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Discrepancy of normal numbers

by

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Introduction. If $\omega = (x_n)$ is a sequence of real numbers in $[0, 1)$ and c_M denotes the characteristic function of the set M, we denote by $D_N(\omega)$ the term

$$\sup_{0 \leqslant u < v < 1} \left| \frac{1}{N} \sum_{1 \leqslant n \leqslant N} c_{[u,v)}(x_n) - (v-u) \right|$$

and call it the *discrepancy of the sequence* (x_n).

A real number α in $[0, 1)$ is called *normal in the scale of* 10, if

$$\lim D_N\big((\{10^n \alpha\})\big) = 0 \quad \text{for} \quad N \to \infty.$$

We write $D(N, \alpha)$ for $D_N\big((\{10^n \alpha\})\big)$ and call it the *discrepancy of the number* α.

It is well known that α is normal in the scale of 10, iff the sequence $(10^n \alpha)$ is uniformly distributed modulo 1. This is equivalent to the condition that, when α has a decimal representation of $\alpha = 0.a_1 a_2 \ldots$, for all k each of the 10^k sequences of k digits occurs as a subblock of $a_1, a_2, \ldots$ with a limiting relative frequency of 10^{-k}.

We will consider infinite decimals having the form

$$\omega(f) = 0.f(1)f(2)\ldots$$

where each $f(n)$ is represented as a decimal and the digits of $f(1)$ are succeeded by those of $f(2)$, and so on.

Davenport and Erdős [2] have shown that $\omega(f)$ is normal to the base 10 for any polynomial $f(n)$ all of whose values for $n = 1, 2, \ldots$ are positive integers. This means $\lim D\big(N, \omega(f)\big) = 0$ for $N \to \infty$. A more precise estimation of $D\big(N, \omega(f)\big)$ has been established by Schoissengeier [4].

In this paper we prove

THEOREM 1. *Let $f(n)$ be a polynomial in n with rational coefficients, f not constant and let (d_n) be a bounded sequence of rational numbers, such that*

$f(n)+d_n$ is a positive integer for all $n \geqslant 1$. Then

$$D(N, \omega) = O\left(\frac{1}{\log N}\right) \quad \textit{for} \quad \omega = 0.(f(1)+d_1)(f(2)+d_2)\ldots$$

In particular, ω is normal to the base 10.

To prove this, we use an estimation of trigonometric sums by H. Weyl. The main ideas of this proof can be applied to other numbers, which leads to the following result:

THEOREM 2. *Let $\delta \in (0, 1]$ and $f\colon [1, \infty) \to \boldsymbol{R}$ be differentiable two times, f' monotone, f'' continuous and $c_1 x^\delta < f(x) < c_2 x^\delta$, $c_3 x^{\delta-1} < f'(x) < c_4 x^{\delta-1}$, $|f''(x)| < c_5 x^{\delta-2}$ for certain constants $c_1, c_2, c_3, c_4, c_5 > 0$ and for all sufficiently large x. Furthermore let (d_n) be a bounded sequence of real numbers and $f(n)+d_n$ be a positive integer for all n. Then*

$$D(N, \omega) = O\left(\frac{1}{\log N}\right) \quad \textit{for} \quad \omega = 0.(f(1)+d_1)(f(2)+d_2)\ldots$$

In particular, ω is normal to the base 10.

In this way many normal numbers can be constructed, such as

$$0.[a][a2^\sigma][a3^\sigma]\ldots \quad \text{for} \quad a > 0,\ 0 < \sigma \leqslant 1.$$

The estimations of $D(N, \omega)$ in Theorem 1 and Theorem 2 cannot be improved. We show:

THEOREM 3. *Let $f(n)$ be a linear polynomial with rational coefficients, $f(n) \geqslant 1$ for $n = 1, 2, \ldots$ and $\omega = 0.[f(1)][f(2)]\ldots$ Then*

$$D(N, \omega) > K\frac{1}{\log N} \quad \textit{for all } N \textit{ and a constant } K > 0.$$

Furthermore

THEOREM 4. *Let $\delta \in (0, 1)$, $f\colon [1, \infty) \to \boldsymbol{R}$ satisfy the conditions in Theorem 2 and $\omega = 0.[f(1)][f(2)]\ldots$ Then*

$$D(N, \omega) > K\frac{1}{\log N}$$

for an infinite number of N's and a constant $K > 0$.

Notations. Let F, G be functions defined on subsets of $[1, \infty)$ taking values in $[0, \infty)$. Then we write $F \ll G$ instead of $F = O(G)$ and define $F \sim G$ to denote $G \ll F \ll G$.

For any function $f\colon \boldsymbol{N} \to [1, \infty)$ we consider the decimal

$$\omega([f]) = 0.[f(1)][f(2)]\ldots[f(n)][f(n+1)]\ldots$$

formed by writing the decimal expressions of the numbers $[f(1)], [f(2)], \ldots$

successively at the right side of the radix point. For more convenient notation we separate $[f(n)]$ and $[f(n+1)]$ by commas.

For any positive integers n, l let $T(n)$ denote the sum of the numbers of digits of $[f(1)], \ldots, [f(n)]$ and let u_l be the least positive integer x for which the number of digits of $[f(x)]$ is at least l. When f is strictly increasing, $f(u_l-1) < 10^{l-1} \leqslant f(u_l)$ for l sufficiently large.

An example will illustrate these definitions: Let $f(x) = \frac{1}{2}(x^4+6)$ and $\omega = \omega([f])$. Then $\omega = 0.3, 11, 43, 131, 315, 651, 1203, \ldots$ and $u_1 = 1$, $u_2 = 2$, $u_3 = 4$, $u_4 = 7$; $T(1) = 1$, $T(2) = 3$, $T(3) = 5$, $T(4) = 8$, $T(5) = 11$, $T(6) = 14$, $T(7) = 18$.

Let $\omega([f]) = \sum_{i \in \boldsymbol{N}} a_i 10^{-i}$ be the decimal representation of $\omega([f])$, where all a_i are digits, which means $0 \leqslant a_i \leqslant 9$. We will call the integers in the intervall $[0, 9]$ *digits*. In order to prove Theorems 1 and 2 it will be necessary to calculate the frequency of occurrences of certain blocks of digits as subblocks of the sequence (a_i). Let $k \in \boldsymbol{N}$, $b_1, b_2, \ldots, b_k$ be digits and $B = (b_1, \ldots, b_k)$ be the finite sequence formed by $b_1, \ldots, b_k$. We call B a *block of digits* having length k and define $\beta(B) = b_1 10^{k-1} + b_2 10^{k-2} + \ldots + b_k$. For $1 \leqslant S \leqslant T$ let $N(B, \omega, S, T)$ be the number of subblock-occurrences of B in $(a_S, a_{S+1}, \ldots, a_T)$, which is the number of i's satisfying $S \leqslant i \leqslant T-k+1$ and $(a_i, a_{i+1}, \ldots, a_{i+k-1}) = B$. For $k \leqslant l$, $u_l \leqslant v < u_{l+1}$ we write $N(B, v)$ instead of $N(B, \omega, T(u_l-1)+1, T(v))$. Obviously $N(B, v)$ is the number of subblock-occurrences of B in the sequences of digits formed by $[f(u_l)]$, $[f(u_l+1)], \ldots, [f(v)]$. There are two possibilities for B to occur as subblock of this sequence:

(a) B is subblock of $[f(u)]$ for a certain u $(u_l \leqslant u \leqslant v)$ and therefore, B does not straddle any comma in $[f(u_l)], \ldots, [f(v)]$. Let $N_1(B, v)$ denote the number of those subblock-occurrences of B;

(b) B is subblock of $[f(u)], [f(u+1)]$ for a certain u $(u_l \leqslant u < v)$, straddling the comma between $[f(u)]$ and $[f(u+1)]$. Let $N_2(B, v)$ denote the number of those subblock-occurrences of B.

Evidently $N(B, v) = N_1(B, v) + N_2(B, v)$ and

$$N(B, \omega, S, U) = N(B, \omega, S, T) + N(B, \omega, T, U) + O(k),$$

when $S \leqslant T \leqslant U$ and B is a block of digits having length k. Let f, ω be as in the example given above and let $v = 6$, $B = (31)$. Then $u_3 \leqslant v < u_4$, $[f(u_3)], \ldots, [f(v)] = 131,\ 315,\ 651$ and hence $N_1(B, v) = 2$, $N_2(B, v) = 0$, $N(B, v) = 2$; $N_1(B, \omega, 1, 14) = N_2(B, \omega, 1, 14) = 2$, $N(B, \omega, 1, 14) = 4$.

Proof of Theorem 1. Let $g \geqslant 1$; $\alpha_0, \alpha_1, \ldots, \alpha_g$ be rational numbers and

$$f(x) = \alpha_g x^g + \ldots + \alpha_1 x + \alpha_0.$$

Without loss of generality we may assume that $f(x)$ is monotonically increasing. Then

(1) $f(x) \sim x^g$, $f'(x) \sim x^{g-1}$; $f^{-1}(x) \sim x^{1/g}$, $(f^{-1})'(x) \sim x^{(1/g)-1}$

and $(f^{-1})''(x) \sim x^{(1/g)-2}$ when $g > 1$.

Thus, for l sufficiently large obviously $u_l < u_{l+1}$, $f(u_l - 1) < 10^{l-1} \leqslant f(u_l)$, the number of digits of $f(x)$ is l if and only if $u_l \leqslant x < u_{l+1}$; $T_l \sim l10^{l/g}$.

In order to prove Theorem 1 we first show

LEMMA 1. *Let $u, v, k, l, n \in \mathbf{N}$, $b \in \mathbf{Z}$ and $0 \leqslant u < v$,*

$$10^{l-1} \leqslant f(u) \leqslant f(v) < 10^l, \quad k \leqslant n \leqslant l, \quad 0 \leqslant b < 10^n.$$

Then

(2) $$\sum_{\substack{u \leqslant x < v \\ [f(x)] \equiv b + t(10^n)}} \sum_{0 \leqslant t < 10^{n-k}} 1 = 10^{-k}(v-u) + O\big((v-u)(10^{-n\varepsilon} + 10^{n-k-l})\big)$$

for an $\varepsilon > 0$. ε and the implicit O-constant do not depend on u, v, k, l, n, b.

Proof. Let $S(u, v)$ denote the expression on the left side of (2). When $u_0 = [f^{-1}(\frac{1}{2}10^{l-1})]$, obviously $S(u, v) = S(u_0, v) - S(u_0, u)$ and so it suffices to show (2) for $u = u_0$, $10^{l-1} \leqslant f(v) < 10^l$. In this case

(3) $$v - u \sim 10^{l/g}.$$

The proof of (2) splits into two cases.

Case A. $g = 1$ or $n > l(1 - 1/4g)$. Let

$$U = [(f(u) - b)10^{-n}] + 1, \quad V = [(f(v-1) - b)10^{-n}] - 1$$

and

$$d(Y) = f^{-1}(Y10^n + b + 10^{n-k}) - f^{-1}(Y10^n + b) \quad \text{for} \quad U - 1 \leqslant Y \leqslant V + 1.$$

Then

(4) $$S(u, v) = \sum_{Y \in \mathbf{N}} \sum_{\substack{u \leqslant x < v \\ [f(x) - b] = Y10^n + t}} \sum_{0 \leqslant t < 10^{n-k}} 1$$

$$= \sum_{U \leqslant Y \leqslant V} \big(d(Y) + O(1)\big) + O\big(d(U-1) + d(V+1)\big).$$

For Y fixed we have

$$d(Y) = 10^{n-k}(f^{-1})'(Y10^n + b) + \tfrac{1}{2}10^{2n-2k}(f^{-1})''\big(Y10^n + b + z(Y)\big),$$

where $z(y) \in [0, 10^{n-k}]$. Using (1) and $Y \sim 10^{l-n}$ we get

(5) $(f^{-1})''\big(Y10^n + b + z(Y)\big) \ll 10^{l/g - 2l}$; $d(Y) \ll 10^{l/g - l + n - k}$

and $V - U \ll 10^{l-n}$.

As $d(Y)$ is monotonically decreasing, we get

$$\sum_{U \leqslant Y \leqslant V} d(Y) = \int_U^V d(Y) + O\big(d(U)\big)$$

$$= 10^{-k}\big(f^{-1}(V10^n + b) - f^{-1}(U10^n + b)\big) + O(10^{l/g - l + n - 2k}) + O\big(d(U)\big).$$

As $U10^n + b = f(u) + O(10^n)$ we have

$$f^{-1}(U10^n + b) = u + O(10^{l/g - l + n})$$

and, similarly,

$$f^{-1}(V10^n + b) = v + O(10^{l/g - l + n}).$$

With (4), (5) we obtain

$$S(u, v) = 10^{-k}(v - u) + O(10^{l/g + n - l - k} + 10^{l-n}).$$

Now, (2) follows from (3) and $l - n \leqslant (l/g) - \varepsilon n$ when $g = 1$ or $n > l(1 - 1/(4g))$.

Case B. $n \leqslant l(1 - 1/(4g))$ and $g \geqslant 2$. We choose $N \in \mathbf{N}$ in such a way that all coefficients of Nf are integers. Furthermore let $F = Nf$, $T = N10^{n-k}$, $M = N10^n$, $B = Nb$ and $e(z) = e^{2\pi i z}$ for all real z. So we get

(6) $$S(u, v) = \sum_{\substack{u \leqslant x < v \\ F(x) \equiv B + t(M)}} \sum_{0 \leqslant t < T} 1 = \sum_{0 \leqslant t < T} \sum_{u \leqslant x < v} \frac{1}{M} \sum_{1 \leqslant h \leqslant M} e\left((F(x) - B - t)\frac{h}{M}\right)$$

$$= (v - u)\frac{T}{M} + \frac{R}{M},$$

where

$$R = \sum_{1 \leqslant h < M} e\left(-B\frac{h}{M}\right)\frac{e(-Th/M) - 1}{e(-h/M) - 1} \sum_{u \leqslant x < v} e\left(F(x)\frac{h}{M}\right).$$

Using

$$|e^{iw} - e^{iz}| = 2\sin\left|\frac{w - z}{2}\right|$$

and writing $E(m, h, t)$ instead of $\Big|\sum_{1 \leqslant x \leqslant m} e\big(F(x + u - 1)(h/t)\big)\Big|$ we get

(7) $$R \ll \sum_{1 \leqslant h < M} \left(\sin \pi\frac{h}{M}\right)^{-1} \left|\sum_{u \leqslant x < v} e\left(F(x)\frac{h}{M}\right)\right|$$

$$= \sum_{t \mid M} \sum_{\substack{1 \leqslant h < t \\ (h, t) = 1}} E(v - u, h, t)\left(\sin \pi\frac{h}{t}\right)^{-1}$$

$$= \sum_{t \mid M} \sum_{\substack{1 \leqslant h < t \\ (h, t) = 1}} \left(\sin \pi\frac{h}{t}\right)^{-1} \left[\frac{v - u}{t}\right] E(t, h, t)$$

$$+\sum_{\substack{t\mid M\\(h,t)=1,t<v-u}}\sum_{1\leqslant h<t}\left(\sin\pi\frac{h}{t}\right)^{-1}E\left(t\left\{\frac{v-u}{t}\right\},h,t\right)$$

$$+\sum_{\substack{t\mid M\\(h,t)=1,t>v-u}}\sum_{1\leqslant h<t}\left(\sin\pi\frac{h}{t}\right)^{-1}E(v-u,h,t).$$

For further calculation we use an estimation of trigonometric sums (see [5]):

Let $p_0, p_1, \ldots, p_g$ be real numbers, $g>1$; $p_g = a/q$, $(a, q) = 1$ for some $a, q \in N$ and

$$p(x) = p_g x^g + \ldots + p_1 x + p_0.$$

Then

$$\Big|\sum_{1\leqslant x\leqslant M} e(p(x))\Big| \ll M^{1+\varepsilon}(M^{-1}+q^{-1}+qM^{-g})^{1/K} \quad \text{where} \quad K = 2^{g-1}.$$

Choosing ε sufficiently small, but fixed, we get

(8) $$\Big|\sum_{1\leqslant x\leqslant M} e(p(x))\Big| = O(M^{1-\sigma}) \quad \text{when} \quad M^{1/8} < q < M^{g-1/8}$$

for an $\sigma = \sigma(g) > 0$. The implicit O-constant in (8) depends on g only.

Let h, t, a, q be integers, $1 \leqslant h < t$, $(h, t) = 1$, $t\mid M$ and $a/q = \alpha_g N\frac{h}{t}$, $(a, q) = 1$. Then $t(\alpha_g N)^{-1} \leqslant q \leqslant t$ because $(h, t) = 1$.

For $t > v-u$ and l sufficiently large we get

$$(v-u)^{1/8} < t^{1/8} < t(\alpha_g N)^{-1} < q; \qquad q \leqslant N10^n \ll 10^{l-l/4g} \ll (v-u)^{g-1/4}$$

and therefore $q \leqslant (v-u)^{g-1/8}$. Hence (8) yields

(9) $$E(v-u, h, t) \ll (v-u)^{1-\sigma} \quad \text{for} \quad t > v-u.$$

For $t^{1-\sigma} < s < t$ we have

$$s^{1/8} < t < t^{(1-\sigma)(g-1/8)} < s^{g-1/8}$$

(σ can be choosen arbitrarily small) and therefore $E(s, h, t) \ll s^{1-\sigma}$. So we have $E(s, h, t) \ll t^{1-\sigma}$ for $s \leqslant t$. From this, (7) and (9) we deduce, using $\sum_{1\leqslant h<t}\left(\sin\pi\frac{h}{t}\right)^{-1} \ll t\log t$:

$$R \ll \sum_{t\mid M}(v-u)t^{1-\sigma}\log t + \sum_{\substack{t\mid M\\t<v-u}} tt^{1-\sigma}\log t + \sum_{\substack{t\mid M\\t>v-u}} t(v-u)^{1-\sigma}\log t$$

$$\ll \log M\left((v-u)M^{1-\sigma} + (v-u)^{1-\sigma}M\log M\right).$$

As $(v-u)^{-\sigma} \ll 10^{-\sigma(l/g)}$ and $n \leqslant l$ we get $R \ll (v-u)10^{n-n\varepsilon}$ for an $\varepsilon > 0$. Now (2) follows immediately from (6) and the proof of Lemma 1 is completed. ∎

Now we prove Theorem 1 for $w = w([f]) = 0.[f(1)][f(2)]\ldots$ Let $B = (b_1, b_2, \ldots, b_k)$ be a block of digits and

$$b = \beta(B) = b_1 10^{k-1} + b_2 10^{k-2} + \ldots + b_k.$$

When $k \leqslant l$, $u_l \leqslant v < u_{l+1}$ we get, using Lemma 1

(10) $$N_1(B, v) = \sum_{0\leqslant j\leqslant l-k}\ \sum_{u_l\leqslant x\leqslant v}\ \sum_{\substack{0\leqslant t<10^{l-k-j}\\ [f(x)]\equiv b10^{l-k-j}+t(10^{l-j})}} 1$$

$$= 10^{-k}(l-k+1)(v-u_l+1) + \sum_{0\leqslant j\leqslant l-k} O(10^{l/g-(l-j)\varepsilon} + 10^{l/g-k-j})$$

$$= l(v+1-u_l)10^{-k} + O(10^{l/g-k\varepsilon}).$$

Furthermore

$$N_2(B, v) \leqslant N_2(B, u_{l+1}-1) \leqslant \sum_{1\leqslant j<k}\ \sum_{u_l\leqslant x<u_{l+1}} 1 + O(1),$$

where the last summation is taken only over those x for which $b_1, b_2, \ldots, b_{k-j}$ are the last digits in the decimal representation of $[f(x)]$ and $b_{k-j+1}, b_{k-j+2}, \ldots, b_k$ are the first digits in the decimal representation of $[f(x+1)]$. For $b_{k-j+1} \neq 0$ let

$$A_j = [f^{-1}(b_{k-j+1}10^{l-1} + \ldots + b_k 10^{l-j} - 1)],$$

$$B_j = [f^{-1}(b_{k-j+1}10^{l-1} + \ldots + b_k 10^{l-j} + 10^{l-j} - 1)].$$

Using (2) and $B_j - A_j \ll 10^{l/g-j}$ we get

(11) $$N_2(B, v) \ll \sum_{\substack{1\leqslant j<k\\ b_{k-j+1}\neq 0}}\Big(\sum_{\substack{A_j\leqslant x\leqslant B_j\\ [f(x)]\equiv b_1 10^{k-j-1}+\ldots+b_{k-j}(10^{k-j})}} 1\Big) + O(1)$$

$$= \sum_{\substack{1\leqslant j<k\\ b_{k-j+1}\neq 0}}\left(10^{-(k-j)}(B_j - A_j) + O\left((B_j - A_j)(10^{-(k-j)\varepsilon} + 10^{-l})\right)\right)$$

$$\ll 10^{l/g-\varepsilon k}.$$

(10) and (11) yield

(12) $$N(B, v) = 10^{-k}l(v+1-u_l) + O(10^{l/g-\varepsilon k}).$$

For $T \in N$ we may choose $n, u \in N$ in such a way that $k < n/2$, $u_n \leqslant u < u_{n+1}$, $T(u-1) < T \leqslant T(u)$. Then $T = T(u) + O(n)$ and we obtain

(13) $$N(B, \omega, 1, T) = \sum_{1\leqslant l<n} N(B, u_{l+1}-1) + N(B, u) + O(kn) + O(kT_k)$$

$$= \sum_{1\leqslant l<n}\left(10^{-k}l(u_{l+1}-u_l) + O(10^{l/g-\varepsilon k})\right)$$

$$+ 10^{-k}n(u-u_n) + O(10^{n/g-\varepsilon k} + kn + k^2 10^{k/g})$$

$$= (10^{-k}T + O(10^{n/g-\varepsilon k}).$$

For any interval $I \subset [0, 1)$ let

$$\Delta_T(I) = \frac{1}{T}\sum_{0 \leqslant t < T} c_I(\{10^t\omega\}) - |I|, \quad D_T(I) = |\Delta_T(I)|,$$

where c_I denotes the characteristic function of I and $|I|$ is the length of I. When $I = [x, y)$ we write $D_T(x, y)$ for $D_T(I)$ and $\Delta_T(x, y)$ for $\Delta_T(I)$. Let $k \geqslant 1$; $c_1, c_2, \ldots, c_k$ be digits,

$$B = (c_1, \ldots, c_k) \quad \text{and} \quad \gamma = \beta(B)10^{-k} = \sum_{1 \leqslant i \leqslant k} c_i 10^{-i}, \quad I = [\gamma, \gamma + 10^{-k}).$$

When $k = O(\log T)$ we get for sufficiently large T

$$\sum_{0 \leqslant t < T} c_I(\{10^t\omega\}) = \sum_{\substack{1 \leqslant t \leqslant T \\ (c_1, \ldots, c_k) = (a_t, \ldots, a_{t+k-1})}} 1 = N(B, \omega, 1, T) + O(k)$$

and because of $T \sim n10^{n/g}$, $n \sim \log T$, (13) we obtain

$$(14) \quad D_T(\gamma, \gamma + 10^{-k}) = \left|\frac{1}{T}N(B, \omega, 1, T) - 10^{-k} + O(k/T)\right| \ll 10^{-\varepsilon k}\frac{1}{\log T}.$$

Now let $\beta \in [0, 1)$, $h \in \boldsymbol{N}$, $\beta 10^h \in \boldsymbol{N}$, $\beta = \sum_{1 \leqslant k \leqslant h} b_k 10^{-k}$ be the decimal representation of β and

$$\beta_{k,j} = \sum_{1 \leqslant i \leqslant k} b_i 10^{-i} + j10^{-j} \quad \text{for} \quad 1 \leqslant k \leqslant h,\ 0 \leqslant j \leqslant b_k.$$

As $\beta_{1,0} = 0$, $\beta_{k,b_k} = \beta_{k+1,0}$ for $1 \leqslant k < h$ we get from (14) when $h = O(\log T)$

$$D_T(0, \beta) = \sum_{1 \leqslant k \leqslant h}\sum_{0 \leqslant j < b_k} D_T(\beta_{k,j}, \beta_{k,j+1}) \ll \sum_{1 \leqslant k \leqslant h} 10^{-\varepsilon k}\frac{1}{\log T} \ll \frac{1}{\log T}.$$

Finally let γ be any real number in $[0, 1)$ and $T \in \boldsymbol{N}$. Let $h = [\log\log T]$ and choose α, β in $[0, 1)$ with $\alpha \leqslant \gamma \leqslant \beta$, $\beta - \alpha = 10^{-h}$, $\alpha 10^h$, $\beta 10^h \in \boldsymbol{Z}$. Then

$$\Delta_T(0, \gamma) \leqslant \frac{1}{T}\sum_{0 \leqslant t < T} c_{[0,\beta)}(\{10^t\omega\}) - \alpha = \Delta_T(0, \beta) + 10^{-h}$$

and, similarly,

$$\Delta_T(0, \gamma) \geqslant \Delta_T(0, \alpha) - 10^{-h}.$$

Therefore we obtain

$$D_T(0, \gamma) \leqslant \max\big(D_T(0, \alpha), D_T(0, \beta)\big) + 10^{-h} \ll \frac{1}{\log T} + 10^{-h} \ll \frac{1}{\log T}.$$

So Theorem 1 is proved for $\omega = 0.[f(1)][f(2)]\ldots$

Let (d_n) be a bounded sequence, $f_1(n) = f(n) + d_n \in \boldsymbol{N}$ for each $n \in \boldsymbol{N}$ and $\omega_1 = 0.\, f_1(1) f_1(2)\ldots$ Obviously (1) holds for f_1 instead of f. Using Lemma 1

one can prove

$$|N(B, \omega, 1, T) - N(B, \omega_1, 1, T)| = O(10^{n/g - \varepsilon k})$$

when B, T, k, n are as in (13). So (13) is valid when f is replaced by f_1. From this, Theorem 1 follows immediately. ∎

Proof of Theorem 2. In this case we have

$$f(x) \sim x^\delta, \quad f'(x) \sim x^{\delta - 1}; \quad f^{-1}(x) \sim x^{1/\delta}, \ (f^{-1})'(x) \sim x^{(1/\delta) - 1}$$

and

$$(f^{-1})''(x) \sim x^{(1/\delta) - 2}.$$

So (2) can be proved for f. (Note that $\delta \leqslant 1$ and consider case A; g just has to be replaced by δ.)

Theorem 2 now can be obtained similarly to Theorem 1. The proof is a straightforward application of the ideas used in the proof of Theorem 1. So we omit the details here. ∎

To show Theorems 3 and 4 it suffices to estimate the difference of the numbers of subblock-occurrences of certain blocks of digits:

LEMMA 2. *Let $\omega \in [0, 1)$ and B_1, B_2 be blocks of digits having same length. If*

$$(15) \qquad |N(B_1, \omega, 1, T) - N(B_2, \omega, 1, T)| > K\frac{T}{\log T}$$

holds for a constant $K > 0$ and an infinite number of [resp. almost all] $T \in \boldsymbol{N}$, then

$$D(T, \omega) > \frac{C}{\log T}$$

for an infinite number [resp. almost all] $T \in \boldsymbol{N}$ and a constant $C > 0$. K and C may depend on B_1, B_2, ω.

Proof. Let k be the length of B_1 and B_2; $\alpha_i = \beta(B_i)10^{-k}$ and $I_i = [\alpha_i, \alpha_i + 10^{-k})$ for $i = 1, 2$. When $T \geqslant k$ and T satisfies (15) we have

$$D(T - k + 1, \omega) \geqslant \max_{i=1,2}\left|\frac{1}{T}\sum_{0 \leqslant j \leqslant T-k} c_{I_i}(\{10^j\omega\}) - 10^{-k}\right|$$

$$= \max_{i=1,2}\left|\frac{1}{T}N(B_i, \omega, 1, T) - 10^{-k}\right|$$

$$> \frac{1}{2T}|N(B_1, \omega, 1, T) - N(B_2, \omega, 1, T)| > \frac{C}{\log(T - k + 1)}$$

for a constant $C > 0$. ∎

Proof of Theorem 3. Let C, D be rational numbers, $f(x) = Cx + D$, $C > 0$, $C + D > 1$. We choose k and two blocks B_1, B_2 of digits with same length k satisfying the following conditions: $\beta(B_1) = (10[f(1)]+1)10^h + r$ for some integers h, r with $h \geq 0$, $0 \leq r < 10^h$ (which means that B has a form like $([f(1)], 1, \ldots)$), $\beta(B_1) = CN$ for an integer N, B_2 is the block consisting of k 0's.

For any block B of k digits and $1 \leq j \leq l-k$, $10^{j-1} \leq m < 10^j$ we define

(16) $$U(j, m, B) = ([f^{-1}] - c_N(f^{-1}))(m10^{l-j} + (\beta(B)+1)10^{l-k-j}) - ([f^{-1}] - c_N(f^{-1}))(m10^{l-j} + \beta(B)10^{l-k-j}),$$

where c_N denotes the characteristic function of N.

$U(j, m, B)$ is the number of all x, for which $10^{l-1} \leq [f(x)] < 10^l$ and $[f(x)]$ has a decimal representation of the form $(m, B, \ldots)$.

Let $v_l = u_{l+1} - 1$ for $l \geq 1$. We deduce

(17) $$N_1(B, v_l) = \sum_{1 \leq j \leq l-k} \sum_{10^{j-1} \leq m < 10^j} U(j, m, B) + \Delta_0(B, l) + O(1),$$

where

$$\Delta_0(B, l) = \begin{cases} \frac{1}{C}10^{l-k} & \text{when} \quad \beta(B) \geq 10^{k-1}, \\ 0 & \text{when} \quad \beta(B) < 10^{k-1}. \end{cases}$$

Now let $T \in N$. There exist $v, n \geq 1$, such that $u_n \leq v < u_{n+1}$, $T = T(v) + O(n)$. Let $[f(v)] = 10^{n-1}z_1 + 10^{n-2}z_2 + \ldots + z_n$ be the decimal representation of $[f(v)]$ and, for $1 \leq j \leq n-k$:

$$A_j = 10^{j-1}z_1 + 10^{j-2}z_2 + \ldots + z_j,$$

$$\gamma_j = z_{j+1}10^{k-1} + z_{j+2}10^{k-2} + \ldots + z_{j+k}.$$

Then

(18) $$N_1(B, v) = \sum_{1 \leq j \leq n-k} \Big(\Big(\sum_{10^{j-1} \leq m < A_j} U(j, m, B)\Big) + \Delta_j(B)\Big) + \Delta_0(B) + O(1),$$

where $\Delta_j(B)$ [resp. $\Delta_0(B)$] denotes the number of all integers x, for which $10^{l-1} \leq [f(x)] \leq v$ and $[f(x)]$ has a decimal representation of $(A_j, B, \ldots)$ [resp. $(B, \ldots)$]. Hence

$$\Delta_j(B) = \begin{cases} U(j, A_j, B) & \text{when} \quad \beta(B) < \gamma_j, \\ v - f^{-1}(A_j10^{n-j} + \beta(B)10^{n-k-j}) + O(1) & \text{when} \quad \beta(B) = \gamma_j, \\ 0 & \text{when} \quad \beta(B) > \gamma_j \end{cases}$$

and

$$\Delta_0(B) \geq 0, \quad \Delta_0(B) = 0 \quad \text{if} \quad \beta(B) < 10^{k-1}.$$

Because of $f^{-1}(Y) = (1/C)(Y-D)$ we have

$$\Delta_j(B) < \frac{1}{C}10^{n-k-j} + O(1) \quad \text{for} \quad 1 \leq j \leq n-k,\ 10^{j-1} \leq m < 10^j.$$

$\beta(B_2) < \beta(B_1)$ implies

$$0 \leq \Delta_j(B_1) \leq \Delta_j(B_2) + O(1)$$

and therefore

(19) $$\Delta_j(B_2) - \Delta_j(B_1) \leq \frac{1}{C}10^{n-k-j} + O(1).$$

As $\beta(B_1) - \beta(B_2) = CN$, we get

(20) $$U(j, m, B_1) = U(j, m, B_2)$$

for $1 \leq j \leq l-k$, $10^{j-1} \leq m < 10^j$ and (17) yields

(21) $$N_1(B_1, v_l) = N_1(B_2, v_l) + \frac{1}{C}10^{l-k} + O(1).$$

Obviously $N_2(B_2, v_l) = 0$. Let $B_1 = (b_1, \ldots, b_k)$ and $i < k$ satisfying $\beta((b_1, \ldots, b_i)) = [f(1)]$. Let M denote the set of all integers m for which $u_l \leq m \leq v_l = u_{l+1} - 1$ and the decimal representation of $[f(m)]$ has the form

$$[f(m)] = (b_{i+1}10^{l-1} + \ldots + b_k10^{l-(k-i)}) + \ldots + (b_1 10^{i-1} + \ldots + b_i).$$

For all $x \in N$

$$[f(1 + (10^i N)x)] = [f(1)] + 10^i CNx \equiv [f(1)](10^i),$$

and for sufficiently large l there exists a positive integer t for which

$$[f(1 + tN10^i)] = [f(1)] + CNt10^i \in f(M).$$

Hence M is not empty and contains at least

$$10^{-i}N^{-1}\Big(\frac{1}{C}10^{l-(k-i)} + O(1)\Big) + O(1)$$

elements.

As $N_2(B_1, v_l) \geq \operatorname{card}(M) + O(1)$, we get $N_2(B_1, v_l) > K10^l$ for a constant $K > 0$. So we obtain from (18), (19), (20), (21) and $\Delta_0(B_1) \geq \Delta_0(B_2)$

$$\begin{aligned} N(B_1, \omega, 1, T) - N(B_2, \omega, 1, T) &\geq \sum_{1 \leq l < n} (N_1(B_1, v_l) - N_1(B_2, v_l) + N_2(B_1, v_l)) \\ &\quad + \sum_{1 \leq j \leq n-k} (\Delta_j(B_1) - \Delta_j(B_2)) + O(n) \\ &\geq \sum_{1 \leq l < n} \Big(K10^l + \frac{1}{C}10^{l-k}\Big) - \sum_{1 \leq j \leq n-k} \frac{1}{C}10^{n-k-j} + O(n) > L10^n \end{aligned}$$

for a constant $L > 0$ and n sufficiently large.

Now Theorem 3 follows from Lemma 2. ■

Proof of Theorem 4. Let $\delta\in(0,1)$; $f(x)$ be defined for real $x\geqslant 1$; $f\geqslant 1$, and $f(x)\sim x^{\delta}$, f' monotone, $f'(x)\sim x^{\delta-1}$, $|f''(x)|\ll x^{\delta-2}$, f'' continuous and $\omega=\omega([f])=0.[f(1)][f(2)]\ldots$ In this case, f' is monotonically decreasing. We consider the two blocks of digits $B_1=(1)$, $B_2=(0)$ of the same length 1.

Let $n\geqslant 2$, $T=T(u_n-1)$; $\Delta=N(B_1,\omega,1,T)-N(B_2,\omega,1,T)$ and

$$\Delta_l=N(B_1,u_l-1)-N(B_2,u_l-1)\quad\text{for}\quad 2\leqslant l\leqslant n.$$

We get

$$\Delta_l=\sum_{1\leqslant j\leqslant l}\sum_{10^{j-1}\leqslant m<10^j}\big(f^{-1}(m10^{l-j}+2\cdot 10^{l-j-1})-f^{-1}(m10^{l-j}+10^{l-j-1})$$
$$-(f^{-1}(m10^{l-j}+10^{l-j-1})-f^{-1}(m10^{l-j}))+O(1)\big)$$
$$+f^{-1}(2\cdot 10^{l-1})-f^{-1}(10^{l-1})+O(1).$$

Now $(f^{-1})'$ is increasing and so the first sum is not negative. Hence

$$10^{l/\delta}\ll 10^{(l-1)(1/\delta-1)}\cdot 10^{l}\ll f^{-1}(2\cdot 10^{l-1})-f^{-1}(10^{l-1})$$

yields

$$\Delta=\sum_{1\leqslant l\leqslant n}\Delta_l>K\sum_{1\leqslant l\leqslant n}10^{l/\delta}>L10^{n/\delta}\quad\text{for constants } K,L>0.$$

Theorem 4 now follows from $10^{n/\delta}\sim T/\log T$ and Lemma 2. ∎

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