On index formulas of Siegel units
in a ring class field

by

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Introduction. Let \( \Sigma = Q(\sqrt{d}) \) be an imaginary quadratic number field with discriminant \( d \), and let \( K \) be an abelian extension over \( \Sigma \) with rational conductor \( f \) such that \( K \) is contained in the ring class field \( N_f \) modulo \( f \) over \( \Sigma \). In the previous paper [5], we studied the several properties on Siegel unit \( \delta_K(C) \) in \( K \) and gave an index formula related to \( \delta_K(C) \). The purpose in the present study is to give other kinds of index formulas related to \( \delta_K(C) \), using the results in [5].

In Section 1, we shall summarize the properties of \( \delta_K(C) \) and some preliminary facts in the theory of group algebra. In Section 2, we shall give two index formulas related to \( \delta_K(C) \) (Theorems 1, 2). The unit group \( \delta_K \) dealt with in Theorem 1 is an analog of the group of cyclotomic units in a real abelian number field studied by Leopoldt ([17]). The main task in this paper is to deduce Theorem 2 from Theorem 1, and it will be done in Section 3. The unit group \( \delta_K \) dealt with in Theorem 2 is fairly larger than \( \delta_K \). The method used in Section 3 is mostly based on the fundamental facts in Section 1, and partially similar to that of Gillard ([22]). Of course these two groups are quite different from those in [2]. As an application of Theorem 2, a formula of Schertz ([3], Satz 3.3) can be proved under the restriction that \( K \subset N_f \) (Corollary 2). Moreover, using the results on Siegel unit in Section 1, we shall construct three unit groups \( \delta_K, \delta_K^*, \delta_K^{**} \) such that \( \delta_K \subset \delta_K^* \) and \( \delta_K \subset \delta_K^{**} \subset \delta_K^{*} \), and give the more refined index formulas for them (Propositions 1, 2, 3). Proposition 3 permits us in some special case to express the class number quotient \( h_K/h \) just as the index of \( \delta_K^{**} \) in the group of the whole units in \( K \).

1. Notations and preliminary. Let \( Q \) be the field of rational numbers, \( Z \) the ring of rational integers and \( \Sigma = Q(\sqrt{d}) \) an imaginary quadratic number field with discriminant \( d \). Let \( K \) be an abelian extension over \( \Sigma \) with rational conductor \( f \) such that \( K \) is contained in the ring class field \( N_f \) modulo \( f \) over \( \Sigma \). We denote by \( O_f \) the order in \( \Sigma \) with conductor \( f \) (\( O_1 \) means the
maximal order in \( \Sigma \) and by \( \mathcal{R}(f) \) the ring ideal class group modulo \( f \), i.e. the group of the equivalent classes of proper \( O_f \)-ideals in \( \Sigma \) by the usual relation.

Let \( \sigma: \mathcal{R}(f) \to \text{Gal}(N_f/\Sigma) \) be the isomorphism from \( \mathcal{R}(f) \) to the galois group of \( N_f \) over \( \Sigma \) via Artin's reciprocity law, and let \( \mathcal{S} \) be the subgroup of \( \mathcal{R}(f) \) whose image by \( \sigma \) is equal to the galois group of \( N_f \) over \( K \).

Hereafter, for each number field \( K \), we denote by \( h_\Sigma \), \( E_\Sigma \), \( \mu_\Sigma \) and \( w_\Sigma \) respectively the class number of \( K \), the unit group of \( K \), the torsion part of \( E_\Sigma \), and the number of elements in \( \mu_\Sigma \). (When \( - \equiv \Sigma \), the subscript \(-\) is omitted from these notations.)

Let \( C \) be a class in \( \mathcal{R}(f) \) and let \( \alpha_f \) be an \( O_f \)-ideal in \( C^{-1} \). Then there exists an element \( \alpha \) in \( \Sigma \) such that \((\alpha_f O_f)^{1} = (\alpha)\) as a principal \( O_f \)-ideal. We define the Siegel unit \( \delta_f(C) \) by

\[
\delta_f(C) = \alpha^{12} \left( \frac{\Delta(\alpha_f)}{\Delta(O_f)} \right)^{h_\Sigma}
\]

Herein \( \Delta(\ ) \) means the usual lattice function expressed by using the Dedekind eta-function as follows:

\[
\Delta(m) = \left( \frac{2\pi}{\omega_1} \right)^{12} \eta \left( \frac{\omega_1}{\omega_2} \right)^{24},
\]

where \( m = [\omega_1, \omega_2] \) is a 2-dimensional complex lattice with \( \mathbb{Z} \)-basis \( \{\omega_1, \omega_2\} \), \( \text{Im}(\omega_1/\omega_2) > 0 \). \( \delta_f(C) \) depends only on the class \( C \), not on the choices of \( \alpha \) and \( \alpha_f \), and is a unit in \( N_f \). As is well known, \( \delta_f(C_1) \delta_f(C_2) = \delta_f(C_1 C_2) \delta_f(C_2) \) for any \( C_1, C_2 \) in \( \mathcal{R}(f) \), and \( \delta_f(C_0) = 1 \) for the unit class \( C_0 \) in \( \mathcal{R}(f) \).

We define \( \delta_k(C) \) as the relative norm of \( \delta_f(C) \) w.r.t. \( N_f/K \), i.e.

\[
\delta_k(C) = \prod_C \delta_f(C^C)/\delta_f(C),
\]

where \( C \) runs over all classes in \( \mathcal{S} \).

Remark 1. In his paper [8], Schertz dealt with the similar unit \( \theta_k(C) \), which is defined by:

\[
\theta_k(C) = \beta^{12} \prod_{t \neq 0} \left( \frac{\Delta(\alpha_f b_j)}{\Delta(b_j)} \right)^{w_k}
\]

Herein \( b_j \) ranges over a system of the complete representative \( O_f \)-ideals of all classes in \( \mathcal{S}_k \), \( m_k \) is the least positive rational integer such that \( m_k \neq 0 \pmod{h} \) and \( \beta \) is an element in \( \Sigma \) such that \((\alpha_f O_f)^{w_k \eta_0} = (\beta)\) as a principal \( O_f \)-ideal in \( \Sigma \). As can be easily seen, \( \delta_k(C) = \theta_k(C)^{\frac{w_k \eta_0}{\beta}} \).

However in our arguments, we use \( \delta_k(C) \) instead of \( \theta_k(C) \) for a few reasons. (For example, one should refer to Lemma 6 in Section 3.)

Now let \( l_k \) be the homomorphism from \( \mathcal{R}(f) \) to a subgroup \( \mathcal{R} \) of \( \mathbb{Z} \langle 24 \rangle \) defined in [6]. Herein \( l_k \) is uniquely determined by \( K/\Sigma \), and indeed \( \mathcal{R} \) can be realized by using the \( K \)-admissible \( O_f \)-ideals in \( \Sigma \). Of course \( l_k(C^3) = 1 \pmod{24} \) for any \( C \) in \( \mathcal{R}(f) \). Then the following lemma holds.

**Lemma 1** ([6], Prop. 1). Let \( C_1 \) and \( C_2 \) be any two classes in \( \mathcal{R}(f) \) and let \( n \) be the least positive rational integer such that

\[
n(l_k(C_1)-1)(l_k(C_2)-1) \equiv 0 \pmod{24}.
\]

Then \( (\delta_k(C_1) \delta_k(C_2))/\delta_k(C_1) \delta_k(C_2) \) is contained in \( E_k^{24} \).

In the rest of this section, we shall provide some preliminary facts from the theory of group algebra.

Let \( G \) be the galois group of \( K \) over \( \Sigma \), and let \( g \) be its order. We denote by \( \mathbb{Z} \langle g \rangle \) (resp. \( \mathbb{Q} \langle g \rangle \)) the group ring of \( G \) over \( \mathbb{Z} \) (resp. \( \mathbb{Q} \)) and by \( \mathcal{A}(K/\Sigma) \) the character group of \( G \). For any \( \chi \) in \( \mathcal{A}(K/\Sigma) \), we denote by \( \chi \) the block to which \( \chi \) belongs, i.e.

\[
\chi = \{ \xi \in \mathcal{A}(K/\Sigma); \langle \xi \rangle = \langle \chi \rangle \}.
\]

Herein \( \langle \cdot \rangle \) means the cyclic group generated by \( \cdot \). Every character in \( \chi \) possesses the common kernel \( U_2 \) and common order \( g_2 \). We often use the same notation \( \chi \), identifying it with the \( Q \)-irreducible character of \( G \) to which \( \chi \) belongs. Let \( K_2 \) be the intermediate field of \( K/\Sigma \) associated with \( U_2 \). Then \( K_2/\Sigma \) is a cyclic extension of degree \( g_2 \) and its galois group \( G_2 \) is isomorphic to \( G/\Sigma U_2 \). If \( K_2 \subseteq K_2 \), i.e. \( U_2 \subseteq U_2 \), then we denote this relation by \( \chi \subseteq \chi \). Let \( \chi \) be a total order in the finite set \( \{ \chi \}; \chi \in \mathcal{A}(K/\Sigma) \}, satisfying the condition that \( \chi \leq \chi \) whenever \( \chi \leq \chi \) (cf. [2]). Hereafter we keep this ordering.

For any \( \psi \), \( \chi \) such that \( \psi \neq \chi \), we denote by \( U_2 \chi \) the subgroup of \( G_2 \) which corresponds to the subfield \( K_2 \), and we let

\[
v_{2, \chi} = \sum_{\gamma \in U_2 \chi} \gamma.
\]

The following lemma is well known as a result of Martinet (cf. [2]) and very useful in our later arguments.

**Lemma 2.** Let \( \mathcal{A}_2 \) be the ideal in \( \mathbb{Z} \langle g_2 \rangle \) generated by \( \{v_{2, \chi}; \chi \neq \chi\} \). Then

\[
\mathcal{A}_2 = P_{g_2}(\mathcal{S}_2) \mathbb{Z} \langle g_2 \rangle,
\]

where \( \mathcal{S}_2 \) is a generator of \( G_2 \) and \( P_{g_2}(X) \) is the \( g_2 \)-th cyclotomic polynomial.

The idempotent \( e_2 \) corresponding to the \( Q \)-irreducible character \( \chi \) is given by

\[
e_2 = 1 \sum_{\gamma \in \gamma(S)} \chi(S^{-1}) S.
\]
As is well known, $\sum_{j=1}^k e_j = 1$ in $Q[G]$ and $Q[G]e_k = Q(\zeta_{2k})$, where $\zeta_{2k}$ is a primitive $2k$-th root of unity. Moreover, we define $\tilde{\alpha}_k$ in $Q[G_2]$ and $\gamma_k$ in $Z[G_2]$ respectively by

$$\gamma_k = \frac{1}{g_2} \frac{1}{s \in G_2} \tilde{\alpha}(\bar{S}^{-1}, \bar{S}) \quad \text{and} \quad \gamma_k = \prod_{\nu} \langle 1 - S_\nu^{2\nu} \rangle,$$

where $\tilde{\alpha}_k$ is a generator of $G_2$ and $p$ runs over all prime divisors of $g_2$. Of course $\gamma_k$ depends on the choice of $\tilde{\alpha}_k$, but as an ideal in $Z[G_2]$, $\gamma_k Z[G_2]$ is uniquely determined independently of the choice of $\tilde{\alpha}_k$. Plainly $\gamma_k$ is the restriction of $e_k$ to $Q[G_2]$. For each $\tilde{\alpha}_k$, it is easy to see that

$$\gamma_k \tilde{\alpha}_k = \gamma_k.$$

2. Statement of results. Let the notations be the same as in the preceding sections. For any $\tilde{\alpha}_k$, $K_2$ is contained in the ring class field $N_{K_2}$ modulo $g_2$ over $\Sigma$. Of course $g_2$ is a divisor of $f$. Let $\tilde{\alpha}_k$ be the subgroup of $K_2$ which corresponds to $K_2$ and $C_2$ a fixed class in $K_2$ such that the coset $C_2 \tilde{\alpha}_k$ is a generator of $K_2$. Then we may take $K_2 \tilde{\alpha}_k$ as a generator of $C_2$. We define two unit $e_2$ and $\tilde{\alpha}_2$ in $K_2$ respectively by

$$e_2 = \delta_{K_2}(C_{2}), \quad \text{and} \quad \tilde{\alpha}_2 = \tilde{\alpha}_{K_2}(C_{2}).$$

Then $N_{K_2 K_2}(e_2) = 1$ for any $\tilde{\alpha}(\tilde{\alpha} \neq \tilde{\alpha})$. Herein $N_{K_2 K_2}$ means the relative norm w.r.t. $K_2/K_2$. Furthermore we define two subgroups $\delta_k$ and $\delta_k$ of $E_k$ as follows:

$$\delta_k = \mu_k \prod_{\tilde{\alpha}_k} \delta_k, \quad \text{where} \quad \delta_k = \mu_k \tilde{\alpha}_k \tilde{\alpha}_{K_2}(C_{2}),$$

$$\delta_k = \mu_k \prod_{\tilde{\alpha}_k} \delta_k, \quad \text{where} \quad \delta_k = \mu_k \tilde{\alpha}_k \tilde{\alpha}_{K_2}(C_{2}).$$

Then we have the following index formulas.

Theorem 1. $(E_k; \delta_k) = M_1 h_K^2$, where

$$M_1 = \frac{w}{w_k} (24h)^{a_1} \sqrt{g_2^{d_2}} \prod_{\nu = 1}^{g_2(1)} P_{g_2}(1).$$

Herein $D_2$ means the discriminant of the cyclotomic field $Q(\zeta_{2k})$.

Theorem 2. $(E_k; \delta_k) = M_1 h_K^2$, where

$$M_2 = \frac{w}{w_k} (24h)^{a_1} \sqrt{g_2^{d_2}} \prod_{\nu = 1}^{g_2(1)} P_{g_2}(1).$$

Herein $\varphi(\ )$ means the Euler function.

Theorem 1 is an analog of Leopoldt's formula on cyclotomic units in a real abelian number field ([7]), and its proof can be accomplished by the almost similar way to that in [7]. Now it is evident that $\delta_k \subseteq \delta_k$. Then in order to prove Theorem 2, we have only to prove that

$$(\delta_k; \delta_k) = \prod_{\nu = 1}^{g_2(1)} \frac{g_2^{d_2}}{g_2(1)}.$$

We shall give a complete proof for the formula (2.1) in the next section. In the rest of this section we shall give several index formulas which can be derived from Theorems 1 and 2.

Taking into consideration the fact that $N_{K_2 K_2}(e_2) = 1$ for any $\tilde{\alpha}(\tilde{\alpha} \neq \tilde{\alpha})$ and the result of Lemma 2, we have

$$\delta_k = \langle \delta_k \rangle, \quad 0 \leq j \leq \varphi(g_2) - 1 \rangle_{K_2},$$

$$= \langle \delta_k(C_2); 1 \leq j \leq \varphi(g_2) \rangle_{K_2},$$

where $\delta_k(C_2)$ means $\delta_k(C_{2})$. Then we have the following:

Corollary 1. $(g - 1)$ units $\{\delta_k(C_2); \tilde{\alpha} \neq 1, 1 \leq i_2 \leq \varphi(g_2)\}$ constitute a maximal system of the independent units in $K_2$ and generate $\delta_k$ in Theorem 1.

In Theorem 2, it is evident that

$$\delta_k = \langle \delta_k \rangle, \quad 0 \leq j \leq \varphi(g_2) - 1 \rangle_{K_2}.$$

On the other hand, by Lemma 2 and Lemma 6 (in § 3), we have

$$\delta_k = \prod_{\tilde{\alpha}_k} \langle \delta_k \rangle, \quad 0 \leq j_2 \leq \varphi(g_2) - 1 \rangle_{K_2}.$$

Then we have the following:

Corollary 2. $(g - 1)$ units $\{\delta_k(C_2); \tilde{\alpha} \neq 1, 1 \leq i_2 \leq \varphi(g_2)\}$ constitute a maximal system of the independent units in $K_2$ and generate $\delta_k$ in Theorem 2.

Replacing $\delta_k(C_2)$ in Corollary 2 by $\delta_2(C_2)$ (Remark 1), we have the similar formula as follows:

Corollary 2'. $(g - 1)$ units $\{\delta_2(C_2); \tilde{\alpha} \neq 1, 1 \leq i_2 \leq \varphi(g_2)\}$ generate a subgroup $\Theta_2$ of finite index in $E$, for which the following index formula holds:

$$(E; \Theta_2) = \frac{w}{w_k} (24h)^{a_1} \sqrt{g_2^{d_2}} \prod_{\nu = 1}^{g_2(1)} \frac{m_2^{d_2}}{g_2(1)}.\sqrt{D_2} \prod_{\nu = 1}^{g_2(1)} P_{g_2}(1)).$$

Herein $m_2$ is the least natural number such that $m_2 \equiv \bar{S}_2 \equiv 0 \pmod{h}$.

Corollary 2' is essentially equal to the formula of Schertz ([8], Satz 3.3 under the restriction that $K \subseteq N_2$). Namely $\langle x(C_2); 1 \leq i_2 \leq \varphi(g_2) \rangle_{K_2}$
constitute an integral basis of the principal ideal \((g_2 - 1)\) in \(\mathbb{Z}[g_2]\), whose absolute norm is equal to \(P_{\mathfrak{q}}(1)\) (see Satz 1.7 in [8]).

Now we are going to deduce the more refined index formulas from Theorems 1 and 2. Namely, using Lemma 1, we shall construct three large unit groups \(\delta_2^2, \delta_2^2, \delta_2^2\) such that \(\delta_2^2 \subseteq \delta_2^2\) and \(\delta_2^2 \subseteq \delta_2^2\).

First we note that \(\delta_2^2\) is the 24th power in \(E_{K_2}\) with the exception of the case where \(g_2 = 2\). Indeed if \(g_2\) is odd, then it is always possible to choose \(C_2\) as a square of some class in \(\mathbf{A}(f_2)\) and hence by Lemma 1, \(\delta_2^2\) is in \(E_{K_2}^{24}\). If \(g_2\) is even but not equal to 2, then there exists a prime divisor \(p\) of \(g_2\) such that \(g_2/p = 2\). Then by the following equality

\[
\delta_2^2(C_2) \delta_2^2 = \frac{\delta_2^2(C_2) \delta_2^2(C_2^{p_2})}{\delta_2^2(C_2^{p_2 + 1})}
\]

and Lemma 1, we may conclude that \(\delta_2^2\) is contained in \(E_{K_2}^{24}\).

In the case where \(g_2 = 2\), we have

\[
\delta_2^2 = \delta_2^2(C_2) \delta_2^2 = \frac{\delta_2^2(C_2)}{\delta_2^2(C_2^2)},
\]

and hence by Lemma 1

\[
\delta_2^2 \in E_{K_2}^{24}\text{ with } n_2 = 1, 2, 3.
\]

Here \(n_2 = 3\) occurs in the only case where \(\Sigma \neq \mathbb{Q}(\sqrt{-3})\) and \(K_2 = \mathbb{Q}(\sqrt{-3})\), and \(n_2 = 2\) occurs in the only case where \(\Sigma = \mathbb{Q}(\sqrt{-3})\) and \(K_2 = \mathbb{Q}(\sqrt{-4})\). Moreover it always holds that \(\prod_{\varphi = 2} \delta_2^2 = \mu_{K_2}\). Hence for each \(\mathfrak{p}\), we may choose a unit \(\tilde{\eta}_2\) in \(E_{K_2}\) such that

\[
\begin{cases}
\delta_2 = \tilde{\eta}_2^{24} & \text{for } g_2 \neq 2, \\
\delta_2 = \tilde{\eta}_2^{24} & \text{for } g_2 = 2.
\end{cases}
\]

Remark 2. Units \(\tilde{\eta}_2\) in the equation \(2.2\) can be explicitly given by using the values of the Dedekind eta-function (see the proof of Proposition 1 in [6]). Of course \(N_{K_2/K_2}(\tilde{\eta}_2) \in \mu_{K_2}\) for any \(\varphi \neq \mathbf{A}\).

We define \(\delta_2^2\) by

\[
\delta_2^2 = \mu_{K_2}\prod_{\varphi} \tilde{\eta}_2^2 \text{ where } \delta_2^2 = \mu_{K_2}\tilde{\eta}_2^2.
\]

Then we have the following:

**Proposition 1.** \((E_{K_2}; \delta_2^2) = M_1^2 h_1\), where

\[
M_1^2 = \sqrt{g_2 - 2} \prod_{\varphi \neq 1} P_{\mathfrak{q}}(1).
\]

**Corollary 3.** In Proposition 1, if \(K/\Sigma\) is a cyclic extension, then

\[
M_1^2 = \prod_{\varphi \neq 1} \frac{\varphi(g_2)}{|D_{K_2}|}.
\]

**Corollary 4.** In Proposition 1, if \(K/\Sigma\) is a cyclic extension of degree \(p^a\) with a prime number \(p\), then

\[
M_1^2 = \frac{p^{a(g_2 - 1)} - 1}{p^{a(g_2 - 1)} - 1}.
\]

**Proof.** When \(K/\Sigma\) is a cyclic extension, it always holds that

\[
\sqrt{g_2 - 2} \prod_{\varphi \neq 1} P_{\mathfrak{q}}(1)^{1/|D_{K_2}|} = 1
\]

(Hasse [4]). Hence Corollaries 3 and 4 follow.

Remark 3. Proposition 1 can be used for the numerical determination of \(h_2/h\) and a system of the fundamental units in \(K\). For this subject the similar methods to those used in the paper of G. Gras and M.-N. Gras ([3]) can be applied (see also [5]).

Next we have

\[
\langle \alpha \rangle = \left\langle \alpha, \delta_2^{\mathfrak{p}, -1}, \delta_2^{\mathfrak{p}, 1}, \delta_2^{\mathfrak{p}, -2} \right\rangle_{K_2} \subseteq \langle \delta_2, \delta_2, \delta_2 \rangle_{K_2} \subseteq \langle \delta_2, \delta_2, \delta_2 \rangle_{K_2}.
\]

Moreover since

\[
\delta_2^{\mathfrak{p}, -1} = \frac{\delta_2^{\mathfrak{p}, 1}}{\delta_2^{\mathfrak{p}, 2}} \delta_2^{\mathfrak{p}, 2} = \delta_2^{\mathfrak{p}, 1} = \delta_2^{\mathfrak{p}, 2},
\]

we have

\[
\left\langle \alpha \right\rangle \subseteq \left\langle \alpha, \delta_2^{\mathfrak{p}, -1}, \delta_2^{\mathfrak{p}, 1}, \delta_2^{\mathfrak{p}, -2} \right\rangle_{K_2} = \left\langle \alpha, \delta_2^{\mathfrak{p}, -1}, \delta_2^{\mathfrak{p}, 1}, \delta_2^{\mathfrak{p}, -2} \right\rangle_{K_2}.
\]

with some units \(\eta_{\mathfrak{p}, i}\) \((i = 2, 3, \ldots, \varphi(g_2))\) in \(E_{K_2}\). Herein \(n_2\) is the least positive rational integer such that \(\eta_{\mathfrak{p}, i}(C_2) - 1) \equiv 0 \pmod{24}\) (by Lemma 1).

Namely \(n_2\) is one of 1, 2, 3, 5. Furthermore,

\[
\delta_2 = \eta_{2, 1}^{24},
\]

with some unit \(\eta_{\mathfrak{p}, 1}\) in \(E_{K_2}\) (by Remark 1). For the simplicity we write

\[
a(\mathfrak{p}) = 24n_2(\# \delta_1, h).
\]

We define a subgroup \(\delta_2^2\) of \(E_2\) by

\[
\delta_2^2 \subseteq \mu_{K_2}\prod_{\varphi \neq 1} \tilde{\eta}_2^2, \text{ where } \delta_2^2 = \langle \eta_{\mathfrak{p}, i}; 1 \leq i \leq \varphi(g_2) \rangle_{K_2}.
\]
Then we have the following

**Proposition 2.** \((E_k; \delta_k^*) = M_k \frac{h_k}{h}\), where

\[
M_k = \frac{w}{w_k} \sqrt{g^{a_1} - 1} \frac{\prod_{\gamma < 1} a(\gamma)}{g^{w(\gamma)}} .
\]

Especially when \((g, h) = 1\) then \(# \delta_k\) must be divisible by \(h\), and when \(g = \text{odd}\) in addition, then \(\delta_k\) is the 24h-th power of a unit in \(E_k\). (Note that \(\{A(\delta)/A(\delta)\}\) is contained in \(N^{24}_{\delta}\) \([1], \text{p. } 41)\). Therefore we have the following

**Proposition 3.** Assume that \(g\) is odd and \((g, h) = 1\). Then for each \(\gamma_i \in \mu_k\), \(\delta_i = \eta_i^{24a}\) with some unit \(\eta_i\) in \(E_k\), and \((g - 1)\) units \([\eta_i^{24a} : \gamma_i \neq 1, 0 \leq i \leq \varphi(g) - 1\] generate a subgroup \(\delta_k^*\) of \(E_k\), for which the following index formula holds:

\[
(E_k; \delta_k^*) = \frac{h_k}{h} \sqrt{g^{a_1} - 1} \prod_{\gamma < 1} \frac{P_{\gamma}(1)}{g^{w(\gamma)}} .
\]

**Corollary 5.** If \(K/\mathbb{Q}\) is a cyclic extension in addition to the conditions of Proposition 3, then

\[
(E_k; \delta_k^*) = \frac{h_k}{h} .
\]

### 3. Proof of Theorem 2

The purpose in this section is to prove the formula (2.1). For this purpose we shall provide several lemmas in advance.

The following lemma is a well-known fact in the group theory.

**Lemma 3.** Let \(A \rightarrow A^n\) be a homomorphism defined on an abelian group \(A\) and let \(A_0\) be its kernel. Let \(B\) be a subgroup of \(A\) with finite index and \(B_0 = A_0 \cap B\). Then

\[
(A:B) = (A^n:B^n)(A_0:B_0) .
\]

Now let \(L: E_k \rightarrow R^0\) be the usual homomorphism from \(E_k\) to the \(g\)-dimensional Euclidian space \(R^0\) defined by

\[
L(e) = (\ldots, \log |e|^{24}, \ldots = 0 \quad \text{for any } e \in E_k .
\]

Then the kernel of \(L\) is \(\mu_k\) and the image \(L(E_k)\) of \(E_k\) is a lattice in \(R^0\), which can be seen as a \(\mathbb{Z}[G]\)-module.

**Lemma 4.** \(L(\tilde{\delta}_k) = \bigoplus_{\eta \neq 1} L(\tilde{\delta})\) (dir. sum.), and for each \(\eta\), \(L(\tilde{\delta}_k)\) is a lattice of free rank \(\varphi(g)\).

Proof. Let \(u\) be any element in \(L(\tilde{\delta}_k) \cap \sum_{\eta \neq 1} L(\tilde{\delta})\) and we let

\[
u = L(\tilde{\delta}_k) = \sum_{\eta \neq 1} L(\tilde{\delta}_k), \quad \text{where } \nu_1 \text{ and } \nu_2 \text{ are suitable elements in } \mathbb{Z}[G]_k\text{ and } \mathbb{Z}[G_j]\text{ respectively. Then}

\[
u = L(\tilde{\delta}_k) = \sum_{\eta \neq 1} L(\tilde{\delta}_k) = \sum_{\eta \neq 1} L(\tilde{\delta}_k).
\]

Taking action of \(g\) on the both sides of the above equation, we see that \(\tilde{\delta}_k\) is in \(\mu_k\), i.e. \(u = 0\). Because \(\tilde{\delta}_k = \tilde{\delta}_k\), and \(L(\tilde{\delta}_k) = \sum_{\eta \neq 1} \tilde{\delta}_k \equiv 1\)

by (1.1)). The second assertion is evident by Theorem 1.

**Lemma 5.** \(L(\tilde{\delta}_k) \cap \sum_{\eta \neq 1} L(\tilde{\delta}) = \{0\}\).

Proof. Let \(u\) be any element in \(L(\tilde{\delta}_k) \cap \sum_{\eta \neq 1} L(\tilde{\delta})\) and we let

\[
u = L(\tilde{\delta}_k) = \sum_{\eta \neq 1} L(\tilde{\delta}_k), \quad \text{where } \nu_1 \text{ and } \nu_2 \text{ are suitable elements in } \mathbb{Z}[G]_k\text{ and } \mathbb{Z}[G_j]\text{ respectively. Then}

\[
u = L(\tilde{\delta}_k) = \sum_{\eta \neq 1} L(\tilde{\delta}_k) = \sum_{\eta \neq 1} L(\tilde{\delta}_k).
\]

By taking the relative norm \(N_{K/K}\) of both sides, we have

\[
u = L(\tilde{\delta}_k) = \sum_{\eta \neq 1} L(\tilde{\delta}_k) = \sum_{\eta \neq 1} L(\tilde{\delta}_k).
\]

Therefore \(\nu\) must be in \(\mu_k\), and \(u = 0\).

**Lemma 6.** For any two \(\tilde{\delta}, \tilde{\gamma}\) such that \(\tilde{\gamma} \neq \tilde{\delta}\),

\[
N_{K/K}\delta(\tilde{\delta}) = \sum_{\eta \neq 1} L(\tilde{\delta}_k) .
\]

Proof. From the definition \(\delta_k = N_{K/K}\delta_k(\tilde{\delta}_k(C_2))\), and hence

\[
N_{K/K}\delta(\tilde{\delta}) = N_{K/K}\delta_k(\tilde{\delta}_k(C_2)) .
\]

Then in the case where \(\delta_k = \delta_0\), then assertion is trivial. For the case where \(\delta_k \neq \delta_0\), we can apply the results of Lemma 3 in [6].

**Proof of (2.1).** By Lemma 3, \((\delta_k; \tilde{\delta}_k) = (L(\tilde{\delta}_k); L(\tilde{\delta})_k)\). Now let \(\tilde{\delta} \neq \tilde{\delta}_k \neq \tilde{\delta}_1 \neq \tilde{\delta}_2 \ldots\) be the arrangement by the ordering \(<\) in Section 1. Then

\[
(L(\tilde{\delta}_k); L(\tilde{\delta})) = \sum_{\tilde{\delta}_k \neq \tilde{\delta}_1 \neq \tilde{\delta}_2 \ldots} \sum_{\tilde{\delta}_1 \neq \tilde{\delta}_2 \neq \tilde{\delta}_3 \ldots} \sum_{\tilde{\delta}_2 \neq \tilde{\delta}_3 \neq \tilde{\delta}_4 \ldots} \cdots .
\]

\[
= \prod_{\tilde{\delta}_k \neq \tilde{\delta}_1 \neq \tilde{\delta}_2 \ldots} \sum_{\tilde{\delta}_2 \neq \tilde{\delta}_3 \neq \tilde{\delta}_4 \ldots} \sum_{\tilde{\delta}_3 \neq \tilde{\delta}_4 \neq \tilde{\delta}_5 \ldots} \cdots .
\]

\[
= \prod_{\tilde{\delta}_k \neq \tilde{\delta}_1 \neq \tilde{\delta}_2 \ldots} \sum_{\tilde{\delta}_2 \neq \tilde{\delta}_3 \neq \tilde{\delta}_4 \ldots} \sum_{\tilde{\delta}_3 \neq \tilde{\delta}_4 \neq \tilde{\delta}_5 \ldots} \cdots .
\]
and each $\tilde{x}$-factor of the above last product can be calculated as follows:

Let $x: \mathcal{L}(\tilde{e}_2) \to \mathcal{L}(\tilde{e}_2)^r$ be a homomorphism defined by

$$\mathcal{L}(e)^r = \mathcal{L}(e^{r/2}) \quad \text{for } e \in \tilde{e}_2.$$

Using the notations of Lemma 3, we let

$$A = \mathcal{L}(\tilde{e}_2) \quad \text{and} \quad B = \mathcal{L}(\tilde{e}_2) \oplus \mathcal{L}(\tilde{e}_2) \cap \sum_{\tilde{e}_2} \mathcal{L}(\tilde{e}_2).$$

Of course $(A : B)$ is finite. Since $P_{\tilde{e}_2}(\tilde{e}_2) = 0$, we have

$$A^r = \mathcal{L}\{\langle e^{r/2} \rangle; \ 0 \leq i \leq \varphi(g) - 1 \rangle K_{\tilde{e}_2} \}.$$

As can be easily seen, $\mathcal{L}(\tilde{e}_2) \cap \sum_{\tilde{e}_2} \mathcal{L}(\tilde{e}_2)^r = \langle 0 \rangle$. Hence

$$B^r = \mathcal{L}\{\langle e^{r/2} \rangle; \ 0 \leq i \leq \varphi(g) - 1 \rangle K_{\tilde{e}_2} \}.$$

Now let $f(X)$ be a polynomial in $\mathbb{Z}[X]$ such that $\deg(f) < \varphi(g)$, and we assume that $\tilde{e}_2^{r/2} \mathbb{Z}[\tilde{e}_2]$ is contained in $\mu_{K_{\tilde{e}_2}}$. Then $\tilde{e}_2^{r/2} \mathbb{Z}[\tilde{e}_2]$ is a priori in $\mathbb{Z}$. Since $\{e \in \mathbb{Z} ; \ 0 \leq i \leq \varphi(g) - 1 \}$ forms a system of independent units in $K$, $f(X)$ must be constantly equal to 0. Hence

$$A_0 = \{\mathcal{L}(e^{r/2}); f(X) \in P_{\tilde{e}_2}(X) \mathbb{Z}[X] \}.$$

Furthermore, by Lemma 2 and Lemma 6, we have

$$A_0 = \mathcal{L}(\tilde{e}_2) \cap \sum_{\tilde{e}_2} \mathcal{L}(\tilde{e}_2) = B,$$

and hence $B_0 = A_0$. Therefore we have

$$(A : B) = (A^r : B^r)$$

$$= (\mathbb{Z}[\tilde{e}_2] \tilde{e}_2 : \mathbb{Z}[\tilde{e}_2] \tilde{e}_2)$$

$$= (\mathbb{Z}[\tilde{e}_2] : \mathbb{Z}[\tilde{e}_2] \mathbb{Z}[\tilde{e}_2])$$

$$= \frac{g^{r/2}}{|D|}.$$

Thus our proof has been accomplished.

References
