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Chapter 15 of Ramanujan's Second Notebook: Part 2, Modular forms

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Chapter 15 has a character different from most of the other chapters in the second notebook because it contains very diverse topics. Moreover, the subjects are examined at highly different levels of sophistication. In another paper [5], we have examined the first seven sections of Chapter 15, which contain several beautiful and original asymptotic expansions. In this paper, we describe Sections 8–14, where modular forms, in particular, Eisenstein series, are at center stage.

Perhaps the most interesting theorem in Chapter 15 is found in (8.3) below. This undoubtedly new result gives an inversion formula for a certain modified theta-function. It may be surprising that an exact formula of this type exists.

Entry 11 is a beautiful and new reciprocity formula reminiscent of some of the formulas in Chapter 14.

Section 12 contains several results found in Ramanujan's famous paper [22], [23, pp. 136–162]. We mention, in particular, Entry 12(x) which is equivalent to the very interesting identity

$$\sum_{k=0}^{n} \sigma_1(2k+1)\sigma_3(n-k) = \tfrac{1}{240}\sigma_5(2n+1), \qquad n \geqslant 0,$$

where $\sigma_\nu(m) = \sum_{d|m} d^\nu$, $m \neq 0$, and $\sigma_3(0) = 1/240$. Ramanujan states this identity without proof in [22], [23, p. 146] and indicates that he has two proofs, one of which is elementary. We have not been able to find an elementary proof in the literature nor can we produce one ourselves. All of the results in Section 13 can also be found in [22].

Entry 14 offers a new recursion formula for Eisenstein series. It is quite

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distinct from the most well-known recursion formula for Eisenstein series which was discovered by Ramanujan in [22], [23, p. 140].

At the beginning of Section 8, Ramanujan remarks that "If $F(h)$ in XV 1 terminates we do not know how far the result is true. But from the following and similar ways we can calculate the error in such cases." To illustrate these cryptic remarks, Ramanujan indicates a method for calculating the error in the asymptotic expansion

$$\sum_{k=1}^{\infty}\frac{1}{e^{k^2x}-1}=\frac{\pi^2}{6x}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\zeta\left(\frac{1}{2}\right)+\frac{1}{4}+o(1), \tag{8.1}$$

as x tends to $0+$, which is the case $p=2$, $q=m=n=1$ in [5, § 6, Theorem 1]. In fact, he indicates that the equalities

$$\int_0^{\infty}\sum_{k=1}^{\infty}e^{-k^2x}\cos(ax)\,dx=\sum_{k=1}^{\infty}\frac{k^2}{a^2+k^4} \tag{8.2}$$

$$=\frac{\pi}{2\sqrt{2a}}\,\frac{\sinh(\pi\sqrt{2a})-\sin(\pi\sqrt{2a})}{\cosh(\pi\sqrt{2a})-\cos(\pi\sqrt{2a})}$$

can be used to deduce the following exact formula extending (8.1).

ENTRY 8. *If* $x>0$, *then*

$$\sum_{k=1}^{\infty}\frac{1}{e^{k^2x}-1}=\frac{\pi^2}{6x}+\frac{1}{2}\sqrt{\frac{\pi}{x}}\zeta\left(\frac{1}{2}\right)+\frac{1}{4} \tag{8.3}$$

$$+\sqrt{\frac{\pi}{2x}}\sum_{k=1}^{\infty}\frac{1}{\sqrt{k}}\left\{\frac{\cos(\pi/4+2\pi\sqrt{\pi k/x})-e^{-2\pi\sqrt{\pi k/x}}\cos(\pi/4)}{\cosh(2\pi\sqrt{\pi k/x})-\cos(2\pi\sqrt{\pi k/x})}\right\}.$$

This is truly a remarkable formula. The left side can be construed as a modification of the theta-function

$$\theta(x)=1+2\sum_{k=1}^{\infty}\frac{1}{e^{k^2x}}.$$

Thus, Entry 8 is an analogue of the inversion formula for $\theta(x)$.

Before proving Entry 8, we first establish (8.2).

The first equality easily follows from inverting the order of summation and integration on the left side and using a well-known integral evaluation [13, p. 477].

To prove the second equality, we shall expand the right side into partial fractions. An elementary calculation shows that the nonzero zeros of $\cosh(\pi\sqrt{2a})-\cos(\pi\sqrt{2a})$ are at $a=\pm k^2 i$, $1\leqslant k<\infty$, and that they are simple. Thus, if R_{z_0} denotes the residue of the function on the far right side of (8.2) at a simple pole z_0, we find that

$$R_{\pm k^2 i}=\mp i/2.$$

Thus, for some entire function $g(a)$,

$$\frac{\pi}{2\sqrt{2a}}\,\frac{\sinh(\pi\sqrt{2a})-\sin(\pi\sqrt{2a})}{\cosh(\pi\sqrt{2a})-\cos(\pi\sqrt{2a})}=\frac{i}{2}\sum_{k=1}^{\infty}\left\{-\frac{1}{a-k^2 i}+\frac{1}{a+k^2 i}\right\}+g(a)$$

$$=\sum_{k=1}^{\infty}\frac{k^2}{a^2+k^4}+g(a).$$

Letting a tend to ∞ on both sides above, we find that $g(a)$ tends to 0. Hence, $g(a)$ is a bounded entire function, and so by Liouville's theorem $g(a)$ is constant. Clearly, this constant is zero. Hence, the proof of the second equality in (8.2) is complete.

A different proof of the second equality in (8.2) may be found in a paper of Glaisher [9].

Proof of Entry 8. Setting $x=\pi y$, we restate (8.3) in the form

$$\sum_{k=1}^{\infty}\frac{1}{e^{\pi k^2 y}-1}-\left(\frac{\pi}{6y}+\frac{\zeta(1/2)}{2\sqrt{y}}+\frac{1}{4}\right)=R, \tag{8.4}$$

where

$$R=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{\sqrt{ky}}\left\{\frac{\cos(2\pi\sqrt{k/y})-\sin(2\pi\sqrt{k/y})-e^{-2\pi\sqrt{k/y}}}{\cosh(2\pi\sqrt{k/y})-\cos(2\pi\sqrt{k/y})}\right\} \tag{8.5}$$

$$=\frac{2}{\pi y}\sum_{k=1}^{\infty}\frac{\pi}{2\sqrt{2a}}\left\{\frac{\sinh(\pi\sqrt{2a})-\sin(\pi\sqrt{2a})}{\cosh(\pi\sqrt{2a})-\cos(\pi\sqrt{2a})}-1\right\},$$

where $a\equiv a_k=2k/y$.

For brevity, set

$$\psi(u)=\sum_{k=1}^{\infty}e^{-\pi k^2 u},\quad u>0.$$

Thus, by (8.2) and (8.5), with $a=2k/y$,

$$R=2\sum_{k=1}^{\infty}\left\{\int_0^{\infty}\psi(u)\cos(2\pi ku/y)\frac{du}{y}-\frac{1}{4\sqrt{ky}}\right\} \tag{8.6}$$

$$=2\sum_{k=1}^{\infty}\left\{\int_0^{\infty}\psi(uy)\cos(2\pi ku)\,du-\frac{1}{4\sqrt{ky}}\right\}.$$

Now, for $y>0$,

$$\sum_{k=1}^{\infty}\psi(ky)=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}e^{-\pi j^2 ky}=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}e^{-\pi j^2 ky}=\sum_{j=1}^{\infty}\frac{1}{e^{\pi j^2 y}-1}; \tag{8.7}$$

By (8.6) and (8.7), the proposed formula (8.4) now becomes

$$(8.8)\quad \sum_{k=1}^{\infty} \psi(ky) - \frac{\pi}{6y} - \frac{\zeta(1/2)}{2\sqrt{y}} - \frac{1}{4} = 2\sum_{k=1}^{\infty} \left\{ \int_0^{\infty} \psi(uy)\cos(2\pi ku)\,du - \frac{1}{4\sqrt{ky}} \right\}.$$

Let $0 < \varepsilon < 1$. Applying the Poisson summation formula [29, p. 443], we deduce that

$$(8.9)\quad \sum_{k=1}^{\infty} \psi(ky) = \int_{\varepsilon}^{\infty} \psi(uy)\,du + 2\sum_{k=1}^{\infty} \int_{\varepsilon}^{\infty} \psi(uy)\cos(2\pi ku)\,du.$$

From (8.2),

$$\int_0^{\infty} \psi(uy)\,du = \pi/(6y).$$

Hence, by (8.9),

$$(8.10)\quad \sum_{k=1}^{\infty} \psi(ky) - \pi/(6y) = \lim_{\varepsilon\to 0+} 2\sum_{k=1}^{\infty} \int_{\varepsilon}^{\infty} \psi(uy)\cos(2\pi ku)\,du.$$

By (8.8) and (8.10), it remains to prove that

$$(8.11)\quad -\frac{1}{4} - \frac{\zeta(1/2)}{2\sqrt{y}} = \lim_{\varepsilon\to 0+} 2\sum_{k=1}^{\infty} \left\{ \int_0^{\varepsilon} \psi(uy)\cos(2\pi ku)\,du - \frac{1}{4\sqrt{ky}} \right\}.$$

From [30, p. 22, equation (2.6.3)], for $u > 0$,

$$\psi(uy) = -\frac{1}{2} + \frac{1}{2\sqrt{uy}} + \frac{1}{\sqrt{uy}}\psi\left(\frac{1}{uy}\right).$$

Therefore,

$$(8.12)\quad \int_0^{\varepsilon} \psi(uy)\cos(2\pi ku)\,du$$

$$= -\frac{1}{2}\int_0^{\varepsilon} \cos(2\pi ku)\,du + \frac{1}{2}\int_0^{\varepsilon} \frac{\cos(2\pi ku)}{\sqrt{uy}}\,du + \int_0^{\varepsilon} \frac{\cos(2\pi ku)}{\sqrt{uy}}\psi\left(\frac{1}{uy}\right)du.$$

The first term on the right side of (8.12) gives to (8.11) the contribution

$$(8.13)\quad \lim_{\varepsilon\to 0+} -2\sum_{k=1}^{\infty} \frac{\sin(2\pi k\varepsilon)}{4\pi k} = \lim_{\varepsilon\to 0+} \tfrac{1}{2}(\varepsilon - [\varepsilon] - \tfrac{1}{2}) = -\tfrac{1}{4},$$

where we have used a familiar Fourier series [30, p. 15, equation (2.1.7)]. The

third expression on the right side of (8.12) contributes to (8.11)

$$(8.14)\quad \lim_{\varepsilon\to 0+} 2\sum_{k=1}^{\infty} \int_0^{\varepsilon} \frac{\cos(2\pi ku)}{\sqrt{uy}}\psi\left(\frac{1}{uy}\right)du = 0,$$

which can be seen after two integrations by parts. By (8.11)–(8.14), it remains to prove that

$$(8.15)\quad -\frac{\zeta(1/2)}{2} = \lim_{\varepsilon\to 0+} \sum_{k=1}^{\infty} \left\{ \int_0^{\varepsilon} \frac{\cos(2\pi ku)}{\sqrt{u}}\,du - \frac{1}{2\sqrt{k}} \right\}.$$

Now [13, p. 395],

$$\int_0^{\infty} \frac{\cos(2\pi ku)}{\sqrt{u}}\,du = \sqrt{\frac{2}{\pi k}}\int_0^{\infty} \cos u^2\,du = \frac{1}{2\sqrt{k}}.$$

Using this in (8.15), we find that (8.15) becomes

$$(8.16)\quad \zeta(1/2) = \lim_{\varepsilon\to 0+} 2\sum_{k=1}^{\infty} \int_{\varepsilon}^{\infty} \frac{\cos(2\pi ku)}{\sqrt{u}}\,du.$$

We shall again apply the Poisson summation formula. Let $0 < \varepsilon < 1 < N$ and suppose N is *not* an integer. Then

$$(8.17)\quad \sum_{\varepsilon<k<N} \frac{1}{\sqrt{k}} - \int_{\varepsilon}^{N} \frac{du}{\sqrt{u}} = 2\sum_{k=1}^{\infty} \int_{\varepsilon}^{N} \frac{\cos(2\pi ku)}{\sqrt{u}}\,du.$$

The left side of (8.17) may be written as

$$\int_{\varepsilon}^{N} \frac{d([u]-u)}{\sqrt{u}} = \frac{[u]-u}{\sqrt{u}}\Bigg|_{\varepsilon}^{N} + \frac{1}{2}\int_{\varepsilon}^{N} \frac{[u]-u}{u^{3/2}}\,du.$$

Using this in (8.17) and letting N tend to ∞, we deduce that

$$(8.18)\quad \sqrt{\varepsilon} + \frac{1}{2}\int_{\varepsilon}^{\infty} \frac{[u]-u}{u^{3/2}}\,du = 2\sum_{k=1}^{\infty} \int_{\varepsilon}^{\infty} \frac{\cos(2\pi ku)}{\sqrt{u}}\,du,$$

where letting N tend to ∞ inside the summation sign is justified by two integrations by parts. Combining (8.16) and (8.18), we see that we must show that

$$\zeta(1/2) = \frac{1}{2}\int_0^\infty \frac{[u]-u}{u^{3/2}}\,du.$$

But this last formula follows immediately from a well-known representation for $\zeta(s)$ found in Titchmarsh's treatise [30, p. 14, equation (2.1.5)]. Hence, the proof of (8.3) is complete.

In the sequel, we shall set

$$F_{m,n}(x) = \sum_{j,k=1}^\infty j^m k^n e^{-jkx}, \tag{9.1}$$

where $x > 0$ and m and n are nonnegative integers. Without loss of generality, assume that $m \geqslant n$. In [5, § 6, Theorem 1], an asymptotic expansion is given for $F_{m,n}(x)$ as x tends to $0+$. Ramanujan begins Section 9 with the special case $p = q = 1$, $m \neq n$ of [5, § 6, Theorem 1]. He then defines, for $|q| < 1$,

$$L = 1-24\sum_{k=1}^\infty \frac{kq^k}{1-q^k}, \qquad M = 1+240\sum_{k=1}^\infty \frac{k^3 q^k}{1-q^k}, \qquad N = 1-504\sum_{k=1}^\infty \frac{k^5 q^k}{1-q^k}.$$

The functions L, M, and N were thoroughly studied in a famous paper [22], [23, pp. 136–162] of Ramanujan, where L, M, and N are denoted by P, Q, and R, respectively. We now show that L, M, and N are essentially the Eisenstein series of weights 2, 4, and 6, respectively, on the full modular group $\Gamma(1)$. To see this, first let $q = \exp(2\pi i\tau)$, where τ is in the upper half-plane $\mathscr{H}$, and write

$$\Phi_\nu(q) \equiv \sum_{k=1}^\infty \frac{k^\nu q^k}{1-q^k} = \sum_{k=1}^\infty k^\nu \sum_{j=1}^\infty e^{2\pi ijk\tau} = \sum_{r=1}^\infty \sigma_\nu(r) e^{2\pi ir\tau}, \tag{9.2}$$

where we put $jk = r$ and where $\sigma_\nu(r) = \sum_{k|r} k^\nu$. Next recall that the Fourier expansions of the Eisenstein series $E_n(\tau)$, where n is an even positive integer, are given by [26, p. 194]

$$E_2(\tau) = 1-24\sum_{k=1}^\infty \sigma_1(k) e^{2\pi ik\tau} - 3/(\pi y) = 1-24\Phi_1(q) - 3/(\pi y) \tag{9.3}$$

and

$$E_n(\tau) = 1-\frac{2n}{B_n}\sum_{k=1}^\infty \sigma_{n-1}(k) e^{2\pi ik\tau} = 1-\frac{2n}{B_n}\Phi_{n-1}(q), \qquad n > 2, \tag{9.4}$$

where $y = \operatorname{Im}\tau > 0$ and where B_n denotes the nth Bernoulli number. Hence, $L = E_2(\tau)+3/(\pi y)$, $M = E_4(\tau)$, and $N = E_6(\tau)$.

Ramanujan next claims that if $m+n$ is an odd, positive integer, then the function $F_{m,n}(x)$ of (9.1) can be evaluated exactly in terms of L, M, and N. First, observe that, by setting $jk = r$ in the definition of $F_{m,n}(x)$, we obtain

$$F_{m,n}(x) = \sum_{r=1}^\infty r^n \sigma_{m-n}(r) e^{-rx}.$$

Thus, with $x = -2\pi i\tau$, $F_{m,n}(x)$ is essentially an n-fold derivative of an Eisenstein series of even weight if $m-n$ is odd. If $m-n = 1$ and $n \geqslant 1$, then $F_{m,n}(x)$ is clearly a multiple of an n-fold derivative of L. Suppose now that $m-n$ is odd and > 1. By a theorem in Rankin's text [26, p. 199] each modular form of even positive weight can be expressed as a polynomial in $E_4(\tau)$ and $E_6(\tau)$. Thus, $\sum_{r=1}^\infty \sigma_{m-n}(r) e^{-rx}$ can be so expressed, and since $F_{m,n}(x)$ is, up to a factor of ± 1, an n-fold derivative of the function above, then $F_{m,n}(x)$ can be represented as a polynomial in M, N, and their derivatives.

Entry 10 (i) (first part). *For each positive integer $n \geqslant 2$,*

$$-\frac{B_{2n}}{4n}E_{2n}(\tau) = -\frac{B_{2n}}{4n} + \sum_{k=1}^\infty \sigma_{2n-1}(k) e^{2\pi ik\tau}$$

can be expressed as a polynomial in M and N.

This statement was verified in Section 9 where we appealed to Rankin's book [26, p. 199]. See also (14.2) and Entry 14 below.

Entry 10 (i) (second part). *For each positive integer n,*

$$f_n(x) \equiv \sum_{k=1}^\infty \frac{k^{2n} q^k}{(1-q^k)^2} - \delta_n \frac{nL}{6}\left\{-\frac{B_{2n}}{4n} + \sum_{k=1}^\infty \frac{k^{2n-1} q^k}{1-q^k}\right\}$$

can be expressed as a polynomial in M and N. Here $\delta_1 = 1/2$ and $\delta_n = 1$ if $n \geqslant 2$.

Proof. By (9.3) and (9.4),

$$\sum_{k=1}^\infty \frac{k^{2n} q^k}{(1-q^k)^2} = \frac{1}{2\pi i}\frac{d}{d\tau}\left(-\frac{B_{2n}}{4n}E_{2n}^*(\tau)\right),$$

where

$$E_{2n}^*(\tau) = \begin{cases} E_2(\tau)+3/(\pi y) = L, & \text{if } n = 1, \\ E_{2n}(\tau), & \text{if } n > 1. \end{cases}$$

Thus, for $n \geqslant 1$,

$$f_n(x) = F_n(\tau) \equiv \frac{1}{2\pi i}\frac{d}{d\tau}\left(-\frac{B_{2n}}{4n}E_{2n}^*(\tau)\right) + \frac{\delta_n n E_2^*(\tau)}{6}\frac{B_{2n}}{4n}E_{2n}^*(\tau).$$

By the aforementioned theorem in [26, p. 199], it suffices to prove that $F_n(\tau)$

is a modular form on $\Gamma(1)$ of weight $2n+2$. We must therefore show that [28, equation (5), p. 80]

$$F_n(-1/\tau) = \tau^{2n+2} F_n(\tau), \quad \tau \in \mathscr{H}. \tag{10.1}$$

Recall that for $V\tau = (a\tau+b)/(c\tau+d) \in \Gamma(1)$ [27, pp. 50, 68]

$$E_{2n}^*(V\tau) = \begin{cases} (c\tau+d)^2 E_2^*(\tau) - 6\pi^{-1} ic(c\tau+d), & \text{if } n = 1, \\ (c\tau+d)^{2n} E_{2n}(\tau), & \text{if } n > 1. \end{cases} \tag{10.2}$$

By (10.2), if $n > 1$,

$$\begin{aligned} \frac{4n}{B_{2n}} F_n(-1/\tau) &= -\frac{\tau^2}{2\pi i}\left(2n\tau^{2n-1} E_{2n}(\tau) + \tau^{2n} E_{2n}'(\tau)\right) \\ &\quad + \frac{n}{6}\left(\tau^2 E_2^*(\tau) - \frac{6i\tau}{\pi}\right)\tau^{2n} E_{2n}(\tau) \\ &= \tau^{2n+2}\left(-\frac{1}{2\pi i} E_{2n}'(\tau) + \frac{n}{6} E_2^*(\tau) E_{2n}(\tau)\right) \\ &= \tau^{2n+2} \frac{4n}{B_{2n}} F_n(\tau). \end{aligned}$$

This proves (10.1) for $n > 1$. A similar argument can be used for the case $n = 1$.

Alternatively, for $n > 1$, (10.1) follows from the theorem [20, pp. 16, 17] that if $f(\tau)$ is a modular form of weight k, then $f'(\tau) - (2\pi ik/12) E_2^*(\tau) f(\tau)$ is a modular form of weight $k+2$.

Ramanujan did not consider the case $n = 1$ in Entry 10 (i).

ENTRY 11. *If $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$, then*

$$\begin{aligned} &\frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2 \sinh^2(\alpha k)} + \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2 \sinh^2(\beta k)} \\ &\quad - 2\alpha \sum_{k=1}^{\infty} k^2 \operatorname{Log}(1 - e^{-2\alpha k}) - 2\beta \sum_{k=1}^{\infty} k^2 \operatorname{Log}(1 - e^{-2\beta k}) = \frac{\alpha^2+\beta^2}{120} - \frac{\alpha\beta}{72}. \end{aligned}$$

Proof. By an elementary calculation,

$$\begin{aligned} \sum_{k=1}^{\infty} k^2 \operatorname{Log}(1 - e^{-2\alpha k}) &= -\sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=1}^{\infty} k^2 e^{-2\alpha jk} \\ &= -\sum_{j=1}^{\infty} \frac{1}{j}\left\{\frac{e^{-2\alpha j}}{(1-e^{-2\alpha j})^2} + \frac{2e^{-4\alpha j}}{(1-e^{-2\alpha j})^3}\right\} = -\frac{1}{4}\sum_{j=1}^{\infty} \frac{\cosh(\alpha j)}{j \sinh^3(\alpha j)}. \end{aligned} \tag{11.1}$$

With (11.1) as motivation, we define

$$f(z) = \pi\cot(\pi z)\left(\frac{1}{z^2 \sinh^2(\alpha z)} + \frac{2\alpha\cosh(\alpha z)}{z \sinh^3(\alpha z)}\right).$$

We shall integrate f over a suitable rectangle, to be described later, and apply the residue theorem. We write R_z to denote the residue of a specified function at z.

First, f has simple poles at each nonzero integer k with

$$R_k = \frac{1}{k^2 \sinh^2(\alpha k)} + \frac{2\alpha\cosh(\alpha k)}{k \sinh^3(\alpha k)}.$$

By (11.1), the sum of all such residues is equal to

$$2\sum_{k=1}^{\infty} \frac{1}{k^2 \sinh^2(\alpha k)} - 16\alpha \sum_{k=1}^{\infty} k^2 \operatorname{Log}(1 - e^{-2\alpha k}). \tag{11.2}$$

Secondly, let $f_1(z) = p(z)/q(z)$, where $p(z) = \pi\cot(\pi z)$ and $q(z) = z^2\sinh^2(\alpha z)$. The function $f_1(z)$ has double poles at $z = ik\pi/\alpha$, for each nonzero integer k. To calculate the residue at $ik\pi/\alpha$, we shall use a formula from Churchill's text [6, p. 160] for the residue of a double pole. Accordingly,

$$R_{i\pi k/\alpha} = \frac{2p'(i\pi k/\alpha)}{q''(i\pi k/\alpha)} - \frac{2p(i\pi k/\alpha)\, q'''(i\pi k/\alpha)}{3\{q''(i\pi k/\alpha)\}^2}. \tag{11.3}$$

Elementary calculations yield

$$\begin{aligned} p(i\pi k/\alpha) &= \pi\cot(\beta ki), & p'(i\pi k/\alpha) &= -\pi^2\csc^2(\beta ki), \\ q''(i\pi k/\alpha) &= -2\pi^2 k^2, \quad \text{and} & q'''(i\pi k/\alpha) &= 12\alpha\pi ki. \end{aligned}$$

Using these values in (11.3), we find that

$$R_{i\pi k/\alpha} = -\frac{1}{k^2\sinh^2(\beta k)} - \frac{2\coth(\beta k)}{\beta k^3}.$$

Thus, the sum of all such residues is

$$-2\sum_{k=1}^{\infty} \frac{1}{k^2\sinh^2(\beta k)} - \frac{4}{\beta}\sum_{k=1}^{\infty} \frac{\coth(\beta k)}{k^3}. \tag{11.4}$$

Consider a function $F(z) = p(z)/q(z)$, where p and q are analytic at z_0, $p(z_0) \neq 0$, and q has a zero of order 3 at z_0. Then a somewhat lengthy, but routine, exercise shows that

$$R_{z_0} = \frac{3p''(z_0)}{q'''(z_0)} - \frac{3p'(z_0)\, q^{(4)}(z_0)}{2\{q'''(z_0)\}^2} - \frac{3p(z_0)\, q^{(5)}(z_0)}{10\{q'''(z_0)\}^2} + \frac{3p(z_0)\{q^{(4)}(z_0)\}^2}{8\{q'''(z_0)\}^3}. \tag{11.5}$$

Now set $f_2(z) = p(z)/q(z)$, where $p(z) = 2\pi\alpha\cot(\pi z)\cosh(\alpha z)$ and $q(z) = z\sinh^3(\alpha z)$. The function $f_2(z)$ has triple poles at $z = i\pi k/\alpha$, for each nonzero integer k. Elementary calculations yield

$$p(i\pi k/\alpha) = -2(-1)^k\pi\alpha i\coth(\beta k), \qquad p'(i\pi k/\alpha) = 2(-1)^k\pi^2\alpha\operatorname{csch}^2(\beta k),$$

$$p''(i\pi k/\alpha) = 4(-1)^k\pi^3\alpha i\operatorname{csch}^2(\beta k)\coth(\beta k) - 2(-1)^k\pi\alpha^3 i\coth(\beta k),$$

$$q'''(i\pi k/\alpha) = 6(-1)^k\pi\alpha^2 ki, \qquad q^{(4)}(i\pi k/\alpha) = 24(-1)^k\alpha^3,$$

and

$$q^{(5)}(i\pi k/\alpha) = 60(-1)^k\pi\alpha^4 ki.$$

Using these values in (11.5), we find, after much simplification, that

$$R_{i\pi k/\alpha} = \frac{2\beta}{k}\operatorname{csch}^2(\beta k)\coth(\beta k) + \frac{2}{k^2}\operatorname{csch}^2(\beta k) + \frac{2}{\beta k^3}\coth(\beta k).$$

Thus, the sum of all such residues, by (11.1), is equal to

$$(11.6)\quad -16\beta\sum_{k=1}^{\infty} k^2\operatorname{Log}(1-e^{-2\beta k}) + 4\sum_{k=1}^{\infty}\frac{1}{k^2\sinh^2(\beta k)} + \frac{4}{\beta}\sum_{k=1}^{\infty}\frac{\coth(\beta k)}{k^3}.$$

Lastly, f has a pole of order 5 at the origin. We have

$$\begin{aligned} f(z) &= \pi\left(\frac{1}{\pi z} - \frac{\pi z}{3} - \frac{\pi^3 z^3}{45} + \ldots\right)\left\{\frac{1}{z^2}\left(\frac{1}{\alpha z} - \frac{\alpha z}{6} + \frac{7\alpha^3 z^3}{360} + \ldots\right)^2\right. \\ &\quad \left. + \frac{2\alpha}{z}\left(\frac{1}{\alpha z} + \frac{\alpha z}{3} - \frac{\alpha^3 z^3}{45} + \ldots\right)\left(\frac{1}{\alpha z} - \frac{\alpha z}{6} + \frac{7\alpha^3 z^3}{360} + \ldots\right)^2\right\} \\ &= \pi\left(\frac{1}{\pi z} - \frac{\pi z}{3} - \frac{\pi^3 z^3}{45} + \ldots\right)\left(\frac{3}{\alpha^2 z^4} - \frac{1}{3z^2} - \frac{\alpha^2}{15} + \ldots\right). \end{aligned}$$

Hence,

$$(11.7)\qquad R_0 = \pi\left(-\frac{\alpha^2}{15\pi} + \frac{\pi}{9} - \frac{\pi^3}{15\alpha^2}\right) = \frac{\alpha\beta}{9} - \frac{\alpha^2+\beta^2}{15}.$$

Consider next

$$I_N = \frac{1}{2\pi i}\int_{C_N} f(z)\,dz,$$

where C_N is a positively oriented rectangle with sides parallel to the coordinate axes and passing through the points $\pm([\sqrt{N}]+1/2)$ and $i\pi(N+1/2)/\alpha$, where N is a positive integer. Note that C_N is free of poles of f. Estimating the integrand on the vertical and horizontal sides separately,

we find that

$$(11.8)\qquad I_N \ll \sqrt{N}\,e^{-2\alpha[\sqrt{N}]} + (1/\sqrt{N}) = o(1),$$

as N tends to ∞.

Apply the residue theorem to I_N and then let N tend to ∞. Using (11.2), (11.4), (11.6), (11.7), and (11.8), we deduce that

$$\begin{aligned} 0 &= 2\sum_{k=1}^{\infty}\frac{1}{k^2\sinh^2(\alpha k)} - 16\alpha\sum_{k=1}^{\infty} k^2\operatorname{Log}(1-e^{-2\alpha k}) \\ &\quad + 2\sum_{k=1}^{\infty}\frac{1}{k^2\sinh^2(\beta k)} - 16\beta\sum_{k=1}^{\infty} k^2\operatorname{Log}(1-e^{-2\beta k}) + \frac{\alpha\beta}{9} - \frac{\alpha^2+\beta^2}{15}, \end{aligned}$$

which is readily seen to be equivalent to the proposed identity.

Another proof of Entry 11 may be constructed from results in Berndt's paper [3, Theorems 2.2, 2.16] together with (11.1).

For other beautiful theorems in the same spirit as Entry 11, see Chapter 14 in the second notebook [24], [4].

Entry 12. *Let L, M, and N be as defined in Section 9, and recall that $E_n(\tau)$, $n > 2$, and $\Phi_n(q)$ are defined by (9.4) and (9.2), respectively. Define the discriminant function $\Delta(\tau)$ by*

$$\Delta(\tau) = q\prod_{k=1}^{\infty}(1-q^k)^{24}, \qquad q = e^{2\pi i\tau},\ \tau\in\mathscr{H}.$$

Then, for $|q| < 1$,

(i) $M^3 - N^2 = 1728\Delta(\tau)$,

(ii) $E_8(\tau) = M^2$,

(iii) $E_{10}(\tau) = MN$,

(iv) $E_{14}(\tau) = M^2 N$,

(v) $\displaystyle\sum_{k=1}^{\infty}\frac{k^2 q^k}{(1-q^k)^2} = \frac{M-L^2}{288}$,

(vi) $\displaystyle\sum_{k=1}^{\infty}\frac{k^4 q^k}{(1-q^k)^2} = \frac{LM-N}{720}$,

(vii) $\displaystyle\sum_{k=1}^{\infty}\frac{k^6 q^k}{(1-q^k)^2} = \frac{M^2-LN}{1008}$,

(viii) $\displaystyle\sum_{k=1}^{\infty}\frac{k^8 q^k}{(1-q^k)^2} = \frac{LM^2-MN}{720}$,

(ix) $\displaystyle L\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2} = \sum_{k=0}^{\infty}(-1)^k(2k+1)^3 q^{k(k+1)/2}$, *and*

$$\text{(x)}\quad M\sum_{k=1}^{\infty}\frac{(2k-1)q^k}{1-q^{2k-1}}=\sum_{k=1}^{\infty}\frac{(2k-1)^5 q^k}{1-q^{2k-1}}.$$

Proofs of (i)–(viii). Formulas (i)–(iv) are very well known and are special cases of the general theorem [26, p. 199] which we applied in Section 9. In particular, (i)–(iv) can be found in [26, pp. 195, 197, equations (6.1.8), (6.1.9), and (6.1.14)]. These formulas were also derived by Ramanujan in [22], [23, p. 141].

Formulas (v)–(viii) are originally due to Ramanujan, and proofs can be found in his paper [22], [23, pp. 141, 142].

First proof of (ix). This formula is a special case of a general formula established by Ramanujan in Chapter 16 [24, vol. 2, p. 202], [1, Entry 35(i)].

Second proof of (ix). Rearranging in (ix), we find that

$$\sum_{j=1}^{\infty}\sigma(j)q^j\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2}$$
$$=\frac{1}{24}\Big\{\sum_{k=0}^{\infty}(-1)^k(2k+1)q^{k(k+1)/2}-\sum_{k=0}^{\infty}(-1)^k(2k+1)^3q^{k(k+1)/2}\Big\}.$$

Equating coefficients of q^n, $n\geqslant 0$, on both sides, we find that

$$\sigma(n)-3\sigma(n-1)+5\sigma(n-3)-7\sigma(n-6)+\ldots=0, \tag{12.1}$$

if n is not a triangular number, while if $n=r(r+1)/2$ is a triangular number,

$$\begin{aligned}\sigma(n)-3\sigma(n-1)+5\sigma(n-3)-7\sigma(n-6)+\ldots&\\ =\tfrac{1}{24}\{(-1)^r(2r+1)-(-1)^r(2r+1)^3\}&\\ =\tfrac{1}{6}(-1)^{r-1}r(r+1)(2r+1)&\\ =(-1)^{r-1}\sum_{k=1}^{r}k^2.&\end{aligned}\tag{12.2}$$

Thus, formula (ix) is equivalent to the arithmetic identities evinced in (12.1) and (12.2). These identities are due to Glaisher [10] in 1884, although they are really consequences of a formula proven seven years earlier by Halphen [16]. Hence, appealing to the theorem of Glaisher and Halphen, we have shown (ix).

For generalizations of Entry 12(ix), see two additional papers of Glaisher [11], [12]. For further references to the literature, consult Dickson's history [7, p. 289].

Proof of (x). If

$$f(\tau)=\sum_{n=0}^{\infty}a_nq^n,\qquad q=e^{2\pi i\tau},$$

define functions f_∞, f_0, and f_1 by

$$f_\infty(\tau)=f(2\tau)=\sum_{n=0}^{\infty}a_nq^{2n},\qquad f_0(\tau)=f(\tau/2)=\sum_{n=0}^{\infty}a_nq^{n/2},$$

and

$$f_1(\tau)=f((\tau+1)/2)=\sum_{n=0}^{\infty}a_n(-1)^nq^{n/2}.$$

Then Entry 12(x) may be rewritten in the form

$$N_1-N_0=21M(L_1-L_0). \tag{12.3}$$

If $w=(\tau+1)/2$, observe that

$$\frac{1-1/\tau}{2}=\frac{w-1}{2w-1}\in\Gamma(1).$$

Thus, from (10.2), we readily find that

$$L_\infty(\tau+1)=L_\infty(\tau),\quad L_0(\tau+1)=L_1(\tau),\quad L_1(\tau+1)=L_0(\tau),$$
$$L_\infty(-1/\tau)=\tfrac14\tau^2L_0(\tau)+3\tau/(\pi i),\quad L_0(-1/\tau)=4\tau^2L_\infty(\tau)+12\tau/(\pi i),$$
$$L_1(-1/\tau)=\tau^2L_1(\tau)+12\tau/(\pi i),$$
$$N_\infty(\tau+1)=N_\infty(\tau),\quad N_0(\tau+1)=N_1(\tau),\quad N_1(\tau+1)=N_0(\tau),$$
$$N_\infty(-1/\tau)=\tfrac{1}{64}\tau^6N_0(\tau),\quad N_0(-1/\tau)=64\tau^6N_\infty(\tau),\quad N_1(-1/\tau)=\tau^6N_1(\tau).$$

Next, define

$$X_\infty=L_1-L_0,\quad X_0=4L_\infty-L_1,\quad X_1=L_0-4L_\infty,$$
$$Z_\infty=N_1-N_0,\quad Z_0=64N_\infty-N_1,\quad Z_1=N_0-64N_\infty.$$

Then the foregoing equalities readily imply that

$$X_\infty(\tau+1)=-X_\infty(\tau),\quad X_0(\tau+1)=-X_1(\tau),\quad X_1(\tau+1)=-X_0(\tau),$$

$$X_\infty(-1/\tau)=-\tau^2X_0(\tau),\quad X_0(-1/\tau)=-\tau^2X_\infty(\tau),\quad X_1(-1/\tau)=-\tau^2X_1(\tau), \tag{12.4}$$

and

$$Z_\infty(\tau+1)=-Z_\infty(\tau),\quad Z_0(\tau+1)=-Z_1(\tau),\quad Z_1(\tau+1)=-Z_0(\tau),$$

$$Z_\infty(-1/\tau)=-\tau^6Z_0(\tau),\quad Z_0(-1/\tau)=-\tau^6Z_\infty(\tau),\quad Z_1(-1/\tau)=-\tau^6Z_1(\tau). \tag{12.5}$$

Let M_k denote the space of modular forms of weight k on the modular subgroup $\Gamma(2)$. If $S(\tau)=\tau+1$ and $T(\tau)=-1/\tau$, then generators of $\Gamma(2)$ are [8, p. 245]

$$S^2(\tau) \quad \text{and} \quad TS^2T(\tau)=\tau/(-2\tau+1).$$

Using these generators and (12.4), we may easily verify that $X_0, X_\infty \in M_2$. Suppose that k is even. Then [26, pp. 104, 105] $\dim M_k = 1+\frac{1}{2}k$. Moreover, since

$$X_0 = 3-24q^{1/2}+72q+\ldots \tag{12.6}$$

and

$$X_\infty = 48q^{1/2}+192q^{3/2}+\ldots \tag{12.7}$$

are obviously linearly independent in M_2, we conclude that $X_0^{k/2}$, $X_0^{k/2-1}X_\infty, \ldots, X_0X_\infty^{k/2-1}$, $X_\infty^{k/2}$ form a basis for M_k. Now suppose that $f\in M_k$ and that $f(\tau)=o(q^{k/4})$, as q tends to 0. Then from (12.6) and (12.7), $f(\tau)\equiv 0$.

In our situation, we take $k=6$. Clearly, $MX_\infty \in M_6$, and, from (12.5), we may verify that $Z_\infty \in M_6$. From $Z_\infty = 1008q^{1/2}+245952q^{3/2}+\ldots$, (12.7), and the definition of M, we find that $21MX_\infty - Z_\infty = o(q^{3/2})$ as q tends to 0. Hence, $21MX_\infty - Z_\infty \equiv 0$, and (12.3) is proved.

We are very grateful to D. W. Masser for supplying us with the proof above. Another proof of Entry 12(x) based upon the theory of modular forms on $\Gamma_0(2)$ was constructed for us by A. O. L. Atkin.

Entry 12(x) was stated by Ramanujan in [22], [23, p. 146] without proof. Ramanujan indicated that he had two proofs, one of which was elementary, while the other used elliptic functions. However, he provided no hints to either proof. It is very unlikely that the proofs of Masser and Atkin are the same as either of Ramanujan's proofs. In her thesis, Ramamani [21, p. 59] has given a proof of Entry 12(x) that uses the theory of elliptic functions. Entry 12(x) is equivalent to the elegant identity

$$\sum_{k=0}^{n} \sigma_1(2k+1)\sigma_3(n-k) = \tfrac{1}{240}\sigma_5(2n+1), \qquad n\geqslant 0,$$

where $\sigma_3(0)=1/240$. It would be nice to have an elementary proof of this identity and hence of Entry 12(x) as well.

In his paper [22], [23, pp. 136-162], Ramanujan studies

$$\Sigma_{r,s}(n) = \sum_{k=0}^{n} \sigma_r(k)\sigma_s(n-k),$$

where r and s are odd, positive integers and $\sigma_m(0)=\frac{1}{2}\zeta(-m)$. He establishes an asymptotic formula for $\Sigma_{r,s}(n)$ as n tends to ∞ with an error term. He, however, conjectured a better error term [22], [23, p. 136, equation (3)]. This conjecture remained unproved until 1978 when Levitt [19] proved Ramanujan's conjecture in his thesis. In some instances, Ramanujan showed that the error term is identically equal to 0. Levitt [19] established necessary and sufficient conditions for the vanishing of the error term and so showed that the instances of such found by Ramanujan are exhaustive. Such a theorem was also found by Grosjean [14], [15] who has made a systematic study of recursion formulas connected with $\Sigma_{r,s}(n)$.

A nice survey paper on convolutions involving $\sigma_k(n)$ has been written by Lehmer [17]. For other papers in this area, consult [18, section A30].

ENTRY 13. *Let $\Phi_n(q)$ be defined as in Entry 12. Then, for $|q|<1$,*

(i) $691+65520\,\Phi_{11}(q) = 441M^3+250N^2$,

(ii) $3617+16320\,\Phi_{15}(q) = 1617M^4+2000MN^2$,

(iii) $43867-28728\,\Phi_{17}(q) = 38367M^3N+5500N^3$,

(iv) $174611+13200\,\Phi_{19}(q) = 53361M^5+121250M^2N^2$,

(v) $77683-552\,\Phi_{21}(q) = 57183M^4N+20500MN^3$,

(vi) $236364091+131040\,\Phi_{23}(q) = 49679091M^6+176400000M^3N^2+10285000N^4$,

(vii) $657931-24\,\Phi_{25}(q) = 392931M^5N+265000M^2N^3$,

(viii) $3392780147+6960\,\Phi_{27}(q) = 489693897M^7+2507636250M^4N^2+395450000MN^4$,

(ix) $1723168255201-171864\,\Phi_{29}(q) = 815806500201M^6N+881340705000M^3N^3+26021050000N^5$,

and

(x) $7709321041217+32640\,\Phi_{31}(q) = 764412173217M^8 + 5323905468000M^5N^2+1621003400000M^2N^4$.

Note.

$$q\frac{dL}{dq}=\frac{L^2-M}{12}, \qquad q\frac{dM}{dq}=\frac{LM-N}{3}, \qquad q\frac{dN}{dq}=\frac{LN-M^2}{2}.$$

EXAMPLES. Define, for $|q|<1$,

$$\Phi_{r,s}(q) = \sum_{j,k=1}^{\infty} j^r k^s q^{jk}.$$

(Thus, $\Phi_{0,s}(q)=\Phi_s(q)$.) Then

(i) $20736\,\Phi_{4,5}(q) = 15LM^2+10L^3M-20L^2N-4MN-L^5$,

(ii) $1728\,\Phi_{2,7}(q) = 2LM^2-MN-L^2N$,

and

(iii) $3456\,\Phi_{3,6}(q) = L^3M-3L^2N+3LM^2-MN$.

All of the foregoing results may be found in Ramanujan's paper [22], [23, pp. 141, 142], where the method of proof is indicated.

Let ω_1 and ω_2 denote two complex numbers linearly independent over the real numbers. Put $\omega = m\omega_1 + n\omega_2$, where m and n are integers. Recall that the Weierstrass $\wp$-function $\wp(z)$ is defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \neq 0} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right),$$

where the sum is over all pairs of integers $(m, n) \neq (0, 0)$.

In order to prove Entry 14, we shall need the following facts about $\wp(z)$ and Eisenstein series taken from Apostol's text [2, pp. 12, 13], as well as a lemma.

For $n \geqslant 1$, put

$$b(n) = 2(2n+1)\zeta(2n+2)E_{2n+2}(\tau), \tag{14.1}$$

where $E_n(\tau)$ is defined by (9.4). Then, for $n \geqslant 3$,

$$(2n+3)(n-2)b(n) = 3\sum_{k=1}^{n-2} b(k)b(n-1-k). \tag{14.2}$$

(This is a more explicit version of the first part of Entry 10(i).) Furthermore, for $|z|$ sufficiently small,

$$\wp(z) = \frac{1}{z^2} + \sum_{k=1}^{\infty} b(k) z^{2k}, \tag{14.3}$$

where $\omega_1 = 1$ and $\omega_2 = \tau$, with $\tau \in \mathscr{H}$. Lastly, $\wp(z)$ satisfies the two differential equations

$$\{\wp'(z)\}^2 = 4\wp^3(z) - 20b(1)\wp(z) - 28b(2) \tag{14.4}$$

and

$$\wp''(z) = 6\wp^2(z) - 10b(1). \tag{14.5}$$

In fact, (14.2) follows immediately from (14.5).

LEMMA. *We have*

$$\wp^{(4)}(z) = 30\{\wp'(z)\}^2 + 240b(1)\wp(z) + 504b(2).$$

Proof. Differentiating (14.5) twice, we find that

$$\wp^{(4)}(z) = 12\wp'(z)^2 + 12\wp(z)\wp''(z). \tag{14.6}$$

Also, by (14.5),

$$12\wp(z)\wp''(z) = 72\wp^3(z) - 120b(1)\wp(z), \tag{14.7}$$

and by (14.4),

$$72\wp^3(z) = 18\wp'(z)^2 + 360b(1)\wp(z) + 504b(2). \tag{14.8}$$

Substituting (14.8) into (14.7), we have

$$12\wp(z)\wp''(z) = 18\wp'(z)^2 + 240b(1)\wp(z) + 504b(2). \tag{14.9}$$

Substituting (14.9) into (14.6), we complete the proof.

If n is an even positive integer, Ramanujan now defines

$$S_n = \frac{(-1)^{n/2-1}B_n}{2n} + (-1)^{n/2}\sum_{k=1}^{\infty} \frac{k^{n-1}q^k}{1-q^k},$$

where $|q| < 1$ and B_n denotes the nth Bernoulli number. If $n > 1$ and $q = \exp(2\pi i\tau)$, with $\tau \in \mathscr{H}$, then, by (9.4),

$$S_{2n} = \frac{(-1)^{n-1}B_{2n}}{4n}E_{2n}(\tau).$$

Furthermore, from (14.1),

$$S_{2n+2} = \frac{(2n)!}{2(2\pi)^{2n+2}}b(n), \qquad n \geqslant 1. \tag{14.10}$$

In Entry 14, Ramanujan provides a recursion formula for S_{2n+2} which is different from (14.2). It should be remarked that in his paper [22], [23, p. 140, equation (22)], where a different definition of S_n is used, Ramanujan gives a very ingenious proof of (14.2). Rankin [25] has given an elementary proof of (14.2) as well as some other recursion formulas for S_{2n}. His paper also contains other references to the literature. However, the recursion formula of Entry 14, which is incompletely stated by Ramanujan [24, vol. 2, p. 191] in his notebooks, does not appear to have been given elsewhere in the literature.

ENTRY 14. *If n is an even integer exceeding 4, then*

$$-\frac{(n+2)(n+3)}{2n(n-1)}S_{n+2} = -20\binom{n-2}{2}S_4S_{n-2}$$
$$+\sum_{k=1}^{[(n-2)/4]}{}' \binom{n-2}{2k}\{(n+3-5k)(n-8-5k)-5(k-2)(k+3)\}S_{2k+2}S_{n-2k},$$

where the dash ' on the summation sign indicates that if $(n-2)/4$ is an integer, then the last term of the sum is to be multiplied by 1/2.

Proof. First, rewrite Entry 14 in the form

$$\frac{(n+2)(n+3)}{2}\frac{S_{n+2}}{n!} = 20\frac{S_4}{2!}\frac{S_{n-2}}{(n-4)!}$$
$$-\sum_{k=1}^{[(n-2)/4]}{}' \{(n+3-5k)(n-8-5k)-5(k-2)(k+3)\}\frac{S_{2k+2}}{(2k)!}\frac{S_{n-2k}}{(n-2k-2)!},$$

where n is even and at least 6. With $n=2(m+1)$, where $m \geqslant 2$, the last equality may be rewritten as

$$(14.11)\qquad (m+2)(2m+5)b(m+1) = 10b(1)b(m-1) + 10\sum_{k=1}^{[m/2]}{}' k(m-k)b(k)b(m-k) - (2m^2-m)\sum_{k=1}^{[m/2]}{}' b(k)b(m-k),$$

where (14.10) has been employed. Now (14.2) can be written in the form

$$(14.12)\qquad (2m+5)(m-1)b(m+1) = 6\sum_{k=1}^{[m/2]}{}' b(k)b(m-k), \qquad m \geqslant 2,$$

where the dash $'$ on the summation sign indicates that if m is even, the last summand is to be multiplied by 1/2. Using (14.12) in (14.11), we find that

$$(m+2)(2m+5)b(m+1) = 10b(1)b(m-1) + 10\sum_{k=1}^{[m/2]}{}' k(m-k)b(k)b(m-k) - \tfrac{1}{6}(2m^2-m)(2m+5)(m-1)b(m+1).$$

Thus, it remains to show that, for $m \geqslant 2$,

$$(14.13)\qquad \tfrac{1}{30}(2m+5)(m+1)(2m^2-5m+12)b(m+1) = 2b(1)b(m-1) + \sum_{k=1}^{m-1} k(m-k)b(k)b(m-k).$$

Subtracting $2(m+1)b(m+1)$ from both sides of (14.13), we see that (14.13) is equivalent to

$$(14.14)\qquad \tfrac{1}{30}m(m+1)(2m-1)(2m+1)b(m+1) = 2b(1)b(m-1) + \sum_{k=1}^{m-1} k(m-k)b(k)b(m-k) - 2(m+1)b(m+1),$$

for $m \geqslant 2$.

Now observe that the first expression $2b(1)b(m-1)$ on the right side of (14.14) is the coefficient of z^{2m-2} in the power series for $2b(1)\wp(z)$, by (14.3). Also, by (14.3), the latter two expressions on the right side of (14.14) constitute the coefficient of z^{2m-2} in the power series expansion for $\wp'(z)^2/4$. Lastly, the left side of (14.14) is the coefficient of z^{2m-2} in the expansion of $\wp^{(4)}(z)/120$. Thus, (14.14) follows from the lemma above, and this completes the proof.

Differentiating (14.5), we find that $\wp'''(z) = 12\wp(z)\wp'(z)$, which yields another recursion formula for $b(n)$ midway in complexity between (14.2) and (14.14).

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Some euclidean properties for real quadratic fields

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The goal of this work is the determination of all the real quadratic number fields which are euclidean in a strong sense. The new requirement is that in every division the remainder can be taken to be positive.

To be more precise, let $\mathcal{O}$ be the ring of integers in an algebraic number field K (a finite algebraic extension of the field $\boldsymbol{Q}$ of rational numbers). Let $\mathcal{U}$ be a set of orderings (real primes) of K. If $\alpha, \beta \in K$, the statement "$\alpha < \beta$ (relative to $\mathcal{U}$)" means that $\beta - \alpha$ is positive with respect to each of the orderings in $\mathcal{U}$. We write $N\alpha$ as an abbreviation of $N_{K/\boldsymbol{Q}}(\alpha)$, the absolute norm.

DEFINITION 1. K is *euclidean* mod $\mathcal{U}$ if for every $\alpha, \beta \in \mathcal{O}$ with $\alpha \geqslant \beta > 0$ (relative to $\mathcal{U}$), there exist $\varkappa, \varrho \in \mathcal{O}$ satisfying $\alpha = \beta\varkappa + \varrho$, $|N\varrho| < |N\beta|$, and $\varrho \geqslant 0$ (relative to $\mathcal{U}$).

This notion was introduced by Eichler [3] in 1938 and was considered recently in [7]. We have not found any further investigations of it in the literature. If $\mathcal{U}$ is empty then K is euclidean mod $\mathcal{U}$ exactly when K is euclidean in the classical sense.

One can show that if K is euclidean mod $\mathcal{U}$ for some set $\mathcal{U}$, then K must be euclidean. When $K = \boldsymbol{Q}(\sqrt{d})$ is a real quadratic field it is known exactly when the euclidean property holds. There are 16 cases.

THEOREM 2. *Suppose* $K = \boldsymbol{Q}(\sqrt{d})$ *where* $d > 1$ *is a square-free integer. Then* K *is euclidean if and only if* $d = 2, 3, 5, 6, 7, 11, 13, 17, 19, 21, 29, 33, 37, 41, 57,$ *or* 73.

The proof of this theorem is rather difficult, and many mathematicians have contributed to the final result. Davenport [2] used reduction theory of binary quadratic forms to show that if $\boldsymbol{Q}(\sqrt{d})$ is euclidean then d is bounded. Careful calculations are then needed to classify the remaining cases. See [1] and the discussions in [4] and [5].

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