2)\right)$$
$$+ (B_k^{1/2} + NB_k^{-1/2})\left(MB_k^{-1/2} + \log(B_k M^{-1}+2)\right) \ll B_k^{11/12} + vB_k^{-1} + ML.$$

We write

$$U_k = \sum_{v \in \varDelta} H_*(v)^{-1/2} e\left(\varphi(v)\right)$$

where $\varphi(v) = f\left(x(v)\right) - v_1 x_1(v) - v_2 x_2(v)$. Noting that $vML \ll v^2 Z^{-1}$ from (23), we can combine (21) with Lemma 3 to get

(25) $$T^2 \ll MNZ^{-1}\Big(MN + \sum_{1 \leqslant |k| \leqslant Z} |T_k|\Big)$$
$$\ll vZ^{-1} \sum_{1 \leqslant |k| \leqslant Z} |U_k| + v^2 Z^{-1} + vZ^{-1} \sum_{1 \leqslant |k| \leqslant Z} (B_k^{11/12} + vB_k^{-1})$$
$$\ll vZ^{-1} \sum_{1 \leqslant |k| \leqslant Z} |U_k| + x^{1-12\eta} + v^{17/6} x^{-11/16+\varepsilon} + v^{3/2} x^{\varepsilon}$$
$$\ll vZ^{-1} \sum_{1 \leqslant |k| \leqslant Z} |U_k| + x^{1-12\eta}$$

on applying (9), (22), (18) and (19).

We now consider the sum U_k for a fixed k, $1 \leqslant |k| \leqslant Z$. For some integer v_1, $v_1 \ll B_k M^{-1}$, we have

(26) $$U_k \ll (B_k M^{-1} + 1) \sum_{\substack{v_2 \\ (v_1, v_2) \in \varDelta}} H_*(v_1, v_2)^{-1/2} e\left(\varphi(v_1, v_2)\right).$$

The last sum plainly takes the form

(27) $$\sum_{a < v_2 \leqslant b} H_*(v_1, v_2)^{-1/2} e\left(\varphi(v_1, v_2)\right)$$

where $B_k N^{-1} \ll a \leqslant b \ll B_k N^{-1}$.

Now $H_*(v_1, v_2)$ is monotonic as a function of v_2 $(a < v_2 \leqslant b)$. By partial summation, the sum in (27) is

$$\ll MNB_k^{-1} \Big| \sum_{a < v_2 \leqslant c} e\left(\varphi(v_1, v_2)\right)\Big|$$

for some c, $a < c \leqslant b$.

Just as in the proof of Proposition 2 of [20], we find that

$$\left|D_{r,s}\left(\varphi(y_1, y_2) - Ay_1^{1/4} y_2^{1/2}\right)\right| < \eta\, |Ay_1^{(1/4)-r} y_2^{(1/2)-s}|$$

on $\varDelta$, for $r, s = O(1)$. Here $A = A(h, k, \zeta)$,

$$(|hk|\, x)^{1/4} \ll |A| \ll (|hk|\, x)^{1/4}.$$

In particular, $N \ll D_{0,1}\varphi \ll N$ on $\varDelta$. The theory of exponent pairs with $(\varkappa, \lambda) = (1/9, 13/18)$ (that is, $ABA^2B(0, 1)$ in the usual notation) gives

$$\Big|\sum_{a < v_2 \leqslant c} e\left(\varphi(v_1, v_2)\right)\Big| \ll N^{\varkappa}(B_k N^{-1})^{\lambda} + 1.$$

In conjunction with (26) this yields

$$U_k \ll (B_k M^{-1} + 1)\, vB_k^{-1}\{N^{\varkappa}(B_k N^{-1})^{\lambda} + 1\}$$
$$\ll H^{\lambda} x^{\lambda} M^{-\lambda-1} N^{\varkappa-3\lambda} |k|^{\lambda} v + H^{\lambda-1} x^{\lambda-1} M^{1-\lambda} N^{-3\lambda+\varkappa+2} |k|^{\lambda-1} v$$
$$+ N + H^{-1} x^{-1} v^2 N |k|^{-1}.$$

Thus, applying (9),

$$vZ^{-1} \sum_{1 \leqslant |k| \leqslant Z} |U_k| \ll (vx^{1/2})^{\lambda} v^{-\lambda-1} N^{\varkappa-2\lambda+1} (v^2 x^{-1})^{\lambda} v^2 x^{16\eta}$$
$$+ x^{\lambda-1+16\eta} v^{1-\lambda} N^{\varkappa-2\lambda+1} (v^2 x^{-1})^{\lambda-1} v^2 + vN$$
$$\ll v^{2\lambda+1} x^{-\lambda/2+16\eta} N^{-(2\lambda-1-\varkappa)} (1 + v^{-\lambda} x^{\lambda/2}) + vN.$$

Taking into account (8) and (18), (19) we find that

$$vZ^{-1} \sum_{1 \leqslant |k| \leqslant Z} |U_k| \ll v^{2\lambda+1} x^{-\lambda+(1+\varkappa)/4+22\eta} + v^{3/2} \ll v^{22/9} x^{-4/9+22\eta} \ll x^{1-12\eta}.$$

In tandem with (25) this establishes (20), and Lemma 5 is proved.

3. Proof of Proposition 1.

LEMMA 6. *Let $f(n)$ be a complex valued function for $n \geqslant 1$. Let $v \geqslant 2$, $v \leqslant v_1 \leqslant ev$. Let a, b, z be positive numbers satisfying*

$$2 \leqslant a \leqslant b \leqslant z, \quad a^2 \leqslant z, \quad 128az^2 \leqslant v, \quad 2^{18} v_1 \leqslant b^3.$$

The sum

$$\sum_{v<n\leqslant v_1} \Lambda(n) f(n)$$

may be decomposed into $\ll (\log v)^6$ sums S, each of which is either of the form (I) or (II), where:

(I) $$S = \sum_{\substack{M<m\leqslant M_1 \\ v<mn\leqslant v_1}} a_m \sum_{N<n\leqslant N_1} f(mn)$$

or

$$S = \sum_{\substack{M<m\leqslant M_1 \\ v<mn\leqslant v_1}} a_m \sum_{N<n\leqslant N_1} (\log n) f(mn)$$

with $N \geqslant z$, $M_1 \leqslant 2M$, $N_1 \leqslant 2N$, $a_m \ll v^\eta$;

(II) $$S = \sum_{\substack{M<m\leqslant M_1 \\ v<mn\leqslant v_1}} a_m \sum_{N<n\leqslant N_1} b_n f(mn),$$

with $a \leqslant N \leqslant b$, $M_1 \leqslant 2M$, $N_1 \leqslant 2N$, $a_m \ll v^\eta$, $b_n \ll v^\eta$.

Proof. See Section 1 of Heath-Brown [9].

LEMMA 7. *Let $v \geqslant 2$, $0 \leqslant b(n) \leqslant 1$ $(v < n \leqslant ev)$. Let $J \geqslant 1$. Then for some ζ_1, ζ_2 with $x \leqslant \zeta_1, \zeta_2 \leqslant x+y$, and for some v_1, $v \leqslant v_1 \leqslant ev$, we have*

$$\sum_{v<n\leqslant ev} b(n)\left\{\psi\left(\frac{x+y}{n}\right)-\psi\left(\frac{x}{n}\right)\right\}$$
$$\ll \frac{y}{v}\left|\sum_{j=1}^{J}\sum_{v<n\leqslant v_1} b(n) e\left(\frac{\zeta_1 j}{n}\right)\right| + \sum_{j=0}^{\infty} \min\left(\frac{\log 2J}{J}, \frac{J}{j^2}\right)\left|\sum_{v<n\leqslant ev} e\left(\frac{\zeta_2 j}{n}\right)\right|.$$

Proof. Following Heath-Brown [10], Section 2, we use the estimate

$$\psi(\theta) = \sum_{0<|j|\leqslant J} \frac{e(\theta j)}{2\pi i j} + O(g(\theta, J)),$$

where

$$g(\theta, J) = \min\left(1, \frac{1}{J\|\theta\|}\right) = \sum_{j=-\infty}^{\infty} a_j e(\theta j).$$

Here $\|\theta\| = \min_{n\in Z} |\theta - n|$ and

(28) $$a_j \ll \min\left(\frac{\log 2J}{J}, \frac{J}{j^2}\right).$$

Thus

(29) $$\sum_{v<n\leqslant ev} b(n)\left\{\psi\left(\frac{x+y}{n}\right)-\psi\left(\frac{x}{n}\right)\right\}$$
$$= -\sum_{v<n\leqslant ev} b(n) \sum_{0<|j|\leqslant J} \frac{e((x+y)j/n) - e(xj/n)}{2\pi i j}$$
$$+ O\left(\sum_{v<n\leqslant ev} g\left(\frac{x+y}{n}, J\right)\right) + O\left(\sum_{v<n\leqslant ev} g\left(\frac{x}{n}, J\right)\right).$$

We now observe that

$$e\left(\frac{(x+y)j}{n}\right) - e\left(\frac{xj}{n}\right) = \frac{2\pi i j}{n}\int_x^{x+y} e\left(\frac{\theta j}{n}\right) d\theta,$$

so that

(30) $$\sum_{v<n\leqslant ev} b(n) \sum_{0<|j|\leqslant J} \frac{e((x+y)j/n) - e(xj/n)}{2\pi i j}$$
$$= \int_x^{x+y} \sum_{0<|j|\leqslant J}\sum_{v<n\leqslant ev} \frac{b(n)}{n} e\left(\frac{\theta j}{n}\right) d\theta$$
$$= \int_x^{x+y}\int_v^{\infty} t^{-2} \sum_{0<|j|\leqslant J}\sum_{v<n\leqslant \min(t, ev)} b(n) e\left(\frac{\theta j}{n}\right) d\theta$$
$$\ll y v^{-1} \left|\sum_{0<|j|\leqslant J}\sum_{v<n\leqslant v_1} b(n) e\left(\frac{\zeta_1 j}{n}\right)\right|$$

for some v_1, $v \leqslant v_1 \leqslant ev$ and ζ_1, $x \leqslant \zeta_1 \leqslant x+y$.

Moreover, with $\zeta = x$ or $x+y$,

(31) $$\sum_{v<n\leqslant ev} g\left(\frac{\zeta}{n}, J\right) = \sum_{j=-\infty}^{\infty} a_j \sum_{v<n\leqslant ev} e\left(\frac{\zeta j}{n}\right)$$
$$\ll \sum_{j=-\infty}^{\infty} \min\left(\frac{\log 2J}{J}, \frac{J}{j^2}\right)\left|\sum_{v<n\leqslant ev} e\left(\frac{\zeta j}{n}\right)\right|,$$

from (28). The lemma follows easily on combining (29), (30) and (31).

LEMMA 8. *Let $H \geqslant 1$, $N \geqslant 1$, $\Delta > 0$. Let γ be real. The number of solutions of the inequality*

(32) $$|hn^\gamma - kr^\gamma| \leqslant \Delta$$

in integers h, k, n, r satisfying $H < h, k \leqslant 2H$, $N < n, r \leqslant 2N$ *is*

$$< C_1(\gamma)(HN\log^2 2HN + \Delta HN^{2-\gamma}). \tag{33}$$

Proof. This is established in all essentials in the argument following equation (16) of Heath-Brown [10].

LEMMA 9. *Let* $M \geqslant 1$, $N \geqslant 1$, $H \geqslant 1$, $M_1 \leqslant 2M$, $N_1 \leqslant 2N$, $H_1 \leqslant 2H$, $1 \leqslant v \leqslant \zeta$, $v_1 \leqslant ev$. *Let* a_m $(M \leqslant m \leqslant M_1)$, b_n $(N < n \leqslant N_1)$, c_h $(H < h \leqslant H_1)$ *be complex numbers all* $\leqslant 1$ *in modulus. Let*

$$S = \sum_{\substack{M<m\leqslant M_1\\ v<mn\leqslant v_1}} a_m \sum_{N<n\leqslant N_1} b_n \sum_{H<h\leqslant H_1} c_h e\left(\frac{h\zeta}{mn}\right).$$

Then, for any exponent pair $(\varkappa, \lambda)$, *and any* $Q \geqslant 1$,

$$S^2 \ll QH^{1+\eta}(v^{2+\eta}N^{-1} + v^{3+\eta}\zeta^{-1}) + (\log 2N)Q^{-\varkappa}M^{1-2\varkappa+\lambda}\zeta^{\varkappa}N^{2-\varkappa}H^{2+\varkappa}. \tag{34}$$

In particular, for $x \leqslant \zeta \leqslant 2x$, $y \leqslant v < x^{5/8}$,

$$H \leqslant vx^{-1/2+4\eta}, \tag{35}$$

$$vx^{-1/2+10\eta} \leqslant N \leqslant x^{2-40\eta}v^{-3}, \tag{36}$$

we have

$$S \ll vx^{-2\eta}. \tag{37}$$

Proof. Following Heath-Brown [10], Section 4, we begin by noting that

$$|S| \leqslant \sum_{M<m\leqslant M_1}\left|\sum_{\substack{H<h\leqslant H_1, N<n\leqslant N_1\\ v<mn\leqslant v_1}} c_h b_n e\left(\frac{\zeta h}{mn}\right)\right|$$

$$\leqslant \sum_{M<m\leqslant M_1}\sum_{q\leqslant Q}\left|\sum_{\substack{(h,n)\in\mathscr{S}_q\\ v<mn\leqslant v_1}} c_h b_n e\left(\frac{\zeta h}{mn}\right)\right|,$$

where $\mathscr{S}_q$ is the set of (h, n), $N < n \leqslant N_1$, $H < h \leqslant H_1$ for which

$$2HN^{-1}(q-1) \leqslant Qhn^{-1} \leqslant 2HN^{-1}q.$$

Now we apply Cauchy's inequality:

$$|S|^2 \leqslant QM \sum_{M<m\leqslant M_1}\sum_{q\leqslant Q}\sum_{\substack{(n,h),(r,k)\in\mathscr{S}_q\\ v<mn,mr\leqslant v_1}} c_h b_n \bar{c}_k \bar{b}_r e\left(\zeta\frac{(hn^{-1}-kr^{-1})}{m}\right) \tag{38}$$

$$\leqslant QM \sum_{n,r,h,k;(39)}\left|\sum_{\substack{M<m\leqslant M_1\\ v<mn,mr\leqslant v_1}} e(\zeta m^{-1}(hn^{-1}-kr^{-1}))\right|,$$

where $\sum_{n,r,h,k;(39)}$ indicates a sum over quadruples with

$$N < n, r \leqslant N_1, \quad H < h, k \leqslant H_1, \quad |hn^{-1} - kr^{-1}| \leqslant 2HN^{-1}Q^{-1}. \tag{39}$$

The contribution from the quadruples with $hr = kn$ to $\sum_{n,r,h,k;(39)}$ is

$$\ll (HN)^{1+\eta}M, \tag{40}$$

by a divisor argument and a trivial estimate for the exponential sum. The remaining quadruples have $|hn^{-1} - kr^{-1}| \geqslant (2N)^{-2}$, and may be partitioned into $O(\log 2N)$ sets defined by

$$\Delta/2 < |hn^{-1} - kr^{-1}| \leqslant \Delta. \tag{41}$$

Here

$$N^{-2} \ll \Delta \leqslant 2HN^{-1}Q^{-1}. \tag{42}$$

For any quadruple satisfying (41), we have

$$\sum_{\substack{M<m\leqslant M_1\\ v<mn,mr\leqslant v_1}} e(\zeta m^{-1}(hn^{-1}-kr^{-1})) \ll \min(M, (\Delta\zeta M^{-2})^{-1} + (\Delta\zeta M^{-2})^{\varkappa}M^{\lambda}). \tag{43}$$

(The term $(\Delta\zeta M^{-2})^{-1}$ is incorporated to allow for the possibility that

$$\left|\frac{d}{dx}\zeta x^{-1}(hn^{-1}-kr^{-1})\right| < \frac{1}{2}$$

at $x = M$; we then use Lemmas 4.8 and 4.2 of Titchmarsh [21].)

Combining (38), (40) and (43), we obtain (for some Δ satisfying (42))

$$|S|^2 \ll QM^2(HN)^{1+\eta} + (\log 2N)QM\min\{M, (\Delta\zeta M^{-2})^{-1} + (\Delta\zeta M^{-2})^{\varkappa}M^{\lambda}\}\sum_{(n,r,h,k):(41)} 1$$

$$\ll QM^2(HN)^{1+\eta} + QM(\Delta\zeta M^{-2})^{\varkappa}M^{\lambda}\Delta HN^3\log 2N + QM(\Delta\zeta M^{-2})^{-1}\Delta HN^3\log 2N$$

in view of Lemma 8 with $\gamma = -1$. Thus, applying (42),

$$|S|^2 \ll Qv^{2+\eta}H^{1+\eta}N^{-1} + QHM^{1-2\varkappa+\lambda}\Delta^{1+\varkappa}\zeta^{\varkappa}N^3\log 2N + QHv^3\zeta^{-1}\log 2N$$

$$\ll QH^{1+\eta}(v^{2+\eta}N^{-1} + v^{3+\eta}\zeta^{-1}) + QHM^{1-2\varkappa+\lambda}\zeta^{\varkappa}N^3(HN^{-1}Q^{-1})^{1+\varkappa}\log 2N$$

$$\ll QH^{1+\eta}(v^{2+\eta}N^{-1} + v^{3+\eta}\zeta^{-1}) + Q^{-\varkappa}H^{2+\varkappa}M^{1-2\varkappa+\lambda}\zeta^{\varkappa}N^{2-\varkappa}\log 2N$$

as claimed in (34).

Now suppose that $x \leqslant \zeta \leqslant 2x$, $y \leqslant v < x^{5/8}$ and that (35), (36) hold.

Then

(44) $$|S|^2 \ll Qvx^{-1/2+5\eta}(v^2 N^{-1}+v^3 x^{-1}) + Q^{-\varkappa} v^{2+\varkappa} M^{1-2\varkappa+\lambda} N^{2-\varkappa} x^{-1+\varkappa/2+11\eta}.$$

We choose

(45) $$Q = [Nv^{-1} x^{1/2-10\eta}].$$

(Note that $Q \geqslant 1$ from (36).) Then

(46) $$Qv^4 x^{-3/2+5\eta} \leqslant Nv^3 x^{-1} < x^{1-4\eta} \leqslant v^2 x^{-4\eta}$$

from (36); also

(47) $$Q^{-\varkappa} v^{2+\varkappa} M^{1-2\varkappa+\lambda} N^{2-\varkappa} x^{-1+\varkappa/2+11\eta} \ll v^{2+2\varkappa} M^{1-2\varkappa+\lambda} N^{2-2\varkappa} x^{-1+16\eta} \ll v^2 M^{1+\lambda} N^2 x^{-1+16\eta} \ll v^{3+\lambda} N^{1-\lambda} x^{-1+16\eta}.$$

Evidently the optimal choice of $(\varkappa, \lambda)$ is $(1/2, 1/2)$, giving

$$S^2 \ll v^2 x^{-4\eta} + v^{7/2} N^{1/2} x^{-1+16\eta} \ll v^2 x^{-4\eta}$$

on combining (44)–(47) with (36). This completes the proof of Lemma 9.

Proof of Proposition 1. Since

$$N(d) = \left[\frac{x+y}{d}\right] - \left[\frac{x}{d}\right] = \frac{y}{d} - \psi\left(\frac{x+y}{d}\right) + \psi\left(\frac{x}{d}\right),$$

we have to show that

$$\sum_{d \leqslant x^{\varphi}} \Lambda(d)\left\{\psi\left(\frac{x+y}{d}\right) - \psi\left(\frac{x}{d}\right)\right\} \ll y.$$

By a simple splitting up argument, it suffices to show that

$$\sum_{v<d\leqslant ev} \Lambda(d)\left(\psi\left(\frac{x+y}{d}\right) - \psi\left(\frac{x}{d}\right)\right) \ll yL^{-1}$$

for $y \leqslant v \leqslant x^{\varphi}$. (These bounds for v will be used frequently in the proof.)

Here, and later on, we write

(48) $$J = J(v) = [vy^{-1} x^{4\eta}].$$

Applying Lemma 7, it suffices to show that

(49) $$\sum_{v<d\leqslant v_1} \Lambda(d) \sum_{j=1}^{J} e\left(\frac{\zeta_1 j}{d}\right) \ll vL^{-1},$$

and that

(50) $$\sum_{j=0}^{\infty} \min\left(\frac{L}{J}, \frac{J}{j^2}\right)\left|\sum_{v<n\leqslant ev} e\left(\frac{\zeta_2 j}{n}\right)\right| \ll yL^{-1}.$$

Here $x \leqslant \zeta_1, \zeta_2 \leqslant x+y$.

We tackle (50) first, using the estimate

(51) $$\sum_{v<n\leqslant v_1} e\left(\frac{\zeta j}{n}\right) \ll \zeta^{1/2} j^{1/2} v^{-1/2} + \zeta^{-1/2} j^{-1/2} v^{3/2}$$

which follows at once from [21], Theorem 5.9. The sum in (50) is

$$\ll v\frac{L}{J} + L\sum_{j=1}^{J} \frac{1}{J}(x^{1/2} j^{1/2} v^{-1/2} + x^{-1/2} j^{-1/2} v^{3/2}) + \sum_{j>J} \frac{J}{j^2}(x^{1/2} j^{1/2} v^{-1/2} + x^{-1/2} j^{-1/2} v^{3/2})$$
$$\ll yx^{-3\eta} + Lx^{1/2} J^{1/2} v^{-1/2} + LJ^{-1/2} v^{3/2} x^{-1/2}$$
$$\ll yx^{-3\eta} + x^{1/4+3\eta} + vx^{-1/4+\eta} \ll yx^{-3\eta}$$

since $\varphi < 3/4 - \varepsilon$.

Next we prove (49). We apply Lemma 6 with

$$f(n) = \sum_{j=1}^{J} e\left(\frac{\zeta_1 j}{n}\right),$$

taking $a = vx^{-1/2+10\eta}$, $b = 100v^{1/3}$, $z = x^{1/4-6\eta}$. We thus decompose the sum in (49) into $O(L^6)$ sums of the types:

(I) $$\sum_{j=1}^{J} \sum_{\substack{M<m\leqslant M_1 \\ v<mn\leqslant v_1}} a_m \sum_{N<n\leqslant N_1} (\log n)\, e\left(\frac{\zeta_1 j}{mn}\right)$$

with $N > x^{1/4-6\eta}$ (with the logarithmic factor possibly omitted, which would simplify the subsequent argument);

(II) $$\sum_{j=1}^{J} \sum_{\substack{M<m\leqslant M_1 \\ v<mn\leqslant v_1}} a_m \sum_{N<n\leqslant N_1} b_n\, e\left(\frac{\zeta_1 j}{mn}\right),$$

with $vx^{-1/2+10\eta} \leqslant N \leqslant 100v^{1/3}$. We have $a_m, b_n \ll v^{\eta}$.

It remains to show that both sums (I) and (II) are $O(vL^{-7})$.

In the case of a sum (I),

$$\sum_{j=1}^{J} \sum_{\substack{M<m\leqslant M_1 \\ v<mn\leqslant v_1}} a_m \sum_{N<n\leqslant N_1} e\left(\frac{\zeta_1 j}{mn}\right)\log n$$
$$= \int_1^{N_1} \frac{1}{t} \sum_{j=1}^{J} \sum_{M<m\leqslant M_1} a_m \sum_{\max(N,t)\leqslant n\leqslant N_1} e\left(\frac{\zeta_1 j}{mn}\right) dt \ll (\log 2N)\, v^{1+\eta} x^{-2\eta} \ll vL^{-7}$$

in view of Lemma 5.

As for Type II sums, we have

$$vx^{-1/2+10\eta} \leqslant N \leqslant 100v^{1/3} \leqslant x^{2-40\eta} v^{-3},$$

and so such a sum is

$$\ll v^{2\eta} \cdot vx^{-2\eta} \ll vL^{-7},$$

by an application of Lemma 9. This completes the proof of Proposition 1.

4. The linear sieve. We shall prove Proposition 2 by obtaining a suitable upper bound for

$$\sum_{v<p\leqslant ev} N(p),$$

where $x^{\varphi} \leqslant v \leqslant x^{7/10}$. With each such v we associate two parameters $z = z(v)$, $D = D(v)$ such that

$$(ev)^{1/8} < z < v^{1/2}; \quad z^2 \leqslant D < x^{3/4}; \tag{52}$$

z and D will be specified later. Note that

$$\sum_{v<p\leqslant ev} N(p) = \sum_{\substack{v<p\leqslant ev \\ x<mp\leqslant x+y \text{ for some } m}} 1 = \sum_{p\in\mathscr{A}} 1,$$

where

$$\mathscr{A} = \{n:\ v < n \leqslant ev,\ x < mn \leqslant x+y \text{ for some integer } m\}.$$

Let

$$S(\mathscr{A}, z) = \sum_{\substack{m\in\mathscr{A} \\ (m,P(z))=1}} 1,$$

where

$$P(z) = \prod_{p<z} p, \quad V(z) = \prod_{p<z} (1-1/p).$$

Then

$$\sum_{v<p\leqslant ev} N(p) \leqslant S(\mathscr{A}, z) - \sum_{r=2}^{7} \sum_{\substack{z\leqslant p_1<\ldots<p_r \\ p_1\ldots p_r\in\mathscr{A}}} 1. \tag{53}$$

We shall obtain the desired upper bound for the left hand side of (53) by using the linear sieve (Lemma 10, below) to give an upper bound for $S(\mathscr{A}, z)$, and by finding subsums of

$$\sum_{r=2}^{7} \sum_{\substack{z\leqslant p_1<\ldots<p_r \\ p_1\ldots p_r\in\mathscr{A}}} 1 \tag{54}$$

for which an asymptotic formula can be given.

Let

$$\mathscr{A}_d = \{n\in\mathscr{A}:\ d|n\}$$

and write

$$r(\mathscr{A}, d) = |\mathscr{A}_d| - y/d.$$

This is small on average for small d, enabling us to use the following lemma of Iwaniec [12] in the form stated in [11]. For the function $F(s)$, see Halberstam and Richert [8], Chapter 8.

LEMMA 10. *Let $z \geqslant 2$, $D \geqslant z^2$ and $\eta > 0$. Then*

$$S(\mathscr{A}, z) \leqslant yV(z)\{F(s)+E\} + R^+, \tag{55}$$

where $s = (\log D)/(\log z)$ and $E = C_2\eta + O((\log D)^{-1/3})$. Here

$$R^+ = \sum_{(D)} \sum_{\substack{v<D^{\eta} \\ v|P(D^{\eta^2})}} c_{(D)}(v, \eta) \sum_{D_i\leqslant p_i<\min(z, D_i^{1+\eta^7})} r(\mathscr{A}, vp_1\ldots p_t);$$

$\sum_{(D)}$ runs over all subsequences $D_1 \geqslant \ldots \geqslant D_t$ (including the empty subsequence) of the sequence $D^{\eta^2(1+\eta^7)^n}$, $n \geqslant 0$, for which

$$D_1\ldots D_{2l} D_{2l+1}^3 \leqslant D \quad (0 \leqslant l \leqslant (t-1)/2).$$

The coefficients $c_{(D)}(v, \eta)$ are $\leqslant 1$ in modulus.

We have

$$V(z) = \frac{e^{-\gamma}}{\log z}\left\{1 + O\left(\frac{1}{\log z}\right)\right\}$$

([8], Chapter 5, equation (1.3)) and

$$F(s) = 2e^{\gamma}/s \quad \text{for} \quad 0 < s \leqslant 3.$$

We deduce from (52) and (55) that

$$S(\mathscr{A}, z) \leqslant \frac{2y}{\log D}(1 + C_3\eta) + R^+ \tag{56}$$

for any choice of z, $D^{1/3} \leqslant z \leqslant D^{1/2}$. We shall therefore choose D as large as possible subject to being able to prove that

$$R^+ = O(yL^{-2}). \tag{57}$$

Having chosen D, we put

$$z = D^{1/3}, \tag{58}$$

so that as many integers as possible are counted in (54). (It is, apparently, very inefficient to take z any smaller.)

To get (57), it suffices to show that

$$\sum_{A_1<p_1\leqslant B_1}\ldots\sum_{A_t<p_t\leqslant B_t} r(\mathscr{A}, vp_1\ldots p_t) \ll yx^{-\eta} \tag{59}$$

for any intervals $(A_1, B_1], \ldots, (A_t, B_t]$ with

$$t \ll 1,\quad A_i \geqslant 1,\quad B_i \leqslant 2A_i,\quad A_1 \geqslant A_2 \geqslant \ldots \geqslant A_t,$$
$$A_1\ldots A_{2l}A_{2l+1}^3 \leqslant D^{1+\eta^7}\quad (0\leqslant l\leqslant (t-1)/2), \tag{60}$$

and for any v,

$$1 \leqslant v < D^{\eta}. \tag{61}$$

Here we have used the fact that $\sum_{(D)}$ is a sum over $O(1)$ sequences, each with $t \ll 1$, and that $|c_{(D)}(v, \eta)| \leqslant 1$, together with a simple splitting up argument.

We now turn our attention to $r(\mathscr{A}, d)$. We have

$$|\mathscr{A}_d| = \sum_{\substack{k\\ v<kd\leqslant ev}}\sum_{\substack{m\\ x<mkd\leqslant x+y}} 1 = \sum_{v/d<k\leqslant ev/d}\left(\frac{y}{kd}-\psi\left(\frac{x+y}{kd}\right)+\psi\left(\frac{x}{kd}\right)\right),$$

so that

$$|\mathscr{A}_d| = \frac{y}{d}\sum_{v/d<k\leqslant ev/d}\frac{1}{k}-\sum_{v/d<k\leqslant ev/d}\left\{\psi\left(\frac{x+y}{kd}\right)-\psi\left(\frac{x}{kd}\right)\right\}$$
$$=\frac{y}{d}\Big(\int_{v/d}^{ev/d} du/u+O(dv^{-1})\Big)-\sum_{v/d<k\leqslant ev/d}\left\{\psi\left(\frac{x+y}{kd}\right)-\psi\left(\frac{x}{kd}\right)\right\}.$$

Hence

$$r(\mathscr{A}, d) = O(yv^{-1})-\sum_{v/d<k\leqslant ev/d}\left\{\psi\left(\frac{x+y}{kd}\right)-\psi\left(\frac{x}{kd}\right)\right\},$$

and (60) yields

$$\sum_{A_1<p_1\leqslant B_1}\ldots\sum_{A_t<p_t\leqslant B_t} r(\mathscr{A}, vp_1\ldots p_t) = O(yDv^{-1}) \tag{62}$$
$$-\sum_{A_1<p_1\leqslant B_1}\ldots\sum_{A_t<p_t\leqslant B_t}\sum_{v<kvp_1\ldots p_t\leqslant ev}\left\{\psi\left(\frac{x+y}{vkp_1\ldots p_t}\right)-\psi\left(\frac{x}{vkp_1\ldots p_t}\right)\right\}.$$

To be able to deal with the term $O(yDv^{-1})$ we are obliged to impose the condition

$$D \leqslant vx^{-\eta}. \tag{63}$$

(For $x^{\varphi} \leqslant v \leqslant x^{3/5-\varepsilon}$ this turns out to be the most severe constraint on D that is needed.)

LEMMA 11. *Suppose that* (60) *and* (61) *hold. Let*

$$D \leqslant \begin{cases} vx^{-\varepsilon} & (x^{\varphi}\leqslant v\leqslant x^{3/5-\varepsilon}),\\ x^{13/2-\varepsilon}v^{-10} & (x^{3/5-\varepsilon}< v\leqslant x^{0.62}),\\ x^{313/740-\varepsilon}v^{-15/74} & (x^{0.62}< v\leqslant x^{0.661}),\\ x^{11/28-\varepsilon}v^{-11/70} & (x^{0.661}< v\leqslant x^{0.7}).\end{cases} \tag{64}$$

Then

$$S \ll yx^{-\eta}, \tag{65}$$

where

$$S = \sum_{A_1<p_1\leqslant B_1}\ldots\sum_{A_t<p_t\leqslant B_t}\sum_{v<kvp_1\ldots p_t\leqslant ev}\left\{\psi\left(\frac{x+y}{kvp_1\ldots p_t}\right)-\psi\left(\frac{x}{kvp_1\ldots p_t}\right)\right\}.$$

Proof. Suppose first that $v \leqslant x^{3/5-\varepsilon}$. We begin by showing that either

(66) (i) $A_1\ldots A_t < v^{1/2}x^{-\eta}$,

or

(ii) some set $T \subset \{1, \ldots, t\}$ satisfies

$$vx^{-1/2+10\eta} < P = \prod_{i\in T} A_i < x^{2-50\eta}v^{-3}. \tag{67}$$

Suppose that (i) is false. Now (60) yields

$$A_1^3 < D^{1+\eta} < vx^{-\varepsilon/2},$$

hence

$$A_1 < v^{1/3}x^{-\varepsilon/6} \leqslant x^{2-\varepsilon/6}v^{-3}.$$

If $A_1 > vx^{-1/2+10\eta}$ we have (ii). If not, let j be the smallest integer such that

$$A_1\ldots A_j > vx^{-1/2+10\eta}$$

(this j plainly exists since (66) is false). We see that

$$A_1\ldots A_j < vx^{-1/2+10\eta}A_j \ll (vx^{-1/2+10\eta})^2 < x^{2-\varepsilon}v^{-3},$$

so that (ii) holds.

It is now easy to complete the proof of the lemma for $v \leqslant x^{3/5-\varepsilon}$. If (i) holds, then we observe that the sum S may be written in the form

$$S = \sum_{\substack{M<m<C_4(\varepsilon)M\\ v<mk\leqslant ev}} a_m \sum_{C_5(\varepsilon)vM^{-1}<k<C_6(\varepsilon)vM^{-1}}\left\{\psi\left(\frac{x+y}{mk}\right)-\psi\left(\frac{x}{mk}\right)\right\} \tag{68}$$

with

$$(69)\qquad vA_1 \dots A_t = M \leqslant v^{1/2},$$

where $0 \leqslant a_m \ll v^\eta$ by a divisor argument. The double sum in (68) can, in fact, be written in the form of a single sum essentially as in Lemma 7. Thus

$$(70)\qquad S \ll yv^{-1}\Big|\sum_{j=1}^{J} \sum_{\substack{M<m<C_4(\varepsilon)M \\ v<mk\leqslant \varepsilon v}} a_m \sum_{C_5(\varepsilon)vM^{-1}<k<C_6(\varepsilon)vM^{-1}} e\Big(\frac{\zeta_1 j}{mk}\Big)\Big| + x^\eta \sum_{j=0}^{\infty} \min\Big(\frac{L}{J}, \frac{J}{j^2}\Big)\Big|\sum_{v<n\leqslant \varepsilon v} e\Big(\frac{\zeta_2 j}{n}\Big)\Big|$$

with $x \leqslant \zeta_1, \zeta_2 \leqslant x+y$.

Exactly as in the proof of Proposition 1,

$$(71)\qquad x^\eta \sum_{j=0}^{\infty} \min\Big(\frac{L}{J}, \frac{J}{j^2}\Big)\Big|\sum_{v<n<\varepsilon v} e\Big(\frac{\zeta_2 j}{n}\Big)\Big| < yx^{-2\eta}.$$

The inequality (65) may now be deduced from (70), (71) by applying Lemma 4 with $(\varkappa, \lambda) = (1/2, 1/2)$.

Now suppose that (ii) holds. Then S may be written in the form

$$(72)\qquad \sum_{\substack{P<m<C_7(\varepsilon)P \\ v<mn\leqslant \varepsilon v}} a_m \sum_{C_8(\varepsilon)vP^{-1}<n<C_9(\varepsilon)vP^{-1}} b_n\Big\{\psi\Big(\frac{x+y}{mn}\Big)-\psi\Big(\frac{x}{mn}\Big)\Big\}.$$

Here a_m (for example) is the number of ways of writing m in the form $\prod_{i\in T} p_i$ with each factor $p_i \in (A_i, B_i]$. Thus $0 \leqslant a_m \ll v^{\eta/4}$, $0 \leqslant b_m \ll v^{\eta/4}$. In place of (70) we have

$$(73)\qquad S \ll yv^{-1}\Big|\sum_{j=1}^{J} \sum_{\substack{P<m<C_7(\varepsilon)P \\ v<mk\leqslant \varepsilon v}} a_m \sum_{C_8(\varepsilon)vP^{-1}<n<C_9(\varepsilon)vP^{-1}} b_n e\Big(\frac{j\zeta_1}{mn}\Big)\Big| + x^\eta \sum_{j=0}^{\infty} \min\Big(\frac{L}{J}, \frac{J}{j^2}\Big)\Big|\sum_{v<n\leqslant \varepsilon v} e\Big(\frac{j\zeta_2}{mn}\Big)\Big|.$$

In view of (71), (67) and Lemma 9 we have

$$S \ll yv^{-1+\eta/2}vx^{-2\eta} + yx^{-\eta}$$

proving (65).

We now turn our attention to

$$x^{3/5-\varepsilon} < v \leqslant x^{8/13}.$$

We first show that either alternative (ii), above, holds, or:

$$(\text{iii})\qquad A_1 \dots A_t \leqslant x^{0.298}.$$

To do this, we suppose that both (ii) and (iii) are false and ultimately obtain a contradiction. Thus we suppose

$$(74)\qquad A_1 \dots A_t > x^{0.298}.$$

Since

$$A_1^3 < D^{1+\eta} < (x^{2-\varepsilon}v^{-3})^3,$$

we evidently have

$$(75)\qquad A_1 \leqslant vx^{-1/2+10\eta} < x^{0.116}.$$

We must have

$$(76)\qquad A_3 \geqslant x^{5/2-60\eta}v^{-4}.$$

For $A_2A_3 \dots A_t > x^{0.182}$ from (75), so if (76) fails there is a least j with

$$A_2A_3 \dots A_j > vx^{-1/2+10\eta},$$

hence

$$A_2A_3 \dots A_j \leqslant vx^{-1/2+10\eta}A_3 < x^{2-50\eta}v^{-3}.$$

Next, we must have

$$(77)\qquad A_1A_2 \geqslant x^{2-50\eta}v^{-3}.$$

For, if not,

$$A_1A_2 \leqslant vx^{-1/2+10\eta},$$

$$A_1A_2A_3 \ll (A_1A_2)^{3/2} \leqslant x^{0.174},$$

$$A_4A_5 \dots A_t \gg x^{0.124},$$

and indeed

$$A_4A_5 \dots A_t \geqslant x^{2-50\eta}v^{-3}.$$

Since

$$A_7 \ll (A_1 \dots A_6 A_7^3)^{1/9} \ll D^{(1+\eta)/9} \ll x^{5/2-\varepsilon/10}v^{-4}$$

for $v < x^{8/13}$, a similar argument to that used in proving (76) gives

$$(78)\qquad A_4A_5A_6 \geqslant x^{2-50\eta}v^{-3}.$$

Now

$$A_4A_5 \ll A_1A_2 \ll vx^{-1/2+10\eta},$$

hence

(79) $$A_6 \gg x^{5/2-60\eta} v^{-4}$$

and indeed

(80) $$D^{1+\eta} \geqslant A_1 A_2 A_3 A_4 A_5^3 \gg (A_4 A_5 A_6)^2 A_6 \gg x^{13/2-\varepsilon/2} v^{-10}$$

from (78), (79). This contradicts (64), and we have established the truth of (77).

Next, we must have

(81) $$A_2 A_3 \leqslant v x^{-1/2+10\eta}.$$

For, if not,

$$A_2 A_3 \geqslant x^{2-50\eta} v^{-3}, \quad D^{1+\eta} \geqslant A_1 A_3 \cdot A_2 A_3 \cdot A_3 \gg (A_2 A_3)^2 x^{5/2-60\eta} v^{-4}$$

from (76), giving a contradiction as in (80).

Next, we have

$$A_1 A_2 A_5^5 \leqslant A_1 A_2 A_3 A_4 A_5^3 \leqslant D^{1+\eta},$$
$$A_5^5 \leqslant D^{1+\eta}(A_1 A_2)^{-1} < x^{25/2-\varepsilon/2} v^{-20}$$

for $v < x^{8/13}$, by an application of (64) and (77). Thus

(82) $$A_5 < x^{5/2-\varepsilon/20} v^{-4}.$$

By arguments similar to the proof of (76) we find that (81), (82) yield

(83) $$A_2 A_3 A_5 \dots A_t \leqslant v x^{-1/2+10\eta},$$

and hence (74) implies

$$A_1 A_4 > x^{0.182+10\eta}.$$

Combining this with (81), we see that

$$A_2/A_1 \leqslant A_2 A_3/(A_1 A_4) \leqslant v x^{-0.682}.$$

With (75) we get

$$A_1 A_2 \leqslant v x^{-0.682} \cdot v^2 x^{-1+20\eta},$$
$$A_3 A_4 \dots A_t \gg x^{0.298+1.682-20\eta} v^{-3} > v x^{-1/2+\varepsilon}.$$

This evidently contradicts (83), and we have obtained the desired contradiction.

Now let

$$x^{8/13} < v \leqslant x^{0.62}.$$

We show that either (ii) holds, or

(iv) $$A_1 \dots A_t \leqslant x^{313/740-\varepsilon/4} v^{-15/74}.$$

We do this in a rather more elegant way than the preceding argument. Writing

(84) $$U = x^{2-50\eta} v^{-3},$$

we first observe that

(85) $$D \leqslant \min(x^{683/370-\varepsilon} v^{-89/37}, U^3 v^{-1} x^{1/2-\varepsilon/2}).$$

It follows from (60) that $\{1, \dots, t\}$ can be partitioned into two sets T and V such that

(86) $$P = \prod_{i \in T} A_i \leqslant D^{1/2+\eta}, \quad Q = \prod_{i \in V} A_i \leqslant D^{1/2+\eta}$$

(this fact is implicit in [12]). Now it follows from (85) that

(87) $$v x^{-1/2+10\eta} D^{1/2+\eta} \leqslant x^{313/740-\varepsilon/4} v^{-15/74}.$$

Suppose that both alternatives (ii) and (iv) fail. If $P < U$, then $P \leqslant v x^{-1/2+10\eta}$,

$$A_1 \dots A_t = PQ \leqslant v x^{-1/2+10\eta} D^{1/2+\eta}$$

which in view of (87) is a contradiction. Thus $P \geqslant U$ and similarly $Q \geqslant U$.

Let P' be the subproduct of P formed from those A_i that exceed $U/(v x^{-1/2+10\eta})$. Define Q' similarly as a subproduct of Q. Since (ii) fails it is clear that

$$P' \gg U, \quad Q' \gg U.$$

If $P'Q' = A_{j_1} \dots A_{j_r}$ with $j_1 > \dots > j_r$ then, for even j_r,

(88) $$D^{1+\eta} \gg A_1 \dots A_{j_r-1}^3 \gg P'Q' A_{j_r} \gg U^2 (U/v x^{-1/2+10\eta})$$

by definition of P', Q'. It can be seen that (88) contradicts (85), so either (ii) or (iv) holds if j_r is even. The proof is similar but simpler if j_r is odd.

It is now easy to complete the proof of Lemma 11 for $x^{3/5-\varepsilon} < v \leqslant x^{0.62}$. If (ii) holds, then of course we proceed exactly as before, using the expression (72) for S. If (iii) or (iv) holds, then S may be written in the form (68) with

(89) $$v A_1 \dots A_t = M \leqslant \begin{cases} x^{0.298+\eta} & \text{if} \quad v \leqslant x^{8/13}, \\ x^{313/740-\varepsilon/5} v^{-15/74} & \text{if} \quad v > x^{8/13}. \end{cases}$$

The inequality (65) may now be deduced from (70), (71) by applying Lemma 4 with $(\varkappa, \lambda) = (75/238, 132/238) = BA^2 BA^2 BA^2 B(0, 1)$.

Finally we have to consider $x^{0.62} < v \leqslant x^{7/10}$. The hypotheses of the lemma now yield

(90) $$v A_1 \dots A_t \leqslant D x^{2\eta} \leqslant x^{-26\eta} \min(v^{1/2} x^{26\eta}, x v^{-1}, (x^{(1+\varkappa)/2} v^{-\varkappa})^{1/(1+\lambda)})$$

where $(\varkappa, \lambda) = (75/238, 132/238)$ for $v \leqslant x^{0.661}$ and $(\varkappa, \lambda) = (1/4, 13/22) = BABABA^2B(0, 1)$ for $x^{0.661} < v \leqslant x^{7/10}$. The proof may now be completed exactly as in alternatives (iii), (iv) above. This completes the proof of Lemma 11.

We can sharpen Lemma 11 in (approximately) the range

$$x^{0.616} < v \leqslant x^{0.646}$$

by appealing to an alternative estimate for Type II sums, which we now discuss.

4. Trigonometric sums. In this section we give an account of an ingenious method of estimating Type II sums, which is sketched by Iwaniec and Laborde in [13].

Let

$$E(t) = \begin{cases} e^{-1/t} & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t \leqslant 0. \end{cases}$$

It is a simple matter to verify that $E(t)$ has derivatives of all orders and that

$$E^{(j)}(t) = 0 \quad \text{for} \quad t \leqslant 0, \qquad |E^{(j)}(t)| \leqslant C_{10}(j) \quad \text{for} \quad t \leqslant 3.$$

Thus

$$G(t) = E(t-4/5)E(11/5-t)$$

is seen to have derivatives of all orders and to vanish for $t \leqslant 4/5$ and $t \geqslant 11/5$. Moreover,

$$G(t) \geqslant e^{-10} \quad \text{for} \quad 1 \leqslant t \leqslant 2.$$

Given $M \geqslant 2$, the function

$$\varphi(t) = e^{10}G(t/M)$$

satisfies

(91) $$\varphi^{(j)}(t) = 0 \quad \text{for} \quad t \leqslant 4M/5 \text{ and } t \geqslant 11M/5 \ (j \geqslant 0),$$

(92) $$\varphi(t) \geqslant 1 \quad \text{for} \quad M \leqslant t \leqslant 2M,$$

and

(93) $$|\varphi^{(j)}(t)| \leqslant C_{11}(j)M^{-j} \quad (j \geqslant 0,\ t \text{ real}).$$

LEMMA 12. *Let M, $\varphi(t)$ be as above. Let T be a positive number,*

(94) $$T \geqslant M^{1+\varepsilon}.$$

Let

$$\Phi(t) = \sum_{4M/5 \leqslant m \leqslant 11M/5} \varphi(m)e(t/m).$$

Then for $T < t \leqslant 2T$ we have

(95) $$\Phi(t) = (M^3T^{-1})^{1/2} \sum_{L < l \leqslant L_1} b(t, l)e(2\sqrt{lt}) + O(T^{-1/\varepsilon})$$

where $L = T/5M^2$, $L_1 = 5T/M^2$. Here $b(t, l)$ is differentiable with respect to t, and

(96) $$b(t, l) \ll 1, \quad \frac{\partial}{\partial t}b(t, l) \ll T^{-1}$$

for $T < t \leqslant 2T$, $L < l \leqslant L_1$.

Proof. We may apply the Poisson summation formula to $\varphi(x)e(t/x)$ since its second derivative is summable on $\boldsymbol{R}$.

Thus

(97) $$\Phi(t) = \sum_{l=-\infty}^{\infty} \int_{4M/5}^{11M/5} \varphi(m)e(t/m + lm)\,dm.$$

Let $\eta(m) = t/m + lm$. For $l \notin (L, L_1]$, $T < t \leqslant 2T$, $4M/5 \leqslant m \leqslant 11M/5$, we have

(98) $$|\eta'(m)| = |l - tm^{-2}| \gg |l| + TM^{-2}.$$

This is obvious if $l \leqslant 0$. If $0 < l \leqslant L$, we use

$$|l - tm^{-2}| \geqslant T/(11M/5)^2 - T/(5M^2) \gg TM^{-2} \gg |l| + TM^{-2}.$$

If $l > L_1$,

$$|l - tm^{-2}| \geqslant l - 2T/(4M/5)^2 \gg l \gg |l| + TM^{-2}.$$

Note also that $\eta^{(j)}(m) \ll TM^{-(j+1)}$ $(j \geqslant 2)$.

Now let us fix a value of l, $l \notin (L, L_1]$. We write $\varphi_0(m) = \varphi(m)$ and define $\varphi_1, \varphi_2, \ldots$ recursively by

$$\varphi_j(m) = \left(-\frac{\varphi_{j-1}(m)}{2\pi i\eta'(m)}\right)'.$$

Integration by parts gives

(99) $$\int_{4M/5}^{11M/5} \varphi_j(m)e(\eta(m))\,dm = \int_{4M/5}^{11M/5} \frac{\varphi_j(m)}{2\pi i\eta'(m)} 2\pi i\eta'(m)e(\eta(m))\,dm$$
$$= \left[\frac{\varphi_j(m)}{2\pi i\eta'(m)}e(\eta(m))\right]_{4M/5}^{11M/5} + \int_{4M/5}^{11M/5} \varphi_{j+1}(m)e(\eta(m))\,dm$$
$$= \int_{4M/5}^{11M/5} \varphi_{j+1}(m)e(\eta(m))\,dm.$$

It is not difficult to verify by induction on j that for $j \geqslant 1$ we have

$$\varphi_j = \sum{}' c(i, j_1, k_1, \ldots, j_r, k_r, s)\varphi^{(i)}\{\eta^{(j_1)}\}^{k_1} \ldots \{\eta^{(j_r)}\}^{k_r}(\eta')^{-s},$$

where the sum $\sum'$ is over $\leqslant C_{12}(j)$ ordered sets $i, j_1, k_1, \ldots, j_r, k_r, s$,

$$|c(i, j_1, \ldots, s)| \leqslant C_{13}(j), \quad r \geqslant 0, \quad s \geqslant j, \quad j_i \geqslant 2,$$
$$i + j_1 k_1 + \ldots + j_r k_r = s, \quad k_1 + \ldots + k_r = s - j.$$

Recalling (93), it follows at once that, for $j \geqslant 2$,

$$\begin{aligned}\varphi_j(m) &< C_{14}(j) \sum{}' M^{-i} \prod_{t=1}^{r} (TM^{-j_t-1})^{k_t} (TM^{-2}+|l|)^{-s} \\ &\leqslant C_{14}(j) \sum{}' M^{-(i+j_1k_1+\ldots+j_rk_r+k_1+\ldots+k_r)+2s} T^{k_1+\ldots+k_r-s} \left(\frac{TM^{-2}}{TM^{-2}+|l|}\right)^s \\ &< C_{15}(j) M^j T^{-j} (TM^{-2})^2/(TM^{-2}+|l|)^2 \\ &< C_{15}(j) T^{-(j\varepsilon/2)+2} M^{-4}/(TM^{-2}+|l|)^2\end{aligned}$$

in view of (94).

In conjunction with (99), this yields

$$I(t, l) = \int_{4M/5}^{11M/5} \varphi(m) e(\eta(m)) dm \ll \frac{C_{15}(j) T^{-(j\varepsilon/2)+2} M^{-3}}{(TM^{-2}+|l|)^2}.$$

Choosing $j = [8/\eta]+1$, we obtain

$$\sum_{l \notin (L, L_1]} I(t, l) \ll T^{-(2/\varepsilon)} M^{-3} \sum_{l=-\infty}^{\infty} \frac{1}{(TM^{-2}+|l|)^2} \ll T^{-(1/\varepsilon)}. \tag{100}$$

In view of (97) and (100), it remains to show that for $L < l \leqslant L_1$, $T < t \leqslant 2T$ we have

$$I(t, l) = b_1(t, l) e(2\sqrt{lt}), \tag{101}$$

where

$$b_1(t, l) \ll (M^3 T^{-1})^{1/2}, \tag{102}$$

$$\frac{\partial}{\partial t} b_1(t, l) \ll M^{3/2} T^{-3/2}. \tag{103}$$

Since

$$t/m + lm = 2\sqrt{lt} + (\sqrt{t/m} - \sqrt{lm})^2,$$

(101) holds with

$$b_1(t, l) = \int_{4M/5}^{11M/5} \varphi(m) e((\sqrt{t/m} - \sqrt{lm})^2) dm.$$

To prove (102), (103) we change the variable of integration, writing

$$\omega = \sqrt{t/m} - \sqrt{lm} \quad (4M/5 \leqslant m \leqslant 11M/5),$$
$$\omega'(m) = -\left(\frac{t^{1/2}}{2m^{3/2}} + \frac{l^{1/2}}{2m^{1/2}}\right) < 0.$$

Denoting by $m(\omega)$ the inverse function of $\omega(m)$, and writing

$$\omega_1 = \sqrt{T/(11M/5)} - \sqrt{L_1(11M/5)}, \quad \omega_2 = \sqrt{2T/(4M/5)} - \sqrt{L(4M/5)},$$

we have

$$b_1(t, l) = \int_{\omega_1}^{\omega_2} \varphi(m(\omega)) m'(\omega) e(\omega^2) d\omega.$$

Write

$$\Omega(\omega) = \int_0^{\omega} e(\lambda^2) d\lambda.$$

Then integrating by parts,

$$b_1(t, l) = -\int_{\omega_1}^{\omega_2} \Omega(\omega) \left(\varphi'(m(\omega))(m'(\omega))^2 + \varphi(m(\omega)) m''(\omega)\right) d\omega. \tag{104}$$

It is easy to see that

$$|\omega_1| + |\omega_2| \ll T^{1/2} M^{-1/2}, \tag{105}$$

and it is well known that

$$\Omega(\omega) \ll 1 \quad \text{for all real } \omega. \tag{106}$$

It is helpful to compute $m(\omega)$ and its derivatives explicitly. We have

$$(\omega^2 + 4\sqrt{lt})^{1/2} - \omega = 2\sqrt{lm}$$

so that, for $\omega_1 \leqslant \omega \leqslant \omega_2$,

$$m(\omega) = \frac{1}{2l}\{\omega^2 + 2\sqrt{lt} - \omega(\omega^2 + 4\sqrt{lt})^{1/2}\},$$

$$m'(\omega) = \frac{1}{l}\left\{\omega - \frac{\omega^2 + 2\sqrt{lt}}{(\omega^2 + 4\sqrt{lt})^{1/2}}\right\} \ll (M^3 T^{-1})^{1/2}$$

and

$$m''(\omega) = \frac{1}{l}\left\{1 - \frac{2\omega}{(\omega^2 + 4\sqrt{lt})^{1/2}} + \frac{\omega(\omega^2 + 2\sqrt{lt})}{(\omega^2 + 4\sqrt{lt})^{3/2}}\right\} \ll M^2 T^{-1}.$$

Hence we obtain (102) from (104)–(106), (93).

For fixed $\omega\in[\omega_1,\omega_2]$, $l\in(L,L_1]$ consider m, m' and m'' as functions of t. We verify readily that

$$\varphi''(m)\frac{dm}{dt}m'^2 \ll M^2T^{-2},$$

$$\varphi'(m)m'\frac{dm'}{dt} \ll M^2T^{-2},$$

$$\varphi(m)\frac{dm''}{dt} \ll M^2T^{-2},$$

$$\varphi'(m)\frac{dm}{dt}m'' \ll M^2T^{-2}.$$

Differentiation of (104) with respect to t therefore yields (103), and the proof of Lemma 12 is complete.

LEMMA 13. *Let* $H\geq 1$, $N\geq 2H^{2\varepsilon}$, $S(\theta)=\sum_{0<h\leq H}e(h\theta)$. *Then*

(107) $$\sum_{N<n_1,n_2\leq 2N}\int_0^1|S(\alpha n_2)S(-\alpha n_1)|\,d\alpha \ll N^{1+2\eta}(N+H).$$

Proof. We have the familiar estimate

(108) $$S(\theta)\ll\min(H,\|\theta\|^{-1}).$$

The contribution to the left hand side of (107) from those α with

(109) $$\|n\alpha\|<N^{-2/\varepsilon}$$

for some n, $N<n\leq 2N$, is at most

(110) $$H^2\sum_{N<n_1,n_2\leq 2N}N^{1-2/\varepsilon}\ll H^2N^{3-2/\varepsilon}\ll N^2,$$

since (109) defines a set having measure $2N^{-2/\varepsilon}$ in $[0,1)$ (for given n).

Write

$$E=\{\alpha\in[0,1):\ \min_{N<n\leq 2N}\|n\alpha\|\geq N^{-2/\varepsilon}\},$$

then from (108) we deduce that

(111) $$\sum_{N<n_1,n_2\leq 2N}\int_E|S(\alpha n_2)S(-\alpha n_1)|\,d\alpha$$
$$\ll\sum_{N<n_1,n_2\leq 2N}\ \sum_{r_1\ll\log N}\ \sum_{r_2\ll\log N}\min(H,2^{r_1})\min(H,2^{r_2})|E(n_1,n_2,r_1,r_2)|.$$

Here

$$E(n_1,n_2,r_1,r_2)=\{\alpha\in[0,1):\ 2^{-r_1-1}<\|\alpha n_j\|\leq 2^{-r_j}\text{ for } j=1,2\}.$$

By symmetry we can confine the summation in (111) to pairs r_1, r_2 with $r_1\geq r_2$. Let $\alpha\in E(n_1,n_2,r_1,r_2)$ and let k_1, k_2 be the nearest integers to αn_1 and αn_2 respectively. Then

$$|\alpha-k_1/n_1|\leq 2^{-r_1}/N,\qquad |\alpha-k_2/n_2|\leq 2^{-r_2}/N,$$
$$|k_1/n_1-k_2/n_2|\leq 2^{-r_2+1}/N.$$

Hence

(112) $$\sum_{N<n_1,n_2\leq 2N}|E(n_1,n_2,r_1,r_2)|$$
$$\ll\sum_{N<n_1,n_2\leq 2N}\ \sum_{0\leq k_1\leq 2N}\ \sum_{\substack{0\leq k_2\leq 2N\\ |k_1n_2-k_2n_1|\leq 2^{-r_2+3}N}}\frac{2^{-r_1}}{N}\ll\frac{2^{-r_1}}{N}N^{2+\eta}(2^{-r_2}N+1),$$

since for given k_1, n_2 the number of possibilities for k_2n_1 is at most $2^{-r_2+4}N+1$; this determines k_2, n_1 with at most $O(N^\eta(2^{-r_2}N+1))$ possibilities, except in the case $k_2=0$. In this case we simply use the bound

$$\sum_{0\leq k_1\leq 2^{-r_2+3}}\frac{2^{-r_1}}{N}\sum_{N<n_1,n_2\leq 2N}1\ll N2^{-r_1}$$

for the contribution to the quadruple sum in (112).

Combining (110), (111), (112),

$$\sum_{N<n_1,n_2\leq 2N}\int_0^1|S(\alpha n_2)S(-\alpha n_1)|\,d\alpha$$
$$\ll N^2+N^\eta\sum_{r_1,r_2\ll\log N}\min(H,2^{r_1})\min(H,2^{r_2})(2^{-(r_1+r_2)}N^2+2^{-r_1}N)$$
$$\ll N^{2+\eta}(\log N)^2+N^\eta(\log N)^2HN.$$

This completes the proof of Lemma 13.

LEMMA 14. *Let* $M\geq 2$, $H\geq 1$, $N\geq 2H^{2\varepsilon}$, $M_1\leq 2M$, $N_1\leq 2N$,

(113) $$\zeta\geq\max((MN^2)^{1+\varepsilon},x).$$

Let a_m, b_n *be complex numbers with modulus* ≤ 1 $(M<m\leq M_1$, $N<n\leq N_1)$. *Let*

$$S=\sum_{M<m\leq M_1}\ \sum_{N<n\leq N_1}\ \sum_{0<h\leq H}a_mb_ne\left(\frac{h\zeta}{mn}\right).$$

Then for any exponent pair $(\varkappa,\lambda)$ *we have*

(114) $$|S|^2\ll M^2(NH)^{1+\eta}$$
$$+(1+HN^{-1})H^{1/2+\lambda}N^{3/2+\lambda-2\varkappa+3\eta}M^{1/2-\varkappa}\zeta^{1/2+\varkappa}+M^{-1/2\varepsilon}N^{3+\eta}H^{2+\eta}.$$

In particular, for $v \ll MN \ll v$, $H \leqslant vx^{-1/2+4\eta}$, $x \leqslant \zeta \leqslant 2x$, $y \leqslant v < x^{3/4}$,

$$vx^{-1/2+7\eta} < N \leqslant x^{-31/300-10\eta} v^{31/75}, \tag{115}$$

we have

$$|S| \ll vx^{-3\eta}.$$

Proof. By Cauchy's inequality,

$$|S|^2 \leqslant M \sum_m \varphi(m) \left| \sum_{N<n\leqslant N_1} \sum_{0<h\leqslant H} b_n e\left(\frac{h\zeta}{mn}\right)\right|^2, \tag{116}$$

since φ majorizes the indicator function of the interval $(M, 2M]$ (see (92)).

Let $\Phi(t)$ be defined as in Lemma 12, then we may rewrite (116) in the form

$$\begin{aligned}|S|^2 &\ll M \sum_{N<n_1,n_2\leqslant N_1} \sum_{0<h_1,h_2\leqslant H} b_{n_1} \bar{b}_{n_2} \Phi\left(\left(\frac{h_1}{n_1}-\frac{h_2}{n_2}\right)\zeta\right) \\ &\leqslant O\left(M^2 (NH)^{1+\eta}\right) + MB,\end{aligned} \tag{117}$$

by a divisor argument. Here

$$B = \sum_{N<n_1,n_2\leqslant N_1} \sum_{\substack{0<h_1,h_2\leqslant H \\ h_1 n_2 \neq h_2 n_1}} b_{n_1} \bar{b}_{n_2} \Phi\left(\frac{(h_1 n_2 - h_2 n_1)\zeta}{n_1 n_2}\right).$$

Now let $c(k, n_1, n_2)$ be the number of solutions of

$$k = h_1 n_2 - h_2 n_1$$

with $0 < h_1, h_2 \leqslant H$. We may write, in the notation of Lemma 13,

$$c(k, n_1, n_2) = \int_0^1 S(\alpha n_2) S(-\alpha n_1) e(-\alpha k)\, d\alpha,$$

and

$$\begin{aligned} B &= \sum_{N<n_1,n_2\leqslant N_1} b_{n_1} \bar{b}_{n_2} \sum_{1\leqslant |k| \leqslant 2HN} c(k, n_1, n_2) \Phi\left(\frac{k\zeta}{n_1 n_2}\right) \\ &= \sum_{N<n_1,n_2\leqslant N_1} b_{n_1} \bar{b}_{n_2} \int_0^1 S(\alpha n_2) S(-\alpha n_1) U(\alpha, n_1 n_2)\, d\alpha. \end{aligned}$$

Here

$$U(\alpha, n) = \sum_{1\leqslant |k| \leqslant 2HN} e(-\alpha k) \Phi\left(\frac{k\zeta}{n}\right) = 2\mathrm{Re} \sum_{1\leqslant k\leqslant 2HN} e(-\alpha k) \Phi\left(\frac{k\zeta}{n}\right).$$

In view of Lemma 13, we deduce that

$$\begin{aligned} |B| &\leqslant \sum_{N<n_1,n_2\leqslant 2N} \int_0^1 |S(\alpha n_2) S(-\alpha n_1)|\, d\alpha \cdot \max_{\substack{0\leqslant\alpha\leqslant 1 \\ N^2<n\leqslant 4N^2}} U(\alpha, n) \\ &\ll N^{1+2\eta}(N+H)|U(\alpha, n)| \end{aligned} \tag{118}$$

for some $\alpha \in [0, 1]$ and $n \in (N^2, 4N^2]$.

For some K, $1/2 < K \leqslant HN$, we have

$$|U(\alpha, n)| \ll \log(2HN) \left| \sum_{K<k\leqslant 2K} e(-\alpha k) \Phi\left(\frac{k\zeta}{n}\right)\right|. \tag{119}$$

Now we apply Lemma 12 with $T = K\zeta/n$: note that $T \gg M^{1+\varepsilon}$ from (113). We obtain

$$\begin{aligned} &\sum_{K<k\leqslant 2K} e(-\alpha k) \Phi\left(\frac{k\zeta}{n}\right) \\ &= \left(\frac{M^3 n}{K\zeta}\right)^{1/2} \sum_{L<l\leqslant L_1} \sum_{K<k\leqslant 2K} b\left(\frac{k\zeta}{n}, l\right) e\left(2\sqrt{\frac{lk\zeta}{n}} - \alpha k\right) + O(KM^{-1/\varepsilon}). \end{aligned} \tag{120}$$

We can 'remove the factor $b(k\zeta/n, l)$ by partial summation'. To be more precise, if $a(1), \ldots, a(r)$ are complex numbers and $b(x)$ is differentiable on $[1, r]$ with $|b(x)| + r|b'(x)| \ll 1$, then

$$\sum_{m=1}^{r} a(m) b(m) = \sum_{m=1}^{r-1} A_m \left(b(m) - b(m+1)\right) + A_r b(r) \ll \max_{m\leqslant r} |A_m|.$$

Here $A_h = \sum_{m=1}^{h} a(m)$. Applying this inequality with

$$b(m) = b\left(\frac{(m+[K])\zeta}{n}, l\right),$$

where

$$b'(m) \ll \zeta n^{-1} \cdot T^{-1} = K^{-1}$$

from (96), we obtain (with $K \leqslant K_1 = K_1(l) \leqslant 2K_1$)

$$\sum_{K<k\leqslant 2K} b\left(\frac{k\zeta}{n}, l\right) e\left(2\sqrt{\frac{lk\zeta}{n}} - \alpha k\right) \ll \left| \sum_{K<k\leqslant K_1} e\left(2\sqrt{\frac{lk\zeta}{n}} - \alpha k\right)\right|. \tag{121}$$

We have reached a sum to which we can apply the theory of exponent pairs, since

$$\frac{d}{dk}\left(2\sqrt{\frac{lk\zeta}{n}} - \alpha k\right) = \left(\frac{l\zeta}{nk}\right)^{1/2} + O(1) \tag{122}$$

and, for $L < l \leqslant L_1$,

(123) $$\frac{l\zeta}{nk} \gg \frac{T\zeta}{M^2N^2K} \gg \frac{\zeta^2}{M^2N^4}; \quad \text{similarly} \quad \frac{l\zeta}{nk} \ll \frac{\zeta^2}{M^2N^4},$$

so that (from (113)) the $O(1)$ term in (122) is smaller in modulus than $\eta(l\zeta/nk)^{1/2}$. Thus

$$\sum_{K<k\leqslant K_1} e\left(2\sqrt{\frac{lk\zeta}{n}}-\alpha k\right) \ll \left(\frac{\zeta}{MN^2}\right)^{\varkappa} K^{\lambda}.$$

Combining this with (117)–(121), and noting that $L \ll K\zeta/M^2N^2$, we have

(124) $$S^2 \ll M^2(NH)^{1+\eta}+MN^{1+2\eta}(N+H)\log(2HN)\left\{\frac{M^3N^2}{K\zeta}\right\}^{1/2}\frac{K\zeta}{M^2N^2}\left\{\frac{\zeta}{MN^2}\right\}^{\varkappa}K^{\lambda}$$
$$+MN^{1+\eta}(N+H)\log(2HN)KM^{-1/\varepsilon}$$
$$\ll M^2(NH)^{1+\eta}+(1+HN^{-1})K^{1/2+\lambda}\zeta^{1/2+\varkappa}M^{1/2-\varkappa}N^{1-2\varkappa+3\eta}$$
$$+M^{-1/2\varepsilon}N^{3+\eta}H^{2+\eta}$$
$$\ll M^2(NH)^{1+\eta}+(1+HN^{-1})H^{1/2+\lambda}\zeta^{1/2+\varkappa}M^{1/2-\varkappa}N^{3/2+\lambda-2\varkappa+3\eta}$$
$$+M^{-1/(2\varepsilon)}N^{3+\eta}H^{2+\eta}$$

as claimed in (114).

Now suppose that $v \ll MN \ll v$, $H \leqslant vx^{-1/2+4\eta}$, $x \leqslant \zeta \leqslant 2x$, $y \leqslant v < x^{3/4}$, and that (115) holds. In particular, (113) holds, since

$$MN^2 \ll vN \ll v^{106/75}x^{-31/300} \ll x^{287/300}.$$

Moreover, $M > v^{1/2} \gg N$ so that the last term in (114) can be absorbed into the first; also $HN^{-1} \ll 1$. Thus, taking

$$\varkappa = 11/53, \quad \lambda = 33/53$$

(compare [10], p. 265) we have

$$|S|^2 \ll M^2(NH)^{1+\eta}+(vx^{-1/2})^{1/2+\lambda}x^{1/2+\varkappa+8\eta}v^{1/2-\varkappa}N^{1+\lambda-\varkappa}$$
$$\ll v^3x^{-1/2+\eta}N^{-1}+v^{75/53}x^{31/212+8\eta}N^{75/53}.$$

We obtain

$$|S| \ll vx^{-3\eta}$$

on inserting the bounds (115). This completes the proof of Lemma 14.

LEMMA 15. *Make all the hypotheses of Lemma 14. Let S_1 be the subsum of S defined by the extra condition of summation*

$$v < mn \leqslant v_1 \quad (\text{where } v_1 \leqslant ev);$$

then for $H \leqslant vx^{-1/2+4\eta}$, $x \leqslant \zeta \leqslant 2x$, $x^{1/2} \leqslant v < x^{3/4}$ and N satisfying (115) we have

$$|S_1| \ll vx^{-2\eta}.$$

Proof. We adapt a device from [5], p. 165: for $A > 0$, $\beta > 0$ and real α,

$$\int_{-A}^{A} e^{it\alpha}\frac{\sin t\beta}{t}\,dt = \begin{cases}\pi+O\left(A^{-1}(\beta-|\alpha|)^{-1}\right) & \text{if } |\alpha| < \beta,\\ O\left(A^{-1}(|\alpha|-\beta)^{-1}\right) & \text{if } |\alpha| > \beta.\end{cases}$$

Putting $\beta = \log u$, we find that for any complex valued function $f(m,n)$ $(M < m \leqslant M_1, N < n \leqslant N_1)$, we have

(125) $$\sum_{\substack{M<m\leqslant M_1\\ mn\leqslant u}}\sum_{N<n\leqslant N_1} f(m,n)$$
$$= \int_{-A}^{A}\sum_{M<m\leqslant M_1}\sum_{N<n\leqslant N_1} f(m,n)(mn)^{it}\frac{\sin(t\log u)}{\pi t}\,dt$$
$$+O\left(A^{-1}\sum_{M<m\leqslant M_1}\sum_{N<n\leqslant N_1}|f(m,n)|\left|\log\frac{mn}{u}\right|^{-1}\right).$$

If u is of the form $u = k+1/2$, where k is an integer, $0 \leqslant k \ll MN$, then

$$\left|\log\frac{mn}{u}\right| \gg \frac{1}{u} \gg \frac{1}{MN}$$

and

$$\sin(t\log u) \ll \min(1, |t|\log 2MN)$$

so that the right hand side of (125) is

(126) $$\ll \int_{-A}^{A}\left|\sum_{M<m\leqslant M_1}\sum_{N<n\leqslant N_1} f(m,n)m^{it}n^{it}\right|\min(1/|t|, \log 2MN)\,dt$$
$$+\frac{MN}{A}\sum_{m,n}|f(m,n)|.$$

We take

$$f(m,n) = \sum_{0<h\leqslant H} a_m b_n e\left(\frac{h\zeta}{mn}\right).$$

The integrand in (126) may then be estimated via Lemma 14. With $A = x^3$, we obtain

$$\sum_{\substack{M<m\leqslant M_1\\ mn\leqslant u}} \sum_{N<n\leqslant N_1} \sum_{0<h\leqslant H} a_m b_n e\left(\frac{h\zeta}{mn}\right)$$

$$\ll \int_{-A}^{A} vx^{-3\eta} \min(1/|t|, \log v)\, dt + v^2 HA^{-1} \ll vx^{-2\eta}.$$

Lemma 15 now follows on subtracting this result for $u = [v_1]+1/2$ and $u = [v]+1/2$.

Lemma 16. *Let*

$$(127)\qquad D = \begin{cases} vx^{-\varepsilon} & (x^{\varphi} \leqslant v \leqslant x^{3/5-\varepsilon}),\\ x^{13/2-\varepsilon}v^{-10} & (x^{3/5-\varepsilon} < v \leqslant x^{0.616}),\\ x^{19/100-\varepsilon}v^{6/25} & (x^{0.616} < v \leqslant x^{6127/9788}),\\ x^{683/370-\varepsilon}v^{-89/37} & (x^{6127/9788} < v \leqslant x^{0.646}),\\ x^{313/740-\varepsilon}v^{-15/74} & (x^{0.646} < v \leqslant x^{0.661}),\\ x^{11/28-\varepsilon}v^{-11/70} & (x^{0.661} < v \leqslant x^{0.7}).\end{cases}$$

Then with $z = D^{1/3}$, we have

$$(128)\qquad S(\mathscr{A}, z) \leqslant \frac{2y}{\log D}(1+\varepsilon).$$

Proof. First of all, for $x^{0.616} \leqslant v \leqslant x^{0.646}$, the part of the proof of Lemma 11 concerned with the range $x^{8/13} < v \leqslant x^{0.62}$ may be repeated verbatim with U replaced by

$$U' = x^{-31/300-20\eta} v^{31/75};$$

we simply substitute Lemma 15 for Lemma 9. Since (127) implies (85) (with U' instead of U) for $x^{0.616} < v \leqslant x^{0.646}$, Lemma 11 holds good with (64) replaced by (127).

With D given by (127), we can combine (65) and (62) to get (59) (under the conditions (60), (61)). (We have (63) with this definition of D.) As explained after (61), this implies (57). Now (128) follows on combining (56) and (57).

In the next section, z and D are as in Lemma 16, and we write

$$\theta = (\log v)/L.$$

5. Asymptotic formulae for almost primes. Let

$$\Delta(y) = \pi(y) - \int_2^y dt/(\log t),$$

then for $2 < a < b$, and monotonic f on $(a, b]$,

$$\sum_{a<p\leqslant b} f(p) - \int_a^b \frac{f(y)}{\log y}\, dy = \int_a^b f(y)\, d\Delta(y)$$

$$= f(b)\Delta(b) - f(a)\Delta(a) + \int_a^b \Delta(y)\, df(y)$$

$$\ll (|f(a)| + |f(b)|)\, b/(\log b)^3.$$

Thus, for $z \leqslant x^{\sigma} < x^{\tau} < v^{1/2}$,

$$(129)\qquad S_2(\sigma, \tau) = \sum_{\substack{x^{\sigma}<p_1\leqslant x^{\tau}\\ v<p_1p_2\leqslant ev}} \frac{1}{p_1p_2} = \sum_{x^{\sigma}<p_1\leqslant x^{\tau}} \frac{1}{p_1} \int_{v/p_1}^{ev/p_1} \frac{1}{t\log t}\, dt\,(1+O(L^{-1}))$$

$$= \sum_{x^{\sigma}<p_1\leqslant x^{\tau}} \frac{1}{p_1 \log v/p_1}(1+O(L^{-1}))$$

$$= \int_{x^{\sigma}}^{x^{\tau}} \frac{1}{t\log(v/t)\log t}\, dt\,(1+O(L^{-1}))$$

$$= \frac{1}{L}\int_{\sigma}^{\tau} \frac{(1+O(L^{-1}))}{\alpha(\theta-\alpha)}\, d\alpha = \frac{1}{\theta L}\left(\log\frac{\tau}{\sigma} + \log\left(\frac{\theta-\sigma}{\theta-\tau}\right)\right)(1+O(L^{-1})).$$

An elaboration of this argument leads to the formula

$$(130)\qquad \sum \frac{1}{p_1p_2p_3} = \frac{(1+O(L^{-1}))}{L} \int_{\sigma_1}^{\theta/3} \int_{\alpha_1}^{(\theta-\alpha_1)/2} \frac{d\alpha_1\, d\alpha_2}{\alpha_1\alpha_2(\theta-\alpha_1-\alpha_2)},$$

where $\sigma_i < \alpha_i \leqslant \tau_i$ $(i = 1, 2)$ and the sum is extended over p_1, p_2, p_3 satisfying

$$x^{\sigma_1} < p_1 \leqslant x^{\tau_1}, \quad x^{\sigma_2} < p_2 \leqslant x^{\tau_2},$$

$$p_1 < p_2 < p_3, \quad v < p_1p_2p_3 \leqslant ev,$$

whenever $0 < \sigma_i < \tau_i < 1$ $(i = 1, 2)$.

Consider, in particular,

$$\varphi \leqslant \theta \leqslant 0.676,$$

$$x^{\sigma} = \max(z,\, vx^{-1/2+\varepsilon}), \qquad x^{\tau} = \max(x^{2-\varepsilon}v^{-3},\, x^{-31/300-\varepsilon}v^{31/75}).$$

It is easily verified that $\sigma < \tau < \theta/2$. We have $x^{\sigma} = z$ precisely when $\theta \leqslant 8/13 - 4\varepsilon/13$. We write

$$T_2(v) = \sum_{\substack{x^{\sigma} < p_1 \leqslant x^{\tau} \\ p_1 < p_2,\, p_1 p_2 \in \mathscr{A}}} 1,$$

$$T_3(1, v) = \sum_{\substack{x^{\sigma} < p_1 \leqslant x^{\tau} \\ p_1 < p_2 < p_3,\, p_1 p_2 p_3 \in \mathscr{A}}} 1,$$

$$T_3(2, v) = \sum_{\substack{z < p_1 \leqslant vx^{-1/2+\varepsilon},\, vx^{-1/2+\varepsilon} < p_2 \leqslant x^{\tau} \\ p_1 < p_2 < p_3,\, p_1 p_2 p_3 \in \mathscr{A}}} 1.$$

We deduce from (53) that

$$(131) \qquad \sum_{v < p \leqslant ev} N(p) \leqslant S(\mathscr{A}, z) - T_2(v) - T_3(1, v) - T_3(2, v).$$

Moreover, since $\tau < \theta/2$,

$$(132) \qquad T_2(v) = \sum_{x^{\sigma} < p_1 \leqslant x^{\tau},\, v < p_1 p_2 \leqslant ev} \; \sum_{x < m p_1 p_2 \leqslant x+y} 1$$

$$= \sum_{x^{\sigma} < p_1 \leqslant x^{\tau},\, v < p_1 p_2 \leqslant ev} \left\{ \frac{y}{p_1 p_2} - \psi\left(\frac{x+y}{p_1 p_2}\right) + \psi\left(\frac{x}{p_1 p_2}\right) \right\}$$

$$= yS_2(\sigma, \tau) - \sum_{x^{\sigma} < m \leqslant x^{\tau}} b_m \sum_{\substack{vx^{-\tau} < n \leqslant evx^{-\sigma} \\ v < mn \leqslant ev}} b_n \left\{ \psi\left(\frac{x+y}{mn}\right) - \psi\left(\frac{x}{mn}\right) \right\}.$$

Here b_n is the indicator function of the prime numbers. By an application of Lemmas 7, 9 and 15, and the inequality (50), we obtain the estimate $O(yx^{-\eta})$ for the sum over m, n in (132). We therefore deduce from (129) and (132) that

$$(133) \qquad T_2(v) = \frac{y}{\theta L}\left(\log\frac{\tau}{\sigma} + \log\left(\frac{\theta - \sigma}{\theta - \tau}\right)\right)(1 + O(L^{-1})).$$

For $\theta \geqslant 3/5$ we have $\tau < \theta/3$. By a slight variant of the argument leading to (133), then,

$$(134) \qquad T_3(1, v) = \frac{y}{L}\int_{\sigma}^{\tau}\int_{\alpha_1}^{(\theta-\alpha_1)/2} \frac{d\alpha_1\, d\alpha_2}{\alpha_1 \alpha_2 (\theta - \alpha_1 - \alpha_2)}(1 + O(L^{-1})).$$

For $\theta \geqslant 8/13$, it is easy to see that, writing $z = x^{\sigma_1}$,

$$(135) \qquad T_3(2, v) = \frac{y}{L}\int_{\sigma_1}^{\theta-1/2+\varepsilon}\int_{\theta-1/2+\varepsilon}^{\tau} \frac{d\alpha_1\, d\alpha_2}{\alpha_1 \alpha_2 (\theta - \alpha_1 - \alpha_2)}(1 + O(L^{-1})).$$

Proof of Proposition 2. We have

$$\sum_{x^{\varphi} < p \leqslant x^{7/10}} N(p)\log p \leqslant \sum_{k=[\varphi L]}^{[7L/10]} \sum_{e^k < p \leqslant e^{k+1}} N(p)(k+1)$$

$$\leqslant \sum_{k=[\varphi L]}^{[7L/10]} (k+1)\left\{ \frac{y \cdot 2(1+\varepsilon)}{\log D(e^k)} - T_2(e^k) - T_3(1, e^k) - T_3(2, e^k) \right\}$$

by Lemma 16 and (131). We now use (133), (134) and (135). It is a straightforward matter to deduce that

$$(136) \qquad \sum_{x^{\varphi} < p \leqslant x^{7/10}} N(p)\log p$$

$$\leqslant yL(1 + C_{16}\varepsilon)\sum_{j=1}^{6} K_j - yL(1 - C_{16}\varepsilon)\Big(\sum_{j=1}^{3} L_j + \sum_{j=1}^{7} M_j\Big).$$

Here

$$K_1 = \int_{13/22}^{3/5} 2\,d\theta = 1/55 < 0.01819,$$

$$K_2 = \int_{3/5}^{0.616} \frac{4\theta}{13 - 20\theta}\,d\theta < 0.04694,$$

$$K_3 = \int_{0.616}^{6127/9788} \frac{200\theta}{19 + 240\theta}\,d\theta < 0.03653,$$

$$K_4 = \int_{6127/9788}^{0.646} \frac{740\theta}{683 - 890\theta}\,d\theta < 0.08078,$$

$$K_5 = \int_{0.646}^{0.661} \frac{1480\theta}{313 - 150\theta}\,d\theta < 0.06749,$$

$$K_6 = \int_{0.661}^{0.7} \frac{280\theta}{11(5 - 2\theta)}\,d\theta < 0.18567,$$

$$L_1 = \int_{13/22}^{3/5} \log\left(\frac{2 - 3\theta}{2\theta - 1}\right)d\theta > 0.00101,$$

$$L_2 = \int_{3/5}^{8/13} \log\left(\frac{13(2-3\theta)}{2(13-20\theta)}\right) d\theta > 0.00476,$$

$$L_3 = \int_{8/13}^{0.676} \log\left(\frac{31(\theta-1/4)}{88(\theta-1/2)(\theta+31/176)}\right) d\theta > 0.00961,$$

$$M_1 = \int_{3/5}^{8/13} \theta \int_{(13/6)-10\theta/3}^{2-3\theta} \int_{\alpha_1}^{(\theta-\alpha_1)/2} \frac{d\alpha_1\, d\alpha_2\, d\theta}{\alpha_1 \alpha_2 (\theta-\alpha_1-\alpha_2)} > 0.00278,$$

$$M_2 = \int_{8/13}^{0.676} \theta \int_{\theta-1/2}^{31\theta/75-31/300} \int_{\alpha_1}^{(\theta-\alpha_1)/2} \frac{d\alpha_1\, d\alpha_2\, d\theta}{\alpha_1 \alpha_2 (\theta-\alpha_1-\alpha_2)} > 0.00789,$$

$$M_3 = \int_{8/13}^{0.616} \theta \int_{(13/6)-10\theta/3}^{\theta-1/2} \int_{\theta-1/2}^{2-3\theta} \frac{d\alpha_1\, d\alpha_2\, d\theta}{\alpha_1 \alpha_2 (\theta-\alpha_1-\alpha_2)}$$

(this is so small that we use only $M_3 > 0$),

$$M_4 = \int_{0.616}^{6127/9788} \theta \int_{(19/300)+2\theta/25}^{\theta-1/2} \int_{\theta-1/2}^{31\theta/75-31/300} \frac{d\alpha_1\, d\alpha_2\, d\theta}{\alpha_1 \alpha_2 (\theta-\alpha_1-\alpha_2)} > 0.00020,$$

$$M_5 = \int_{6127/9788}^{0.646} \theta \int_{(683/1110)-89\theta/111}^{\theta-1/2} \int_{\theta-1/2}^{31\theta/75-31/300} \frac{d\alpha_1\, d\alpha_2\, d\theta}{\alpha_1 \alpha_2 (\theta-\alpha_1-\alpha_2)} > 0.00097,$$

$$M_6 = \int_{0.646}^{0.661} \theta \int_{(313-150\theta)/2220}^{\theta-1/2} \int_{\theta-1/2}^{31\theta/75-31/300} \frac{d\alpha_1\, d\alpha_2\, d\theta}{\alpha_1 \alpha_2 (\theta-\alpha_1-\alpha_2)} > 0.00072,$$

$$M_7 = \int_{0.661}^{0.676} \theta \int_{(11/84)-11\theta/210}^{\theta-1/2} \int_{\theta-1/2}^{31\theta/75-31/300} \frac{d\alpha_1\, d\alpha_2\, d\theta}{\alpha_1 \alpha_2 (\theta-\alpha_1-\alpha_2)} > 0.00028.$$

Thus

$$\sum_{j=1}^{6} K_j - \sum_{j=1}^{3} L_j - \sum_{j=1}^{7} M_j < 0.40738,$$

and Proposition 2 follows from (136).

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