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On index formulas of Siegel units in a ring class field

by

HEIMA HAYASHI (Kumamoto, Japan)

Introduction. Let $\Sigma = Q(\sqrt{d})$ be an imaginary quadratic number field with discriminant d, and let K be an abelian extension over Σ with rational conductor f such that K is contained in the ring class field N_f modulo f over Σ. In the previous paper [6], we studied the several properties on Siegel unit $\delta_K(C)$ in K and gave an index formula related to $\delta_K(C)$. The purpose in the present study is to give other kinds of index formulas related to $\delta_K(C)$, using the results in [6].

In Section 1, we shall summarize the properties of $\delta_K(C)$ and some preliminary facts in the theory of group algebra. In Section 2, we shall give two index formulas related to $\delta_K(C)$ (Theorems 1, 2). The unit group $\hat{\mathcal{E}}_K$ dealt with in Theorem 1 is an analog of the group of cyclotomic units in a real abelian number field studied by Leopoldt ([7]). The main task in this paper is to deduce Theorem 2 from Theorem 1, and it will be done in Section 3. The unit group $\mathcal{E}_K$ dealt with in Theorem 2 is fairly larger than $\hat{\mathcal{E}}_K$. The method used in Section 3 is mostly based on the fundamental facts in Section 1, and partially similar to that of Gillard ([2]). Of course these two groups are quite different from those in [2]. As an application of Theorem 2, a formula of Schertz ([8], Satz 3.3) can be proved under the restriction that $K \subset N_f$ (Corollary 2'). Moreover, using the results on Siegel unit in Section 1, we shall construct three unit groups $\hat{\mathcal{E}}_K^*$, $\mathcal{E}_K^*$, $\mathcal{E}_K^{**}$ such that $\hat{\mathcal{E}}_K \subset \hat{\mathcal{E}}_K^*$ and $\mathcal{E}_K \subset \mathcal{E}_K^* \subset \mathcal{E}_K^{**}$, and give the more refined index formulas for them (Propositions 1, 2, 3). Proposition 3 permits us in some special case to express the class number quotient h_K/h just as the index of $\mathcal{E}_K^{**}$ in the group of the whole units in K.

1. Notations and preliminary. Let Q be the field of rational numbers, Z the ring of rational integers and $\Sigma = Q(\sqrt{d})$ an imaginary quadratic number field with discriminant d. Let K be an abelian extension over Σ with rational conductor f such that K is contained in the ring class field N_f modulo f over Σ. We denote by O_f the order in Σ with conductor f (O_1 means the maximal order in Σ) and by $\mathfrak{K}(f)$ the ring ideal class group modulo f, i.e. the group of the equivalent classes of proper O_f-ideals in Σ by the usual relation. Let σ: $\mathfrak{K}(f) \to \mathrm{Gal}(N_f/\Sigma)$ be the isomorphism from $\mathfrak{K}(f)$ to the galois group of N_f over Σ via Artin's reciprocity law, and let $\mathfrak{H}$ be the subgroup of $\mathfrak{K}(f)$ whose image by σ is equal to the galois group of N_f over K.

Herein after for each number field $-$, we denote by h_-, E_-, μ_- and w_- respectively the class number of $-$, the unit group of $-$, the torsion part of E_- and the number of elements in μ_-. (When $- = \Sigma$, the subscript $-$ is omitted from these notations).

Let C be a class in $\mathfrak{K}(f)$ and let $\mathfrak{a}_f$ be an O_f-ideal in C^{-1}. Then there exists an element α in Σ such that $(\mathfrak{a}_f O_1)^h = (\alpha)$ as a principal O_1-ideal. We define the Siegel unit $\delta_f(C)$ by

$$\delta_f(C) = \alpha^{12}\left(\frac{\Delta(\mathfrak{a}_f)}{\Delta(O_f)}\right)^h.$$

Herein $\Delta(\ )$ means the usual lattice function expressed by using the Dedekind eta-function as follows:

$$\Delta(\mathfrak{m}) = \left(\frac{2\pi}{\omega_2}\right)^{12} \eta\left(\frac{\omega_1}{\omega_2}\right)^{24},$$

where $\mathfrak{m} = [\omega_1, \omega_2]$ is a 2-dimensional complex lattice with $\boldsymbol{Z}$-basis $\{\omega_1, \omega_2\}$, $\mathrm{Im}(\omega_1/\omega_2) > 0$. $\delta_f(C)$ depends only on the class C, not on the choices of $\mathfrak{a}_f$ and α, and is a unit in N_f. As is well known, $\delta_f(C_1)^{\sigma(C_2)} = \delta_f(C_1 C_2)/\delta_f(C_2)$ for any C_1, C_2 in $\mathfrak{K}(f)$, and $\delta_f(C_0) = 1$ for the unit class C_0 in $\mathfrak{K}(f)$.

We define $\delta_K(C)$ as the relative norm of $\delta_f(C)$ w.r.t. N_f/K, i.e.

$$\delta_K(C) = \prod_{C'} \delta_f(CC')/\delta_f(C'),$$

where C' runs over all classes in $\mathfrak{H}$.

Remark 1. In his paper [8], Schertz dealt with the similar unit $\theta_K(C)$, which is defined by:

$$\theta_K(C) = \beta^{12}\prod_{\mathfrak{b}_f}\left(\frac{\Delta(\mathfrak{a}_f\mathfrak{b}_f)}{\Delta(\mathfrak{b}_f)}\right)^{m_K}.$$

Herein $\mathfrak{b}_f$ ranges over a system of the complete representative O_f-ideals of all classes in $\mathfrak{H}$, m_K is the least positive rational integer such that $m_K \# \mathfrak{H} \equiv 0 \pmod{h}$ and β is an element in Σ such that $(\mathfrak{a}_f O_1)^{m_K \# \mathfrak{H}} = (\beta)$ as a principal O_1-ideal in Σ. (1) As can be easily seen, $\delta_K(C) = \theta_K(C)^{(h, \# \mathfrak{H})}$. However in our arguments, we use $\delta_K(C)$ instead of $\theta_K(C)$ for a few reasons. (For example, one should refer to Lemma 6 in Section 3.)

(1) $\# \mathfrak{H}$ means the number of elements in $\mathfrak{H}$.

Now let l_K: $\mathfrak{K}(f) \to \mathfrak{R}$ be the homomorphism from $\mathfrak{K}(f)$ to a subgroup $\mathfrak{R}$ of $(\boldsymbol{Z}/24\boldsymbol{Z})^\times$ defined in [6]. Herein l_K is uniquely determined by K/Σ, and indeed $\mathfrak{R}$ can be realized by using the K-admissible O_1-ideals in Σ. Of course $l_K(C^2) \equiv 1 \pmod{24}$ for any C in $\mathfrak{K}(f)$. Then the following lemma holds.

Lemma 1 ([6], Prop. 1). *Let C_1 and C_2 be any two classes in $\mathfrak{K}(f)$ and let n be the least positive rational integer such that*

$$n\big(l_K(C_1)-1\big)\big(l_K(C_2)-1\big) \equiv 0 \pmod{24}.$$

Then $\big(\delta_K(C_1 C_2)/\delta_K(C_1)\,\delta_K(C_2)\big)^n$ is contained in E_K^{24h}.

In the rest of this section, we shall provide some preliminary facts from the theory of group algebra.

Let G be the galois group of K over Σ, and let g be its order. We denote by $\boldsymbol{Z}[G]$ (resp. $\boldsymbol{Q}[G]$) the group ring of G over $\boldsymbol{Z}$ (resp. $\boldsymbol{Q}$) and by $\mathscr{X}(K/\Sigma)$ the character group of G. For any χ in $\mathscr{X}(K/\Sigma)$, we denote by $\tilde{\chi}$ the block to which χ belongs, i.e.

$$\tilde{\chi} = \{\xi \in \mathscr{X}(K/\Sigma);\ \langle\xi\rangle = \langle\chi\rangle\}.$$

Herein $\langle * \rangle$ means the cyclic group generated by $*$. Every character in $\tilde{\chi}$ possesses the common kernel $U_{\tilde{\chi}}$ and common order $g_{\tilde{\chi}}$. We often use the same notation $\tilde{\chi}$, identifying it with the $\boldsymbol{Q}$-irreducible character of G to which χ belongs. Let $K_{\tilde{\chi}}$ be the intermediate field of K/Σ associated with $U_{\tilde{\chi}}$. Then $K_{\tilde{\chi}}/\Sigma$ is a cyclic extension of degree $g_{\tilde{\chi}}$ and its galois group $G_{\tilde{\chi}}$ is isomorphic to $G/U_{\tilde{\chi}}$. If $K_{\tilde{\psi}} \subseteqq K_{\tilde{\chi}}$, i.e. $U_{\tilde{\psi}} \supseteqq U_{\tilde{\chi}}$ for some two blocks $\tilde{\psi}$ and $\tilde{\chi}$, then we denote this relation by $\tilde{\psi} \leqslant \chi$. Let $\prec$ be a total order in the finite set $\{\tilde{\chi};\ \chi \in \mathscr{X}(K/\Sigma)\}$, satisfying the condition that $\tilde{\psi} \preceq \tilde{\chi}$ whenever $\tilde{\psi} \leqslant \tilde{\chi}$ (cf. [2]). Herein after we keep this ordering.

For any $\tilde{\psi}$, $\tilde{\chi}$ such that $\tilde{\psi} \lneqq \tilde{\chi}$ we denote by $U_{\tilde{\chi},\tilde{\psi}}$ the subgroup of $G_{\tilde{\chi}}$ which corresponds to the subfield $K_{\tilde{\psi}}$, and we let

$$\nu_{\tilde{\chi},\tilde{\psi}} = \sum_{\bar{S} \in U_{\tilde{\chi},\tilde{\psi}}} \bar{S}.$$

The following lemma is well known as a result of Martinet (cf. [2]) and very useful in our later arguments.

Lemma 2. *Let $\mathcal{N}_{\tilde{\chi}}$ be the ideal in $\boldsymbol{Z}[G_{\tilde{\chi}}]$ generated by $\{\nu_{\tilde{\chi},\tilde{\psi}};\ \tilde{\psi} \lneqq \tilde{\chi}\}$. Then*

$$\mathcal{N}_{\tilde{\chi}} = P_{g_{\tilde{\chi}}}(\bar{S}_{\tilde{\chi}})\,\boldsymbol{Z}[G_{\tilde{\chi}}],$$

where $\bar{S}_{\tilde{\chi}}$ is a generator of $G_{\tilde{\chi}}$ and $P_{g_{\tilde{\chi}}}(X)$ is the $g_{\tilde{\chi}}$-th cyclotomic polynomial.

The idempotent $e_{\tilde{\chi}}$ corresponding to the $\boldsymbol{Q}$-irreducible character $\tilde{\chi}$ is given by

$$e_{\tilde{\chi}} = \frac{1}{g}\sum_{S \in G} \tilde{\chi}(S^{-1})\,S.$$

As is well known, $\sum_{\tilde\chi} e_{\tilde\chi} = 1$ in $Q[G]$ and $Q[G]e_{\tilde\chi} \simeq Q(\zeta_{g_{\tilde\chi}})$, where $\zeta_{g_{\tilde\chi}}$ is a primitive $g_{\tilde\chi}$-th root of unity. Moreover, we define $\tilde e_{\tilde\chi}$ in $Q[G_{\tilde\chi}]$ and $\gamma_{\tilde\chi}$ in $Z[G_{\tilde\chi}]$ respectively by

$$\tilde e_{\tilde\chi} = \frac{1}{g_{\tilde\chi}} \sum_{\tilde S \in G_{\tilde\chi}} \tilde\chi(\tilde S^{-1})\tilde S \quad \text{and} \quad \gamma_{\tilde\chi} = \prod_p (1 - \tilde S_{\tilde\chi}^{g_{\tilde\chi}/p}),$$

where $\tilde S_{\tilde\chi}$ is a generator of $G_{\tilde\chi}$ and p runs over all prime divisors of $g_{\tilde\chi}$. Of course $\gamma_{\tilde\chi}$ depends on the choice of $\tilde S_{\tilde\chi}$, but as an ideal in $Z[G_{\tilde\chi}]$, $\gamma_{\tilde\chi} Z[G_{\tilde\chi}]$ is uniquely determined independently of the choice of $\tilde S_{\tilde\chi}$. Plainly $\tilde e_{\tilde\chi}$ is the restriction of $e_{\tilde\chi}$ to $Q[G_{\tilde\chi}]$. For each $\tilde\chi$, it is easy to see that

$$\gamma_{\tilde\chi} \tilde e_{\tilde\chi} = \gamma_{\tilde\chi}. \tag{1.1}$$

2. Statement of results. Let the notations be the same as in the preceding sections. For any $\tilde\chi$, $K_{\tilde\chi}$ is contained in the ring class field $N_{f_{\tilde\chi}}$ modulo $f_{\tilde\chi}$ over Σ. Of course $f_{\tilde\chi}$ is a divisor of f. Let $\mathfrak{H}_{\tilde\chi}$ be the subgroup of $\mathfrak{R}(f_{\tilde\chi})$ which corresponds to $K_{\tilde\chi}$ and $C_{\tilde\chi}$ a fixed class in $\mathfrak{R}(f_{\tilde\chi})$ such that the coset $C_{\tilde\chi}\mathfrak{H}_{\tilde\chi}$ is a generator of $\mathfrak{R}(f_{\tilde\chi})/\mathfrak{H}_{\tilde\chi}$. Then we may take $\mathrm{Res}_{K_{\tilde\chi}} \sigma(C_{\tilde\chi})$ as a generator $\tilde S_{\tilde\chi}$ of $G_{\tilde\chi}$. We define two unit $\varepsilon_{\tilde\chi}$ and $\hat\varepsilon_{\tilde\chi}$ in $K_{\tilde\chi}$ respectively by

$$\varepsilon_{\tilde\chi} = \delta_{K_{\tilde\chi}}(C_{\tilde\chi}) \quad \text{and} \quad \hat\varepsilon_{\tilde\chi} = \varepsilon_{\tilde\chi}^{\gamma_{\tilde\chi}}.$$

Then $N_{K_{\tilde\chi}/K_{\tilde\psi}}(\hat\varepsilon_{\tilde\chi}) = 1$ for any $\tilde\psi$ ($\not\geq \tilde\chi$). Herein $N_{K_{\tilde\chi}/K_{\tilde\psi}}$ means the relative norm w.r.t. $K_{\tilde\chi}/K_{\tilde\psi}$. Furthermore we define two subgroups $\hat{\mathscr{E}}_K$ and $\mathscr{E}_K$ of E_K as follows:

$$\mathscr{E}_K = \mu_K \prod_{\tilde\chi \neq 1} \mathscr{E}_{\tilde\chi}, \quad \text{where} \quad \mathscr{E}_{\tilde\chi} = \mu_{K_{\tilde\chi}} \varepsilon_{\tilde\chi}^{Z[G_{\tilde\chi}]},$$

$$\hat{\mathscr{E}}_K = \mu_K \prod_{\tilde\chi \neq 1} \hat{\mathscr{E}}_{\tilde\chi}, \quad \text{where} \quad \hat{\mathscr{E}}_{\tilde\chi} = \mu_{K_{\tilde\chi}} \hat\varepsilon_{\tilde\chi}^{Z[G_{\tilde\chi}]}.$$

Then we have the following index formulas.

THEOREM 1. $(E_K : \hat{\mathscr{E}}_K) = M_1 \dfrac{h_K}{h}$, *where*

$$M_1 = \frac{w}{w_K}(24h)^{g-1}\sqrt{g^{g-2}} \prod_{\tilde\chi \neq 1} \frac{P_{g_{\tilde\chi}}(1)}{\sqrt{|D_{\tilde\chi}|}}.$$

Herein $D_{\tilde\chi}$ means the discriminant of the cyclotomic field $Q(\zeta_{g_{\tilde\chi}})$.

THEOREM 2. $(E_K : \mathscr{E}_K) = M_2 \dfrac{h_K}{h}$, *where*

$$M_2 = \frac{w}{w_K}(24h)^{g-1}\sqrt{g^{g-2}} \prod_{\tilde\chi \neq 1} \frac{\sqrt{|D_{\tilde\chi}|}\, P_{g_{\tilde\chi}}(1)}{g_{\tilde\chi}^{\varphi(g_{\tilde\chi})}}.$$

Herein $\varphi(\ )$ means the Euler function.

Theorem 1 is an analog of Leopoldt's formula on cyclotomic units in a real abelian number field ([7]), and its proof can be accomplished by the almost similar way to that in [7]. Now it is evident that $\hat{\mathscr{E}}_K \subset \mathscr{E}_K$. Then in order to prove Theorem 2, we have only to prove that

$$(\mathscr{E}_K : \hat{\mathscr{E}}_K) = \prod_{\tilde\chi \neq 1} \frac{g_{\tilde\chi}^{\varphi(g_{\tilde\chi})}}{|D_{\tilde\chi}|}. \tag{2.1}$$

We shall give a complete proof for the formula (2.1) in the next section. In the rest of this section we shall give several index formulas which can be derived from Theorems 1 and 2.

Taking into consideration the fact that $N_{K_{\tilde\chi}/K_{\tilde\psi}}(\hat\varepsilon_{\tilde\chi}) = 1$ for any $\tilde\psi$ ($\not\geq \tilde\chi$) and the result of Lemma 2, we have

$$\begin{aligned}\hat{\mathscr{E}}_{\tilde\chi} &= \langle\langle \hat\varepsilon_{\tilde\chi}^{S_{\tilde\chi}^j};\ 0 \leq j \leq \varphi(g_{\tilde\chi})-1 \rangle\rangle_{K_{\tilde\chi}}\ ^{(2)} \\ &= \langle\langle \hat\delta_{K_{\tilde\chi}}(C_{\tilde\chi}^i);\ 1 \leq i \leq \varphi(g_{\tilde\chi}) \rangle\rangle_{K_{\tilde\chi}},\end{aligned}$$

where $\hat\delta_{K_{\tilde\chi}}(C_{\tilde\chi}^i)$ means $\delta_{K_{\tilde\chi}}(C_{\tilde\chi}^i)^{\gamma_{\tilde\chi}}$. Then we have the following:

COROLLARY 1. $(g-1)$ *units* $\{\hat\delta_{K_{\tilde\chi}}(C_{\tilde\chi}^{i_{\tilde\chi}});\ \tilde\chi \neq 1,\ 1 \leq i_{\tilde\chi} \leq \varphi(g_{\tilde\chi})\}$ *constitute a maximal system of the independent units in K, and generate $\hat{\mathscr{E}}_K$ in Theorem* 1.

In Theorem 2, it is evident that

$$\mathscr{E}_{\tilde\chi} \supset \langle\langle \varepsilon_{\tilde\chi}^{S_{\tilde\chi}^j};\ 0 \leq j \leq \varphi(g_{\tilde\chi})-1 \rangle\rangle_{K_{\tilde\chi}}.$$

On the other hand, by Lemma 2 and Lemma 6 (in § 3), we have

$$\mathscr{E}_{\tilde\chi} \subset \prod_{\tilde\psi \leq \tilde\chi} \langle\langle \varepsilon_{\tilde\psi}^{S_{\tilde\psi}^{j_{\tilde\psi}}};\ 0 \leq j_{\tilde\psi} \leq \varphi(g_{\tilde\psi})-1 \rangle\rangle_{K_{\tilde\psi}}.$$

Then we have the following:

COROLLARY 2. $(g-1)$ *units* $\{\delta_{K_{\tilde\chi}}(C_{\tilde\chi}^{i_{\tilde\chi}});\ \tilde\chi \neq 1,\ 1 \leq i_{\tilde\chi} \leq \varphi(g_{\tilde\chi})\}$ *constitute a maximal system of the independent units in K, and generate $\mathscr{E}_K$ in Theorem* 2.

Replacing $\delta_{K_{\tilde\chi}}(C_{\tilde\chi}^{i_{\tilde\chi}})$ in Corollary 2 by $\theta_{K_{\tilde\chi}}(C_{\tilde\chi}^{i_{\tilde\chi}})$ (Remark 1), we have the similar formula as follows:

COROLLARY 2'. $(g-1)$ *units* $\{\theta_{K_{\tilde\chi}}(C_{\tilde\chi}^{i_{\tilde\chi}});\ \tilde\chi \neq 1,\ 1 \leq i_{\tilde\chi} \leq \varphi(g_{\tilde\chi})\}$ *generate a subgroup Θ_K of finite index in E_K, for which the following index formula holds*:

$$(E_K : \Theta_K) = \frac{w}{w_K}(24)^{g-1}\sqrt{g^{g-2}} \prod_{\tilde\chi \neq 1} \left(\frac{m_{\tilde\chi}}{g_{\tilde\chi}}\right)^{\varphi(g_{\tilde\chi})} \sqrt{|D_{\tilde\chi}|}\, P_{g_{\tilde\chi}}(1).$$

Herein $m_{\tilde\chi}$ is the least natural number such that $m_{\tilde\chi} \# \mathfrak{H}_{\tilde\chi} \equiv 0 \pmod{h}$.

Corollary 2' is essentially equal to the formula of Schertz ([8], Satz 3.3 under the restriction that $K \subset N_f$). Namely $\{\chi(C_{\tilde\chi}^{i_{\tilde\chi}}) - 1;\ 1 \leq i_{\tilde\chi} \leq \varphi(g_{\tilde\chi})\}$

(2) $\langle\langle \varepsilon_1, \varepsilon_2, \ldots \rangle\rangle_K$ means the subgroup of E_K generated by μ_K and units $\varepsilon_1, \varepsilon_2, \ldots$

constitute an integral basis of the principal ideal $(\zeta_{g_{\tilde\chi}}-1)$ in $\mathbf{Z}[\zeta_{g_{\tilde\chi}}]$, whose absolute norm is equal to $P_{g_{\tilde\chi}}(1)$ (see Satz 1.7 in [8]).

Now we are going to deduce the more refined index formulas from Theorems 1 and 2. Namely, using Lemma 1, we shall construct three large unit groups $\hat{\mathscr{E}}_K^*$, $\mathscr{E}_K^*$, $\mathscr{E}_K^{**}$ such that $\hat{\mathscr{E}}_K \subset \hat{\mathscr{E}}_K^*$ and $\mathscr{E}_K \subset \mathscr{E}_K^* \subset \mathscr{E}_K^{**}$.

First we note that $\hat\varepsilon_{\tilde\chi}$ is the $24h$-th power in $E_{K_{\tilde\chi}}$ with the exception of the case where $g_{\tilde\chi}=2$. Indeed if $g_{\tilde\chi}$ is odd, then it is always possible to choose $C_{\tilde\chi}$ as a square of some class in $\mathfrak{R}(f_{\tilde\chi})$ and hence by Lemma 1, $\hat\varepsilon_{\tilde\chi}$ is in $E_{K_{\tilde\chi}}^{24h}$. If $g_{\tilde\chi}$ is even but not equal to 2, then there exists a prime divisor p of $g_{\tilde\chi}$ such that $g_{\tilde\chi}/p=$ even. Then by using the following equality

$$\delta_{K_{\tilde\chi}}(C_{\tilde\chi})^{1-s_{\tilde\chi}^{g_{\tilde\chi}/p}}=\frac{\delta_{K_{\tilde\chi}}(C_{\tilde\chi})\,\delta_{K_{\tilde\chi}}(C_{\tilde\chi}^{g_{\tilde\chi}/p})}{\delta_{K_{\tilde\chi}}(C_{\tilde\chi}^{g_{\tilde\chi}/p+1})}$$

and Lemma 1, we may conclude that $\hat\varepsilon_{\tilde\chi}$ is contained in $E_{K_{\tilde\chi}}^{24h}$.

In the case where $g_{\tilde\chi}=2$, we have

$$\hat\varepsilon_{\tilde\chi}=\delta_{K_{\tilde\chi}}(C_{\tilde\chi})^{1-s_{\tilde\chi}}=\frac{\delta_{K_{\tilde\chi}}(C_{\tilde\chi})^2}{\delta_{K_{\tilde\chi}}(C_{\tilde\chi}^2)},$$

and hence by Lemma 1

$$\hat\varepsilon_{\tilde\chi}^{n_{\tilde\chi}}\in E_{K_{\tilde\chi}}^{24h} \quad \text{with} \quad n_{\tilde\chi}=1,2 \text{ or } 3.$$

Here $n_{\tilde\chi}=3$ occurs in the only case where $\Sigma\neq Q(\sqrt{-3})$ and $K_{\tilde\chi}=\Sigma(\sqrt{-3})$, and $n_{\tilde\chi}=2$ occurs in the only case where $\Sigma\neq Q(\sqrt{-4})$ and $K_{\tilde\chi}=\Sigma(\sqrt{-4})$. Moreover it always holds that $\prod_{g_{\tilde\chi}=2} n_{\tilde\chi}=w_K/w$. Hence for each $\tilde\chi$, we may choose a unit $\hat\eta_{\tilde\chi}$ in $E_{K_{\tilde\chi}}$ such that

$$\text{(2.2)}\qquad \begin{cases}\hat\varepsilon_{\tilde\chi}=\hat\eta_{\tilde\chi}^{24h} & \text{for} \quad g_{\tilde\chi}\neq 2,\\ \hat\varepsilon_{\tilde\chi}^{n_{\tilde\chi}}=\hat\eta_{\tilde\chi}^{24h} & \text{for} \quad g_{\tilde\chi}=2.\end{cases}$$

Remark 2. Units $\hat\eta_{\tilde\chi}$ in the equation (2.2) can be explicitly given by using the values of the Dedekind eta-function (see the proof of Proposition 1 in [6]). Of course $N_{K_{\tilde\chi}/K_{\tilde\psi}}(\hat\eta_{\tilde\chi})\in\mu_{K_{\tilde\psi}}$ for any $\tilde\psi$ ($\lneq\tilde\chi$).

We define $\hat{\mathscr{E}}_K^*$ by

$$\hat{\mathscr{E}}_K^*=\mu_K\prod_{\tilde\chi\neq 1}\hat{\mathscr{E}}_{\tilde\chi}^* \quad \text{where} \quad \hat{\mathscr{E}}_{\tilde\chi}^*=\mu_{K_{\tilde\chi}}\hat\eta_{\tilde\chi}^{\mathbf{Z}[G_{\tilde\chi}]}.$$

Then we have the following:

PROPOSITION 1. $(E_K:\hat{\mathscr{E}}_K^*)=M_1^*\dfrac{h_K}{h}$, *where*

$$M_1^*=\sqrt{g^{g-2}}\prod_{\tilde\chi\neq 1}\frac{P_{g_{\tilde\chi}}(1)}{\sqrt{|D_{\tilde\chi}|}}.$$

COROLLARY 3. *In Proposition 1, if K/Σ is a cyclic extension, then*

$$M_1^*=\prod_{\tilde\chi\neq 1}\frac{g_{\tilde\chi}^{\varphi(g_{\tilde\chi})}}{|D_{\tilde\chi}|}.$$

COROLLARY 4. *In Proposition 1, if K/Σ is a cyclic extension of degree p^ν with a prime number p, then*

$$M_1^*=p^{(p^\nu-1)/(p-1)}.$$

Proof. When K/Σ is a cyclic extension, it always holds that

$$\sqrt{g^{g-2}}\prod_{\tilde\chi\neq 1}P_{g_{\tilde\chi}}(1)\sqrt{|D_{\tilde\chi}|}/g_{\tilde\chi}^{\varphi(g_{\tilde\chi})}=1$$

(Hasse [4]). Hence Corollaries 3 and 4 follow.

Remark 3. Proposition 1 can be used for the numerical determination of h_K/h and a system of the fundamental units in K. For this subject the similar methods to those used in the paper of G. Gras and M.-N. Gras ([3]) can be applied (see also [5]).

Next we have

$$\langle\langle \varepsilon_{\tilde\chi}^{s_{\tilde\chi}^j};\ 0\leqslant j\leqslant\varphi(g_{\tilde\chi})-1\rangle\rangle_{K_{\tilde\chi}}=\langle\langle \varepsilon_{\tilde\chi},\ \varepsilon_{\tilde\chi}^{s_{\tilde\chi}-1},\ \varepsilon_{\tilde\chi}^{s_{\tilde\chi}^j-s_{\tilde\chi}^{j-2}};\ 2\leqslant j\leqslant\varphi(g_{\tilde\chi})-1\rangle\rangle_{K_{\tilde\chi}}.$$

Moreover since

$$\varepsilon_{\tilde\chi}^{s_{\tilde\chi}-1}=\frac{\delta_{K_{\tilde\chi}}(C_{\tilde\chi}^2)}{\delta_{K_{\tilde\chi}}(C_{\tilde\chi})^2} \quad \text{and} \quad \varepsilon_{\tilde\chi}^{s_{\tilde\chi}^j-s_{\tilde\chi}^{j-2}}=\delta_{K_{\tilde\chi}}(C_{\tilde\chi}^2)^{s_{\tilde\chi}^{j-2}(s_{\tilde\chi}-1)},$$

we have

$$\varepsilon_{\tilde\chi}^{n_{\tilde\chi}(s_{\tilde\chi}-1)}=\eta_{\tilde\chi,2}^{24h} \quad \text{and} \quad \varepsilon_{\tilde\chi}^{s_{\tilde\chi}^j-s_{\tilde\chi}^{j-2}}=\eta_{\tilde\chi,j+1}^{24h},$$

with some units $\eta_{\tilde\chi,i}$ ($i=2,3,\ldots,\varphi(g_{\tilde\chi})$) in $E_{K_{\tilde\chi}}$. Herein $n_{\tilde\chi}$ is the least positive rational integer such that $n_{\tilde\chi}(l_{K_{\tilde\chi}}(C_{\tilde\chi})-1)^2\equiv 0 \pmod{24}$ (by Lemma 1). Namely $n_{\tilde\chi}$ is one of 1, 2, 3 and 6. Furthermore,

$$\varepsilon_{\tilde\chi}=\eta_{\tilde\chi,1}^{(\#\mathfrak{H}_{\tilde\chi},h)},$$

with some unit $\eta_{\tilde\chi,1}$ in $E_{K_{\tilde\chi}}$ (by Remark 1). For the simplicity we write

$$a(\tilde\chi)=24n_{\tilde\chi}\frac{h}{(\#\mathfrak{H}_{\tilde\chi},h)}.$$

We define a subgroup $\mathscr{E}_K^*$ of E_K by

$$\mathscr{E}_K^*=\mu_K\prod_{\tilde\chi\neq 1}\mathscr{E}_{\tilde\chi}^*, \quad \text{where} \quad \mathscr{E}_{\tilde\chi}^*=\langle\langle\eta_{\tilde\chi,i};\ 1\leqslant i\leqslant\varphi(g_{\tilde\chi})\rangle\rangle_{K_{\tilde\chi}}.$$

Then we have the following

PROPOSITION 2. $(E_K:\mathscr{E}_K^*) = M_2^*\frac{h_K}{h}$, *where*

$$M_2^* = \frac{w}{w_K}\sqrt{g^{g-2}}\prod_{\tilde\chi\neq 1} a(\tilde\chi)\frac{P_{g_{\tilde\chi}}(1)\sqrt{|D_{\tilde\chi}|}}{g_{\tilde\chi}^{\varphi(g_{\tilde\chi})}}.$$

Especially when $(g_{\tilde\chi}, h) = 1$ then $\# \mathfrak{H}_{\tilde\chi}$ must be divisible by h, and when $g_{\tilde\chi}$ = odd in addition, then $\varepsilon_{\tilde\chi}$ is the $24h$-th power of a unit in $E_{K_{\tilde\chi}}$. (Note that $\{\Delta(a_f^2)/\Delta(O_f)\}$ is contained in N_f^{24} ([1], p. 41)). Therefore we have the following

PROPOSITION 3. *Assume that g is odd and $(g, h) = 1$. Then for each $\tilde\chi$, $\varepsilon_{\tilde\chi} = \eta_{\tilde\chi}^{24h}$ with some unit $\eta_{\tilde\chi}$ in $E_{K_{\tilde\chi}}$, and $(g-1)$ units $\{\eta_{\tilde\chi}^{s^{i_{\tilde\chi}}}; \tilde\chi\neq 1, 0\leqslant i_{\tilde\chi}\leqslant \varphi(g_{\tilde\chi})-1\}$ generate a subgroup $\mathscr{E}_K^{**}$ of E_K, for which the following index formula holds*:

$$(E_K:\mathscr{E}_K^{**}) = \frac{h_K}{h}\sqrt{g^{g-2}}\prod_{\tilde\chi\neq 1}\frac{P_{g_{\tilde\chi}}(1)\sqrt{|D_{\tilde\chi}|}}{g_{\tilde\chi}^{\varphi(g_{\tilde\chi})}}.$$

COROLLARY 5. *If K/Σ is a cyclic extension in addition to the conditions of Proposition 3, then*

$$(E_K:\mathscr{E}_K^{**}) = \frac{h_K}{h}.$$

3. Proof of Theorem 2. The purpose in this section is to prove the formula (2.1). For this purpose we shall provide several lemmas in advance.

The following lemma is a well-known fact in the group theory.

LEMMA 3. *Let $\varkappa$: $A\to A^\varkappa$ be a homomorphism defined on an abelian group A and let A_0 be its kernel. Let B be a subgroup of A with finite index and $B_0 = A_0\cap B$. Then*

$$(A:B) = (A^\varkappa:B^\varkappa)(A_0:B_0).$$

Now let $\mathscr{L}$: $E_K\to \mathbf{R}^g$ be the usual homomorphism from E_K to the g-dimensional Euclidian space $\mathbf{R}^g$ defined by

$$\mathscr{L}(\varepsilon) = (\ldots, \log|\varepsilon^S|^2, \ldots)_{S\in G} \quad \text{for any } \varepsilon \text{ in } E_K.$$

Then the kernel of $\mathscr{L}$ is μ_K and the image $\mathscr{L}(E_K)$ of E_K is a lattice in $\mathbf{R}^g$, which can be seen as a $\mathbf{Z}[G]$-module.

LEMMA 4. $\mathscr{L}(\hat{\mathscr{E}}_K) = \bigoplus_{\tilde\chi\neq 1}\mathscr{L}(\hat{\mathscr{E}}_{\tilde\chi})$ *(dir. sum.), and for each $\tilde\chi$, $\mathscr{L}(\hat{\mathscr{E}}_{\tilde\chi})$ is a lattice of free rank $\varphi(g_{\tilde\chi})$.*

Proof. Let u be any element in $\mathscr{L}(\hat{\mathscr{E}}_{\tilde\chi})\cap\sum_{\tilde\psi\neq 1,\tilde\chi}\mathscr{L}(\hat{\mathscr{E}}_{\tilde\psi})$ and we let $u = \mathscr{L}(\hat\varepsilon_{\tilde\chi}^{t_{\tilde\chi}}) = \sum_{\tilde\psi}\mathscr{L}(\hat\varepsilon_{\tilde\psi}^{t_{\tilde\psi}})$, where $t_{\tilde\chi}$ and $t_{\tilde\psi}$ are suitable elements in $\mathbf{Z}[G_{\tilde\chi}]$ and $\mathbf{Z}[G_{\tilde\psi}]$ respectively. Then

$$\hat\varepsilon_{\tilde\psi}^{t_{\tilde\chi}} = \varrho\prod_{\tilde\psi\neq 1,\tilde\chi}\hat\varepsilon_{\tilde\psi}^{t_{\tilde\psi}} \quad \text{with some } \varrho \text{ in } \mu_K.$$

Taking action of $ge_{\tilde\chi}$ on the both sides of the above equation, we see that $\hat\varepsilon_{\tilde\chi}^{t_{\tilde\chi}}$ is in $\mu_{K_{\tilde\chi}}$, i.e. $u = 0$. Because $\hat\varepsilon_{\tilde\chi}^{ge_{\tilde\chi}} = \varepsilon_{\tilde\chi}^{\tilde\chi g\bar e_{\tilde\chi}} = \hat\varepsilon_{\tilde\chi}^{g}$ and $\hat\varepsilon_{\tilde\psi}^{ge_{\tilde\chi}} = \hat\varepsilon_{\tilde\psi}^{\bar e_{\tilde\psi}ge_{\tilde\chi}} = \hat\varepsilon_{\tilde\psi}^{ge_{\tilde\psi}e_{\tilde\chi}} = 1$ (by (1.1)). The second assertion is evident by Theorem 1.

LEMMA 5. $\mathscr{L}(\hat{\mathscr{E}}_{\tilde\chi})\cap\sum_{\tilde\psi\not\geq\tilde\chi}\mathscr{L}(\mathscr{E}_{\tilde\psi}) = \{0\}$.

Proof. Let u be any element in $\mathscr{L}(\hat{\mathscr{E}}_{\tilde\chi})\cap\sum_{\tilde\psi}\mathscr{L}(\mathscr{E}_{\tilde\psi})$ and we let $u = \mathscr{L}(\hat\varepsilon_{\tilde\chi}^{t_{\tilde\chi}}) = \sum_{\tilde\psi}\mathscr{L}(\varepsilon_{\tilde\psi}^{t_{\tilde\psi}})$, where $t_{\tilde\chi}$ and $t_{\tilde\psi}$ are suitable elements in $\mathbf{Z}[G_{\tilde\chi}]$ and $\mathbf{Z}[G_{\tilde\psi}]$ respectively. Then

$$\hat\varepsilon_{\tilde\chi}^{t_{\tilde\chi}} = \varrho\prod_{\tilde\psi\not\geq\tilde\chi}\varepsilon_{\tilde\psi}^{t_{\tilde\psi}} \quad \text{with some } \varrho \text{ in } \mu_K.$$

By taking the relative norm $N_{K/K_{\tilde\chi}}$ of both sides, we have

$$(\hat\varepsilon_{\tilde\chi}^{t_{\tilde\chi}})^{g/g_{\tilde\chi}}\in\mu_{K_{\tilde\chi}}\prod_{\tilde\psi\not\geq\tilde\chi}E_{K_{\tilde\psi}}.$$

Therefore $\hat\varepsilon_{\tilde\chi}^{t_{\tilde\chi}}$ must be in $\mu_{K_{\tilde\chi}}$, and $u = 0$.

LEMMA 6. *For any two $\tilde\psi$, $\tilde\chi$ such that $\tilde\psi\not\geq\tilde\chi$,*

$$N_{K_{\tilde\chi}/K_{\tilde\psi}}(\mathscr{E}_{\tilde\chi})\subset\prod_{\tilde\xi\leqslant\tilde\psi}\mathscr{E}_{\tilde\xi}.$$

Proof. From the definition $\varepsilon_{\tilde\chi} = N_{N_{f_{\tilde\chi}}/K_{\tilde\chi}}(\delta_{f_{\tilde\chi}}(C_{\tilde\chi}))$, and hence

$$N_{K_{\tilde\chi}/K_{\tilde\psi}}(\varepsilon_{\tilde\chi}) = N_{N_{f_{\tilde\psi}}/K_{\tilde\psi}}N_{N_{f_{\tilde\chi}}/N_{f_{\tilde\psi}}}(\delta_{f_{\tilde\psi}}(C_{\tilde\chi})).$$

Then in the case where $f_{\tilde\chi} = f_{\tilde\psi}$, then assertion is trivial. For the case where $f_{\tilde\chi}\neq f_{\tilde\psi}$, we can apply the results of Lemma 3 in [6].

Proof of (2.1). By Lemma 3, $(\mathscr{E}_K:\hat{\mathscr{E}}_K) = (\mathscr{L}(\mathscr{E}_K):\mathscr{L}(\hat{\mathscr{E}}_K))$. Now let $\tilde\chi_1\not\geq\tilde\chi_2\not\geq\ldots$ be the arrangement by the ordering $\prec$ in Section 1. Then

$$\begin{aligned}
(\mathscr{L}(\mathscr{E}_K):\mathscr{L}(\hat{\mathscr{E}}_K)) &= \Big(\sum_{i\geqslant 1}\mathscr{L}(\mathscr{E}_{\tilde\chi_i});\ \mathscr{L}(\hat{\mathscr{E}}_{\tilde\chi_1})\oplus\sum_{i\geqslant 2}\mathscr{L}(\mathscr{E}_{\tilde\chi_i})\Big)\\
&\quad\times\Big(\mathscr{L}(\hat{\mathscr{E}}_{\tilde\chi_1})\oplus\sum_{i\geqslant 2}\mathscr{L}(\mathscr{E}_{\tilde\chi_i}):\mathscr{L}(\hat{\mathscr{E}}_{\tilde\chi_1})\oplus\mathscr{L}(\hat{\mathscr{E}}_{\tilde\chi_2})\oplus\sum_{i\geqslant 3}\mathscr{L}(\mathscr{E}_{\tilde\chi_i})\Big)\\
&\quad\times\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&= \prod_{\tilde\chi\neq 1}\Big(\sum_{\tilde\psi\leqq\tilde\chi}\mathscr{L}(\mathscr{E}_{\tilde\psi}):\mathscr{L}(\hat{\mathscr{E}}_{\tilde\chi})\oplus\sum_{\tilde\psi\not\geq\tilde\chi}\mathscr{L}(\mathscr{E}_{\tilde\psi})\Big)\\
&= \prod_{\tilde\chi\neq 1}\Big(\mathscr{L}(\mathscr{E}_{\tilde\chi}):\mathscr{L}(\hat{\mathscr{E}}_{\tilde\chi})\oplus\{\mathscr{L}(\mathscr{E}_{\tilde\chi})\cap\sum_{\tilde\psi\not\geq\tilde\chi}\mathscr{L}(\mathscr{E}_{\tilde\psi})\}\Big),
\end{aligned}$$

and each $\tilde\chi$-factor of the above last product can be calcutated as follows: Let $\varkappa$: $\mathscr{L}(\mathscr{E}_{\tilde\chi}) \to \mathscr{L}(\mathscr{E}_{\tilde\chi})^{\varkappa}$ be a homomorphism defined by

$$\mathscr{L}(\varepsilon)^{\varkappa} = \mathscr{L}(\varepsilon^{g_{\tilde\chi}\tilde e_{\tilde\chi}}) \quad \text{for } \varepsilon \text{ in } \mathscr{E}_{\tilde\chi}.$$

Using the notations of Lemma 3, we let

$$A = \mathscr{L}(\mathscr{E}_{\tilde\chi}) \quad \text{and} \quad B = \mathscr{L}(\hat{\mathscr{E}}_{\tilde\chi}) \oplus \{\mathscr{L}(\mathscr{E}_{\tilde\chi}) \cap \sum_{\tilde\psi \gneq \tilde\chi} \mathscr{L}(\mathscr{E}_{\tilde\psi})\}.$$

Of course $(A:B)$ is finite. Since $P_{g_{\tilde\chi}}(\tilde S_{\tilde\chi})\tilde e_{\tilde\chi} = 0$, we have

$$A^{\varkappa} = \mathscr{L}(\langle\langle \varepsilon_{\tilde\chi}^{g_{\tilde\chi}\tilde e_{\tilde\chi}\tilde S_{\tilde\chi}^{i}};\ 0 \leqslant i \leqslant \varphi(g_{\tilde\chi})-1\rangle\rangle_{K_{\tilde\chi}}).$$

As can be easily seen, $\{\mathscr{L}(\mathscr{E}_{\tilde\chi}) \cap \sum_{\tilde\psi \gneq \tilde\chi} \mathscr{L}(\mathscr{E}_{\tilde\psi})\}^{\varkappa} = \{0\}$. Hence

$$B^{\varkappa} = \mathscr{L}(\langle\langle \hat\varepsilon_{\tilde\chi}^{g_{\tilde\chi}\tilde e_{\tilde\chi}\tilde S_{\tilde\chi}^{i}};\ 0 \leqslant i \leqslant \varphi(g_{\tilde\chi})-1\rangle\rangle_{K_{\tilde\chi}}).$$

Now let $f(X)$ be a polynomial in $\mathbf{Z}[X]$ such that $\deg(f) < \varphi(g_{\tilde\chi})$, and we assume that $\varepsilon_{\tilde\chi}^{f(\tilde S_{\tilde\chi})g_{\tilde\chi}\tilde e_{\tilde\chi}}$ is contained in $\mu_{K_{\tilde\chi}}$. Then $\hat\varepsilon_{\tilde\chi}^{f(\tilde S_{\tilde\chi})g_{\tilde\chi}\tilde e_{\tilde\chi}}$ is *a priori* in $\mu_{K_{\tilde\chi}}$. Since $\{\varepsilon_{\tilde\chi}^{S_{\tilde\chi}^{i}};\ 0 \leqslant i \leqslant \varphi(g_{\tilde\chi})-1\}$ forms a system of independent units in $K_{\tilde\chi}$, $f(X)$ must be constantly equal to 0. Hence

$$A_0 = \{\mathscr{L}(\varepsilon_{\tilde\chi}^{f(\tilde S_{\tilde\chi})});\ f(X) \in P_{g_{\tilde\chi}}(X)\mathbf{Z}[X]\}.$$

Furthermore, by Lemma 2 and Lemma 6, we have

$$A_0 \subset \mathscr{L}(\mathscr{E}_{\tilde\chi}) \cap \sum_{\tilde\psi \gneq \tilde\chi} \mathscr{L}(\mathscr{E}_{\tilde\psi}) \subset B,$$

and hence $B_0 = A_0$. Therefore we have

$$\begin{aligned}(A:B) &= (A^{\varkappa}:B^{\varkappa})\\ &= (\mathbf{Z}[G_{\tilde\chi}]\tilde e_{\tilde\chi} : (\gamma_{\tilde\chi})\mathbf{Z}[G_{\tilde\chi}]\tilde e_{\tilde\chi})\\ &= (\mathbf{Z}[\zeta_{g_{\tilde\chi}}] : \chi(\gamma_{\tilde\chi})\mathbf{Z}[\zeta_{g_{\tilde\chi}}])\\ &= \frac{g_{\tilde\chi}^{\varphi(g_{\tilde\chi})}}{|D_{\tilde\chi}|}.\end{aligned}$$

Thus our proof has been accomplished.

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