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On the distribution of multiplicative arithmetical functions

by

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1. Introduction. Throughout the paper, $g(n)$ denotes a strongly multiplicative real valued arithmetical function. That is, for coprime integers m and n, $g(mn) = g(m)g(n)$, and $g(p^k) = g(p)$ for all integers $k \geqslant 1$ and primes p.

We use the notation $A_N\{\ldots\}$ for the number of positive integers $n \leqslant N$ for which the property stated in the dotted space holds.

We say that an arithmetical function $h(n)$ has an *asymptotic distribution* $F(x)$ if, as $N \to +\infty$,

$$F_N(x) = (1/N)A_N\{h(n) \leqslant x\}$$

converges to $F(x)$ at each of its continuity points, and $F(x)$ is a proper distribution function (that is, $F(x)$ is nondecreasing, continuous from the right, and its limits at plus and minus infinity are one and zero, respectively).

Note that in this definition of convergence one may always disregard a denumerable set of points x.

Our interest is the existence of the asymptotic distribution of $g(n)$. Therefore, we may assume, and we do so in the remainder of the paper, that $g(n) \neq 0$. Indeed, if P denotes the collection of those primes p for which $g(p) = 0$, then one can easily see that, as $N \to +\infty$,

$$\lim (1/N)A_N\{g(n) = 0\} = 1$$

whenever

$$\sum_{p \in P} 1/p = +\infty.$$

On the other hand, if the series above is finite then $g(n)$ and $g^*(n)$ have asymptotic distribution concurrently, where $g^*(n)$ is the strongly multiplicative function defined as $g(n)$ if $g(n) \neq 0$, and $g^*(p) = 1$ if $g(p) = 0$.

Now, the combined results of Bakshtys [1] and Galambos [3] on a strongly multiplicative function $g(n)$ reads as follows.

THEOREM 1. *A strongly multiplicative function $g(n)$ has an asymptotic distribution if, and only if, each of the series*

$$\sum^{*}\frac{\log|g(p)|}{p};\quad \sum^{*}\frac{\log^{2}|g(p)|}{p};\quad \sum^{**}\frac{1}{p}$$

converges, where $\sum^{*}$ signifies summation over primes p such that $|\log|g(p)||\leqslant 1$, while all other primes belong to $\sum^{**}$.

Theorem 1 seems to indicate that the existence of the asymptotic distribution of $g(n)$ follows from the existence of the asymptotic distribution of the strongly additive function $f(n)=\log|g(n)|$ (recall that $g(n)\neq 0$). In particular, it appears that Theorem 1 follows from the following result of Erdős and Wintner [2].

THEOREM 2. *A strongly additive arithmetical function $f(n)$ has an asymptotic distribution if, and only if, each of the series*

$$\sum^{*}\frac{f(p)}{p};\quad \sum^{*}\frac{f^{2}(p)}{p};\quad \sum^{**}\frac{1}{p}$$

converges, where $\sum^{*}$ and $\sum^{**}$ are defined as in Theorem 1, with $f(p)$ taking the role of $\log|g(p)|$.

The aim of the present paper is to analyze the relation between Theorems 1 and 2. In addition, we estimate the deviation of the asymptotic distribution $F(x)$ of $g(n)$ from an appropriate symmetric distribution when $F(x)$ itself is not symmetric. The reason for a special interest in symmetric asymptotic distributions is that they are associated with zero mean values of (real valued) multiplicative functions which are usually more difficult to establish than the existence of non-zero mean values.

2. The relation of Theorems 1 and 2. The Erdős–Wintner theorem cannot imply the Bakshtys–Galambos result. Indeed, if $g(p)=\pm 1$ for all p, then $f(n)=\log|g(n)|=0$ for all n, in which case, with or without the Erdős–Wintner theorem, one evidently has that the asymptotic distribution of $f(n)$ exists. If it would now follow that $g(n)$ has an asymptotic distribution as well, one would have as "an evident" result that this particular $g(n)$ has an asymptotic distribution. However, this latter result is a very deep one, and indeed it used to be one of the most celebrated open problems of number theory until it was solved by Wirsing [4] (it, of course, also follows from Theorem 1). Several other examples can be constructed which all speak against the expectation that Theorem 1 would somehow follow from Theorem 2. Indeed, many sieve arguments could be avoided if it would follow that, when the interest is the existence of the density of a set expressible in terms of a multiplicative function $g(n)$, one could always assume that $g(p)>0$.

However, there is a special case when Theorem 2 suffices for establishing the existence of the asymptotic distribution of $g(n)$. This is contained in the following statement.

THEOREM 3. *Let $g(n)$ be a strongly multiplicative function. If*

$$\sum_{g(p)<0} 1/p < +\infty, \tag{1}$$

then the asymptotic distribution of $g(n)$ exists if, and only if, Theorem 2 applies to $f(n)=\log|g(n)|$.

Remark. The emphasis in this statement is that for its conclusion the only tools to be utilized are the concept of an asymptotic distribution and Theorem 2.

Proof. First note that if $g(n)$ has an asymptotic distribution then so does $|g(n)|$, and thus $f(n)=\log|g(n)|$ as well. Consequently, Theorem 2 must apply to $f(n)$.

In proving the converse statement, we introduce the following strongly multiplicative functions. Let

$$g_0(n)=|g(n)|.$$

For each integer $r\geqslant 1$, define $g_r(n)$ as $g(n)$ if, for each prime divisor p of n, $g(p)>0$. On the other hand, denoting by $q_1<q_2<\ldots$ those primes for which $g(q_j)<0$, we define $g_r(q_j)=g(q_j)$ if $j\leqslant r$, and $g_r(q_j)=1$ if $j>r$. Hence,

$$\{g_r(n)\neq g(n)\}\subseteq\{\text{for some } j>r,\ q_j|n\}. \tag{2}$$

Define

$$F_{r,N}(x)=(1/N)A_N\{g_r(n)\leqslant x\},\qquad r\geqslant 0. \tag{3}$$

Then, by (2),

$$|F_N(x)-F_{r,N}(x)|\leqslant (1/N)A_N\{g_r(n)\neq g(n)\}$$
$$\leqslant (1/N)\sum_{j>r}A_N\{q_j|n\}=(1/N)\sum_{j>r}[N/q_j],$$

where $[y]$ signifies the integer part of y. We thus obtained

$$|F_N(x)-F_{r,N}(x)|\leqslant\sum_{j>r}1/q_j, \tag{4}$$

where, on account of (1), the right-hand side tends to zero as $r\to+\infty$ (an empty sum is taken as zero, which occurs above for large r if the set of the q_j is finite). On the other hand, the Erdős–Wintner theorem implies that, as $N\to+\infty$,

$$\lim F_{0,N}(x)=G_0(x) \tag{5}$$

exists for all continuity points of $G_0(x)$, and it is a proper distribution function. We now prove by induction that, for each $r \geqslant 1$, as $N \to +\infty$,

$$\lim F_{r,N}(x) = G_r(x) \tag{6}$$

exists for all continuity points of $G_r(x)$, and $G_r(x)$ is a proper distribution function. We prove (6) by establishing a set of recursive relations, and we then appeal to (5). We write

$$A_N\{g_{r+1}(n) \leqslant x\} = A_N\{g_r(n) \leqslant x,\ q_{r+1} \nmid n\} + A_N\{g_{r+1}(n) \leqslant x,\ q_{r+1} | n\}, \tag{7}$$

where we used the evident fact that $g_{r+1}(n) = g_r(n)$ for all n such that $q_{r+1} \nmid n$. Now, if $q_{r+1}|n$, then there is an integer $k \geqslant 1$ with $n = q_{r+1}^k m$, where m and q_{r+1} are relatively prime, which we abbreviate to $q_{r+1}^k \| n$. Hence,

$$A_N\{g_{r+1}(n) \leqslant x,\ q_{r+1}|n\} = \sum_{k \geqslant 1} A_N\{g_r(n) \geqslant x/g(q_{r+1}),\ q_{r+1}^k \| n\}. \tag{8}$$

Furthermore, in view of the evident relation

$$A_N\{g_r(n) \leqslant x,\ q_{r+1} \nmid n\} = A_N\{g_r(n) \leqslant x\} - A_N\{g_r(n) \leqslant x,\ q_{r+1}|n\},$$

and since $g_r(q_{r+1}) = 1$,

$$A_N\{g_r(n) \leqslant x,\ q_{r+1} \nmid n\} = A_N\{g_r(n) \leqslant x\} - \sum_{k \geqslant 1} A_N\{g_r(n) \leqslant x,\ q_{r+1}^k \| n\}. \tag{9}$$

Notice that the dependence on k in the sums of (8) and (9) appears only through $q_{r+1}^k \| n$. Hence, every $n \leqslant N$ which is a multiple of q_{r+1} gets counted in these sums. Consequently, if we assume that (6) holds for r, then (7), (8) and (9) yield that (6) holds for $r+1$ at the continuity points of $G_r(x)$, and

$$G_{r+1}(x) = \left(1 - \frac{1}{q_{r+1}}\right) G_r(x) + \frac{1}{q_{r+1}} \left\{1 - G_r\left(\frac{x}{g(q_{r+1})}\right)\right\}. \tag{10}$$

However, since the set of discontinuity points of $G_r(x)$ is denumerable, (6) holds for all $r \geqslant 1$ (see the remark in Section 1). We also have from (10) that, because $G_0(x)$ is a proper distribution function, so are all $G_r(x)$. Furthermore, (10) implies that

$$|G_{r+1}(x) - G_r(x)| \leqslant 2/q_{r+1},$$

and thus, in view of (1),

$$F(x) = \lim G_r(x) \quad (r \to +\infty) \tag{11}$$

exists for all x for which each $G_r(x)$ is continuous. In other words, $F(x)$ is defined for all x except on a denumerable set. But for each x for which (11) holds we get from (4), by letting first $N \to +\infty$ and then $r \to +\infty$,

$$\lim F_N(x) = F(x) \quad (N \to +\infty).$$

This last limit relation remains to hold if $F(x)$ is extended to the whole real line by right continuity. Finally, because for continuity points of $G_0(x)$ and $F(x)$,

$$G_0(x) = F(x) - F(-x), \quad x > 0,$$

$F(x)$ is a proper distribution function because $G_0(x)$ is. This completes the proof.

3. Limiting distribution of $g(n)$ and symmetric distributions. We conclude the paper with an estimate. When the series in (1) is divergent, the result of Galambos [3] states that the limiting distribution $F(x)$ of $g(n)$ satisfies the relation

$$F(x) = 1 - F(-x)$$

for all continuity points of $F(x)$. This property is referred to as "$F(x)$ is symmetric". While $F(x)$ is never symmetric under (1), except in the trivial case $g(2) = -1$, we can estimate from (10) that $F(x)$ comes close to being symmetric if the series in (1) is "large". As a matter of fact, putting

$$K = \sum_{g(p) < 0} 1/p \quad \text{and} \quad \Delta(x) = 1 - F(x) - F(-x),$$

we prove the following inequality.

THEOREM 4. *With the preceding notations, if $g(q_j) = -1$ for all j, then*

$$|\Delta(x)| \leqslant e^{-2K}. \tag{12}$$

(The assumption $g(q_j) = -1$ is made only for the simplicity of the inequality (12). The fact that $\Delta(x) \to 0$ as $K \to +\infty$ remains to hold without this assumption.)

Proof. First, put

$$\Delta_r(x) = 1 - G_r(x) - G_r(-x).$$

Then, on account of (10),

$$\Delta_{r+1}(x) = (1 - 2/q_{r+1})\, \Delta_r(x).$$

Consequently,

$$\Delta_{r+1}(x) = \left(1 - \frac{2}{q_{r+1}}\right)\left(1 - \frac{2}{q_r}\right) \cdots \left(1 - \frac{2}{q_1}\right) \Delta_0(x),$$

from which the elementary inequality

$$1 - z \leqslant e^{-z}, \quad z > 0,$$

yields

$$|\Delta_{r+1}(x)| \leqslant e^{-2(K-\varepsilon)},$$

where

$$\varepsilon = \sum_{j > r+1} 1/q_j.$$

Now, by letting $N \to +\infty$ in

$$\{1 - F_N(x) - F_N(-x)\} - \{1 - F_{r+1,N}(x) - F_{r+1,N}(-x)\},$$

we get from (4)

$$|\Delta(x) - \Delta_{r+1}(x)| \leqslant 2\varepsilon,$$

that is,

$$|\Delta(x)| \leqslant |\Delta_{r+1}(x)| + 2\varepsilon \leqslant e^{-2(K-\varepsilon)} + 2\varepsilon.$$

A passage to the limit, as $r \to +\infty$, yields (12).

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On the distribution of the values of Euler's function

by

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1. Introduction. Let $V(x)$ denote the number of distinct values of $\varphi(n)$ not exceeding x, where φ denotes Euler's function. Since every number of the form $p-1$ where p is a prime is a value of $\varphi(n)$, the prime number theorem immediately gives that $V(x)$ is at least of order of magnitude $x/\log x$. On the other hand it is relatively easy to see that $V(x) = o(x)$ since most values of $\varphi(n)$ are divisible by a high power of 2 and most integers are not (see Niven and Zuckerman [8], Th. 11.9).

In 1935, Erdős [2] showed that

$$V(x) = O\left(x/(\log x)^{1-\varepsilon}\right) \tag{1.1}$$

for every $\varepsilon > 0$. This result was improved in 1973 by Erdős and Hall [3] to

$$V(x) = O\left(\frac{x}{\log x}\exp(B\sqrt{\log\log x})\right) \tag{1.2}$$

for every $B > 2\sqrt{2/\log 2}$. In 1976, Erdős and Hall [4] obtained the lower bound

$$V(x) \gg \frac{x}{\log x}\exp\{A(\log\log\log x)^2\} \tag{1.3}$$

for every $A < 1/\log 16$. (For positive $f(x)$, $g(x)$ the notation $f(x) \gg g(x)$ is equivalent to $g(x) = O(f(x))$.) In [4], Erdős and Hall state that they do not know which of (1.2), (1.3) is nearer to the truth about $V(x)$.

In this paper, I prove an upper bound for $V(x)$ of the same shape as the lower bound (1.3).

THEOREM. *For every* $C > (\log 4 - 2\log(2-\log 2))^{-1} = 1.175018095\ldots$,

$$V(x) = O\left(\frac{x}{\log x}\exp\{C(\log\log\log x)^2\}\right). \tag{1.4}$$

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