ACTA ARITHMETICA
XLVI (1986)

Some Ω-theorems for the Riemann zeta-function

by

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1. Introduction. Let $N(t)$ $(t > 0)$ denote the number of zeros $\varrho = \beta + i\gamma$ of $\zeta(s)$ in the rectangle $0 < \beta < 1$, $0 < \gamma < t$. It is well known that

$$N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi e} + \frac{7}{8} + S(t) + O(t^{-1}), \tag{1.1}$$

where

$$S(t) = \pi^{-1}\,\mathrm{Im}\log\zeta(\tfrac{1}{2}+it),$$

the branch of the logarithm being determined in the usual way. The behavior of $S(t)$ is thus closely related to the distribution of the imaginary parts of the zeros. For instance, $S(t)$ has a jump of r when t is the ordinate of r zeros and it is smooth with derivative $\sim \frac{-1}{2\pi}\log t$ elsewhere. Concerning the order of $S(t)$, Bäcklund [1] has proved that

$$S(t) = O(\log t),$$
$$= O(\log t/\log\log t), \quad \text{assuming Riemann Hypothesis (RH).}$$

In the other direction, Selberg ([6], Theorem 9) has proved that

$$S(t) = \Omega_{\pm}\big((\log t)^{1/3}(\log\log t)^{-7/3}\big), \tag{1.2}$$

$$= \Omega_{\pm}\big((\log t/\log\log t)^{1/2}\big), \quad \text{assuming RH.} \tag{1.3}$$

Although there is a considerable gap between the O-results and the Ω-results, there are heuristic arguments suggesting that (1.3) is closer to the truth.

In this paper, we shall prove several new Ω-theorems for $S(t)$ and some related functions by refining the arguments of Selberg in [6]. Corresponding to (1.2), we have

THEOREM 1.

$$S(t) = \Omega_{\pm}\big((\log t/\log\log t)^{1/3}\big). \tag{1.4}$$

More generally, for $\sigma \in [\frac{1}{2}, 1]$, we can define
$$S(\sigma, t) = \pi^{-1} \operatorname{Im} \log \zeta(\sigma + it)$$
so that $S(t) = S(\frac{1}{2}, t)$.

Theorem 1 is a particular case of

THEOREM 2. *There exists a positive constant c such that when $T \to \infty$*
$$\sup_{t \in [T, 2T]} \pm S(\sigma, t) \geqslant c(\log T/\log\log T)^{1/3}$$
$$\text{for } \tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{1}{2} + (\log\log T/\log T)^{1/3},$$
$$\geqslant c\big((\sigma - \tfrac{1}{2})\log T/\log\log T\big)^{1/2}$$
$$\text{for } \tfrac{1}{2} + (\log\log T/\log T)^{1/3} \leqslant \sigma \leqslant \tfrac{1}{2} + (\log\log T)^{-1}.$$

This theorem can be compared with a result of Montgomery ([4], Theorem 1) which says that: for fixed $\sigma > \frac{1}{2}$
$$S(\sigma, t) = \Omega_{\pm}\big((\sigma - \tfrac{1}{2})^{1/2} (\log t)^{1-\sigma} (\log\log t)^{-\sigma}\big).$$

For small values of h, the function $S(t+h) - S(t)$ measures the variation of $S(t)$ over short intervals. There is not much literature about this function. Selberg (unpublished) has proved that, assuming the RH, there exists $c > 0$ such that when $T \to \infty$,
$$\sup_{t \in [T, 2T]} \pm \{S(t+h) - S(t)\} \geqslant c(h \log T)^{1/2}$$
for any $h \in [(\log T)^{-1}, (\log\log T)^{-1}]$.

Without assuming the RH, we shall prove

THEOREM 3. *There exists a positive constant c such that when $T \to \infty$*
$$\sup_{t \in [T, 2T]} \pm \{S(t+h) - S(t)\} \geqslant c(h \log T)^{1/3}$$
for any $h \in [(\log T)^{-1}, (\log\log T)^{-1}]$.

To prove this theorem, we shall need another result concerning $S(t+h) - S(t)$.

THEOREM 4. *Let $a > \frac{1}{2}$, $T^a < H \leqslant T$ and $0 < h < 1$. For any positive integer k, we have*
$$\int_T^{T+H} \{S(t+h) - S(t)\}^{2k}\, dt \tag{1.5}$$
$$= A_k H \{\log(2 + h \log T)\}^k + O\big\{H (ck)^k \{k^k + (\log(2 + h\log T))^{k-1/2}\}\big\},$$
where c is a positive constant and
$$A_k = \frac{(2k)!}{2^k \pi^{2k} k!}. \tag{1.6}$$

Fujii ([2], Main Theorem) has proved, among other things, the same asymptotic formula but with a worse O-term given by $H(ck)^{4k}(\log(2 + h\log T))^{k-1/2}$. The proof of Theorem 3, however, requires the sharper estimate given in (1.5).

The integral of $S(t)$, namely,
$$S_1(t) = \int_0^t S(u)\, du$$
is also an interesting function. Selberg ([6], Theorems 10, 11) has proved that
$$S_1(t) = \Omega_+\big((\log t)^{1/2} (\log\log t)^{-4}\big)$$
and
$$S_1(t) = \Omega_-\big((\log t)^{1/3} (\log\log t)^{-10/3}\big).$$
By extending our method, we obtained the following results.

THEOREM 5.
$$S_1(t) = \Omega_+\big((\log t)^{1/2} (\log\log t)^{-9/4}\big), \tag{1.7}$$
$$S_1(t) = \Omega_-\big((\log t)^{1/3} (\log\log t)^{-4/3}\big)$$
and, assuming RH,
$$S_1(t) = \Omega_{\pm}\big((\log t)^{1/2} (\log\log t)^{-3/2}\big).$$

(1.7) is remarkable since it has already come very close to what we can prove using the RH.

THEOREM 6. *There exists a positive constant c such that when $T \to \infty$*
$$\sup_{t \in [T, 2T]} \pm \{S_1(t+h) - S_1(t)\} \geqslant ch(\log T/\log\log T)^{1/3}, \tag{1.8}$$
$$\geqslant ch(\log T/\log\log T)^{1/2}, \text{ assuming the RH},$$
for any $h \in [0, (\log\log T)^{-1}]$.

Here we note that Theorem 1 is a direct consequence of (1.8) because
$$\sup_{u \in [t, t+h]} \pm S(u) \geqslant h^{-1} \int_t^{t+h} \pm S(u)\, du = h^{-1} \cdot \pm \{S_1(t+h) - S_1(t)\}.$$

The proofs of our theorems (except Theorem 4) follow the same basic idea of Selberg in [6], § 7. Consider, for example, the proof of (1.4). By taking the convolution of $\log \zeta(\frac{1}{2} + it + iu)$ and a kernel $V(u)$, we get (see Lemma 5)
$$\int_{-\infty}^{\infty} S(t+u) V(u)\, du = W(t) + R(t),$$
where $W(t)$ is a Dirichlet series and $R(t)$ is a sum over those zeros of $\zeta(s)$ with $\beta > \frac{1}{2}$ (if there is any). We then choose $V(u)$ to be an approximant of a

Dirac δ-function, so that the integral approximates $S(t)$. To prove the desired Ω-results, we look for values of $t\in[T, 2T]$ for which $W(t)$ (or $-W(t)$) is large and, at the same time, $R(t)$ is of lower order. To this end, we use arguments different from those in Selberg's paper. Our key idea is that (see Lemma 4), if

$$\int_T^{2T}\{W(t)\}^{2k}\,dt \geqslant TM^{2k} \quad\text{and}\quad \int_T^{2T}|R(t)|^{2k+1}\,dt \leqslant T(\tfrac{1}{2}M)^{2k+1},$$

then there exists $t\in[T, 2T]$ for which

$$|W(t)|-|R(t)| \geqslant \tfrac{1}{2}M.$$

So, it becomes a problem of finding good lower and upper estimates for high moments of $W(t)$ and $R(t)$ respectively. The estimation involving $W(t)$ is fairly effective because $W(t)$ is a rather well behaved Dirichlet series (or even a Dirichlet polynomial). However, due to the very limited knowledge about the distribution of the zeros, the estimation of $\int_T^{2T}|R(t)|^{2k+1}\,dt$ is more involved and is the most difficult part of the proof. We do this by means of a zero density theorem. In case RH is true, $R(t)$ will be zero and the conditional Ω-results come out easily.

As an illustration of the above ideas, we shall work out the details of Theorems 2 and 5 (in § § 3 and 5 respectively). In each case, we consider only the Ω_+ part, the proof of the Ω_- part being the same. Theorem 4 is a consequence of a result of Selberg and will be proved in § 4. Finally, in addition to the above ideas, the proof of (1.7) requires a more advantageous choice of $V(u)$ and some careful treatment of the remainder $R(t)$. We shall come to this in § 6.

Throughout this paper, we always assume T to be a large positive number. We shall use p, q to denote primes and use k to denote a positive integer. c and $c_1, c_2, \ldots$ etc. will denote some unspecified absolute constants which may not be the same at each occurrence. The constants implied in the symbols O and $\ll$ are absolute and, in particular, do not depend on σ and k. Finally, as usual, $\varrho=\beta+i\gamma$ denotes a typical complex zero of $\zeta(s)$.

2. Some preparations.

LEMMA 1. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of complex numbers. For any real numbers T and H, we have*

$$\int_T^{T+H}\Big|\sum_n a_n n^{-it}\Big|^2\,dt = H\sum_n |a_n|^2+O\Big(\sum_n n|a_n|^2\Big) \tag{2.1}$$

and

$$\int_T^{T+H}\Big(\sum_n a_n n^{-it}\Big)\overline{\Big(\sum_n b_n n^{-it}\Big)}\,dt = H\sum_n a_n\bar{b}_n+O\Big\{\Big(\sum_n n|a_n|^2\Big)^{1/2}\Big(\sum_n n|b_n|^2\Big)^{1/2}\Big\}. \tag{2.2}$$

Proof. (2.1) is a consequence of a refinement of the Hilbert's inequality given by Montgomery and Vaughan. See [5], p. 577.

(2.2) is a generalization and can be derived from (2.1) as follows. For any two complex numbers u and v, we have

$$u\bar{v} = \tfrac{1}{4}(|u+v|^2-|u-v|^2+i|u+vi|^2-i|u-vi|^2). \tag{2.3}$$

Write

$$U(t)=\sum_n a_n n^{-it} \quad\text{and}\quad V(t)=\sum_n b_n n^{-it}.$$

For any $\lambda>0$,

$$\begin{aligned} U(t)\overline{V(t)} &= \frac{U(t)}{\lambda}\cdot\overline{\lambda V(t)}\\ &= \tfrac{1}{4}\left(\left|\frac{U}{\lambda}+\lambda V\right|^2-\left|\frac{U}{\lambda}-\lambda V\right|^2+i\left|\frac{U}{\lambda}+\lambda Vi\right|^2-i\left|\frac{U}{\lambda}-\lambda Vi\right|^2\right).\end{aligned}$$

Integrate both sides of this equation with respect to t and apply (2.1), we have

$$\begin{aligned}\int_T^{T+H} U(t)\overline{V(t)}\,dt = H\cdot\tfrac{1}{4}\sum_n\Bigg(&\left|\frac{a_n}{\lambda}+\lambda b_n\right|^2-\left|\frac{a_n}{\lambda}-\lambda b_n\right|^2+i\left|\frac{a_n}{\lambda}+\lambda b_n i\right|^2\\ &-i\left|\frac{a_n}{\lambda}-\lambda b_n i\right|^2\Bigg)+\text{remainder term}.\end{aligned}$$

By (2.3), the main term is equal to $H\sum_n a_n\bar{b}_n$. The remainder term, according to (2.1), is

$$\begin{aligned}&\ll \sum_n n\left(\left|\frac{a_n}{\lambda}+\lambda b_n\right|^2+\left|\frac{a_n}{\lambda}-\lambda b_n\right|^2+\left|\frac{a_n}{\lambda}+\lambda b_n i\right|^2+\left|\frac{a_n}{\lambda}-\lambda b_n i\right|^2\right)\\ &\ll \lambda^{-2}\sum_n n|a_n|^2+\lambda^2\sum_n n|b_n|^2\\ &\ll \Big(\sum_n n|a_n|^2\Big)^{1/2}\Big(\sum_n n|b_n|^2\Big)^{1/2},\end{aligned}$$

by taking $\lambda=\big(\sum_n n|a_n|^2/\sum_n n|b_n|^2\big)^{1/4}$. This proves the lemma.

LEMMA 2. *Let $\{a_p\}$ be a sequence of complex numbers and $f(t)$ be the real part or the imaginary part of $\sum_p a_p p^{-it}$. For any real numbers T, H and any positive integer k, we have*

$$\int_T^{T+H}\{f(t)\}^{2k}\,dt = 2^{-2k}\binom{2k}{k}H\sum_p |a_p|^2P(p)+O\Big(k^k\Big(\sum_p p|a_p|^2\Big)^k\Big) \tag{2.4}$$

and

$$\int_T^{T+H} \{f(t)\}^{2k+1}\,dt = O\Big(k^k\big(\sum_p p|a_p|^2\big)^{k+1/2}\Big). \tag{2.5}$$

Here p *denotes* $(p_1, \ldots, p_k)$, $a_p = a_{p_1} \ldots a_{p_k}$ *and* $P(p) = P(p_1, \ldots, p_k)$ *is the number of permutations of* $p_1, \ldots, p_k$.

Proof. Consider the case $f(t) = \operatorname{Re}\sum_p a_p p^{-it}$. Write $\Sigma_t = \sum_p a_p p^{-it}$ so that $f(t) = \frac{1}{2}(\Sigma_t + \overline{\Sigma}_t)$ and

$$\{f(t)\}^{2k} = 2^{-2k}\sum_{m=0}^{2k}\binom{2k}{m}\Sigma_t^{2k-m}(\overline{\Sigma}_t)^m.$$

Therefore

$$\int_T^{T+H}\{f(t)\}^{2k}\,dt = 2^{-2k}\sum_{m=0}^{2k}\binom{2k}{m}I_m, \tag{2.6}$$

where

$$I_m = \int_T^{T+H}\Sigma_t^{2k-m}(\overline{\Sigma}_t)^m\,dt \quad \text{for} \quad m = 0, 1, \ldots, 2k.$$

For any two non-negative integers τ and ν, (2.2) yields

$$\int_T^{T+H}\Sigma_t^\tau(\overline{\Sigma}_t)^\nu\,dt = H\sum_n A_n\overline{B}_n + O\,\{(\sum_n n|A_n|^2)^{1/2}(\sum_n n|B_n|^2)^{1/2}\},$$

where

$$A_n = \begin{cases} a_{p_1}\ldots a_{p_\tau}P(p_1, \ldots, p_\tau) & \text{for } n = p_1\ldots p_\tau,\\ 0 & \text{otherwise,}\end{cases}$$

and similarly for the B_n's.

By uniqueness of integer factorization, we see that

$$\sum_n A_n\overline{B}_n = \begin{cases}\sum_{p_1,\ldots,p_\tau}|a_{p_1}\ldots a_{p_\tau}|^2 P(p_1, \ldots, p_\tau) & \text{for } \tau = \nu,\\ 0 & \text{otherwise.}\end{cases}$$

Moreover,

$$\sum_n n|A_n|^2 \leqslant \tau!\sum_{p_1,\ldots,p_\tau}p_1\ldots p_\tau|a_{p_1}\ldots a_{p_\tau}|^2 = \tau!\,(\sum_p p|a_p|^2)^\tau$$

and

$$\sum_n n|B_n|^2 \leqslant \nu!\,(\sum_p p|a_p|^2)^\nu.$$

Thus

$$I_m = \begin{cases} H\sum_p|a_p|^2P(p) + O\big(k!\,(\sum_p p|a_p|^2)^k\big) & \text{for } m = k,\\ O\big(\sqrt{(2k-m)!\,m!}\,(\sum_p p|a_p|^2)^k\big) & \text{for } m \neq k,\end{cases}$$

and (2.4) follows by substituting this in (2.6). The proof of (2.5) is along the same line. It has no main term because there is no middle term in the binomial expansion of $\{f(t)\}^{2k+1}$. The other case is similarly proved.

Lemma 3. *Let* k *be a large positive integer,* $\lambda = 2k\log k$ *and* $5\lambda \leqslant x^{3/4}$. *Let* $\{a_p\}_{p\leqslant x}$ *be a sequence of complex numbers and* $A \geqslant 1$ *be an absolute constant such that*

$$A^{-1} \leqslant |a_p/a_q| \leqslant A \tag{2.7}$$

for any $p, q \in [\lambda, x^{3/4}]$, $q < p \leqslant 2q$.

Then both

$$\int_T^{2T}\{\operatorname{Re}\sum_{p\leqslant x}a_p p^{-it}\}^{2k}\,dt \quad \text{and} \quad \int_T^{2T}\{\operatorname{Im}\sum_{p\leqslant x}a_p p^{-it}\}^{2k}\,dt$$

$$\geqslant T(eA)^{-2k}k!\,\big(\sum_{\lambda<p\leqslant x^{3/4}}|a_p|^2\big)^k - O\big(k^k\big(\sum_{p\leqslant x}p|a_p|^2\big)^k\big).$$

Proof. In view of (2.4), what we have to prove is that

$$2^{-2k}\binom{2k}{k}\sum_p|a_p|^2P(p) \geqslant (eA)^{-2k}k!\,\big(\sum_{\lambda<p\leqslant x^{3/4}}|a_p|^2\big)^k. \tag{2.8}$$

Define the following sequence of abutting sub-intervals of $(\lambda, x]$:

$$I_j = (2^{j-1}\lambda, 2^j\lambda]; \quad j = 1, 2, \ldots, m,$$

where m satisfies

$$x^{3/4} \leqslant 2^m\lambda < 2x^{3/4}.$$

Let $j = (j_1, \ldots, j_k)$ and let

$$\psi_j = \sum_{p\in I_j}|a_p|^2P(p),$$

where $p \in I_j$ means $p_\nu \in I_{j_\nu}$ for $\nu = 1, 2, \ldots, k$. For a fixed j, if $|a_q|$ is the smallest one among all the $|a_p|$, $p \in I_j$, then

$$\psi_j \geqslant |a_q|^2\sum_{p\in I_j}P(p)$$

$$\geqslant k!\,|a_q|^2\big(\sum_{p\in I_j}1\big)\times(\text{chance of having a } p\in I_j \text{ with all entries distinct}).$$

The chance of picking a $p \in I_j$ with all entries distinct is higher when j has more distinct entries and when I_{j_ν} $(\nu = 1, 2, \ldots, k)$ contains more primes. Since I_1 is the shortest sub-interval, it contains the least number of primes. Indeed, if l is the number of primes in I_1, then

$$l = \frac{\lambda}{\log \lambda} + O\left(\frac{\lambda}{\log^2 \lambda}\right) \geqslant k$$

when k is sufficiently large. The chance of choosing k distinct primes from I_1 is equal to

$$l(l-1)\ldots(l-k+1)\, l^{-k} \geqslant e^{-k^2/l} \geqslant e^{-k}.$$

Thus,

$$\psi_j \geqslant |a_q|^2\, k!\, e^{-k} \Big(\sum_{p \in I_j} 1\Big).$$

The assumption (2.7) guarantees that for any $p, p' \in I_j$

$$A^{-k} \leqslant |a_p/a_{p'}| \leqslant A^k.$$

So,

$$\psi_j \geqslant k!\, e^{-k} A^{-2k} \sum_{p \in I_j} |a_p|^2.$$

Summing over all j with entries in $[1, m]$, we have

$$\sum_p |a_p|^2 P(p) \geqslant \sum_j \psi_j \geqslant k!\, e^{-k} A^{-2k} \sum_p |a_p|^2 = k!\, e^{-k} A^{-2k} \Big(\sum_p |a_p|^2\Big)^k,$$

and (2.8) follows immediately.

The next lemma contains the key idea that leads to the improved Ω-results.

LEMMA 4. *Let k be a positive integer and let $W(t)$ and $R(t)$ be real valued functions satisfying the following conditions*:

(i) $\int_T^{2T} \{W(t)\}^{2k}\, dt \geqslant T M_1^{2k}$,

(ii) $\left|\int_T^{2T} \{W(t)\}^{2k+1}\, dt\right| \leqslant \frac{1}{2} T M_1^{2k+1}$,

(iii) $\int_T^{2T} |R(t)|^{2k+1}\, dt \leqslant T M_2^{2k+1}$,

(iv) $M_1 \geqslant 2M_2$.

We have

$$\sup_{t \in [T, 2T]} \pm \{W(t) + R(t)\} \geqslant \tfrac{1}{2} M_1 - M_2.$$

Proof. Let $W_+(t)$ and $W_-(t)$ be respectively the positive and negative parts of $W(t)$, that is

$$W_+(t) = \max\{W(t), 0\}, \qquad W_-(t) = \min\{W(t), 0\}.$$

Conditions (i) and (ii) imply that $W(t)$ has to be large in both directions. Indeed, by (i) and Cauchy–Schwarz inequality, we have

$$\int_T^{2T} |W(t)|^{2k+1}\, dt \geqslant T M_1^{2k+1}.$$

From the definitions of $W_+(t)$ and $W_-(t)$,

$$\int_T^{2T} |W_+(t)|^{2k+1}\, dt = \tfrac{1}{2} \int_T^{2T} |W(t)|^{2k+1}\, dt + \tfrac{1}{2} \int_T^{2T} \{W(t)\}^{2k+1}\, dt$$

and

$$\int_T^{2T} |W_-(t)|^{2k+1}\, dt = \tfrac{1}{2} \int_T^{2T} |W(t)|^{2k+1}\, dt - \tfrac{1}{2} \int_T^{2T} \{W(t)\}^{2k+1}\, dt.$$

Thus, in view of (ii), both

$$\int_T^{2T} |W_+(t)|^{2k+1}\, dt \quad \text{and} \quad \int_T^{2T} |W_-(t)|^{2k+1}\, dt \quad \geqslant \tfrac{1}{4} T M_1^{2k+1}.$$

Let

$$\int_T^{2T} |W_+(t)|^{2k+1}\, dt = T M_3^{2k+1} \tag{2.9}$$

so that $M_3 \geqslant \frac{1}{2} M_1$. Write $m = 2k+1$. By Cauchy–Schwarz inequality, (2.9) and (iii), we have

$$\int_T^{2T} |W_+(t)|^{m-\nu} |R(t)|^{\nu-1}\, dt \leqslant T M_3^{m-\nu} M_2^{\nu-1} \quad \text{for} \quad \nu = 1, 2, \ldots, m.$$

Hence

$$\begin{aligned} T(M_3^m - M_2^m) &\leqslant \int_T^{2T} \big(|W_+(t)|^m - |R(t)|^m\big)\, dt \\ &= \int_T^{2T} \big(|W_+(t)| - |R(t)|\big)\Big(\sum_{\nu=1}^{m} |W_+(t)|^{m-\nu} |R(t)|^{\nu-1}\Big)\, dt \\ &\leqslant \sup_{t \in [T, 2T]} \big(|W_+(t)| - |R(t)|\big)\Big(\sum_{\nu=1}^{m} T M_3^{m-\nu} M_2^{\nu-1}\Big) \\ &\leqslant \sup_{t \in [T, 2T]} \{W(t) + R(t)\}\, T(M_3^m - M_2^m)(M_3 - M_2)^{-1}. \end{aligned}$$

That is,

$$\sup_{t\in[T,2T]} \{W(t)+R(t)\} \geqslant M_3 - M_2 \geqslant \tfrac{1}{2}M_1 - M_2.$$

Similarly,

$$\sup_{t\in[T,2T]} -\{W(t)+R(t)\} \geqslant \tfrac{1}{2}M_1 - M_2.$$

LEMMA 5. *Suppose* $\sigma\in[\frac{1}{2}, 2]$. *Let* $V(x+iy)$ *be an analytic function in the horizontal strip*: $\sigma-2\leqslant y\leqslant 0$ *satisfying the growth condition*

$$\sup_{\sigma-2\leqslant y\leqslant 0} |V(x+iy)| \ll (|x|\log^2|x|)^{-1}. \tag{2.10}$$

For any $t\neq 0$, *we have*

$$\int_{-\infty}^{\infty} \log\zeta(\sigma+i(t+u))\,V(u)\,du = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{\log n}\hat{V}\left(\frac{-\log n}{2\pi}\right)n^{-\sigma-it} + 2\pi\sum_{\beta>\sigma}\int_0^{\beta-\sigma} V(\gamma-t-i\alpha)\,d\alpha + O(|t|^{-1}). \tag{2.11}$$

Proof. Let Z be a large number. From the rectangle with vertices at $\sigma\pm iZ$, $2\pm iZ$, we remove the stretch joining σ and 1 and all those horizontal stretches joining $\sigma+i\gamma$ to $\beta+i\gamma$ for any zero $\varrho=\beta+i\gamma$ lying inside. Designate this domain by R. From the definition of $\log\zeta(s)$, we see that $\log\zeta(z)V(-t+i\sigma-iz)$ is a single-valued analytic function in R. Its value at the upper and lower side of a cut differ by $2\pi i V(\gamma-t-i(\alpha-\sigma))$, $\sigma\leqslant\alpha\leqslant\beta$. Using Cauchy's theorem on R, we have

$$\int_{-Z-t}^{Z-t} \log\zeta(\sigma+i(t+u))\,V(u)\,du \tag{2.12}$$

$$= \int_{-Z-t}^{Z-t} \log\zeta(2+i(t+u))\,V(i(\sigma-2)+u)\,du$$

$$+2\pi \sum_{\substack{\sigma<\beta\\ -Z<\gamma\leqslant Z}} \int_{\sigma}^{\beta} V(\gamma-t-i(\alpha-\sigma))\,d\alpha - 2\pi\int_{\min(1,\sigma)}^{1} V(-t-i(\alpha-\sigma))\,d\alpha$$

$$+i\int_{\sigma}^{2}\{\log\zeta(\alpha+iZ)\,V(Z-t-i(\alpha-\sigma)) - \log\zeta(\alpha-iZ)\,V(-Z-t-i(\alpha-\sigma))\}\,d\alpha.$$

It is well known that (for example, from Theorem 9.6 (B) of [7])

$$\int_{\sigma}^{2} |\log\zeta(\alpha+it)|\,d\alpha = O(\log|t|). \tag{2.13}$$

Thus, in view of the growth condition (2.10), the last integral in (2.12) tends to zero as $Z\to\infty$. Since $\log\zeta(2+i(t+u)) = \sum_{n=2}^{\infty}\frac{\Lambda(n)}{\log n}n^{-2-i(t+u)}$, and the Dirichlet series converges absolutely, we obtained from (2.12), by letting $Z\to\infty$, that

$$\int_{-\infty}^{\infty} \log\zeta(\sigma+i(t+u))\,V(u)\,du \tag{2.14}$$

$$= \sum_{n=2}^{\infty}\frac{\Lambda(n)}{\log n}\int_{-\infty}^{\infty} n^{-2-i(t+u)}\,V(i(\sigma-2)+u)\,du$$

$$+2\pi\sum_{\sigma<\beta}\int_0^{\beta-\sigma} V(\gamma-t-i\alpha)\,d\alpha - 2\pi\int_{\min(1-\sigma,0)}^{1-\sigma} V(-t-i\alpha)\,d\alpha.$$

Finally, by Cauchy's theorem and (2.10), we can shift the path of integration so that

$$\int_{-\infty}^{\infty} n^{-2-i(t+u)}\,V(i(\sigma-2)+u)\,du = \int_{-\infty}^{\infty} n^{-\sigma-i(t+u)}\,V(u)\,du = n^{-\sigma-it}\,\hat{V}\left(\frac{-\log n}{2\pi}\right).$$

This together with (2.14) proves our lemma.

The next lemma gives an upper estimate to a sum involving the zeros of $\zeta(s)$. Its proof makes use of the following zero density theorem of Selberg ([6], Theorem 1):

$$N(\sigma, T+H) - N(\sigma, T) \ll H(H/\sqrt{T})^{(1/2-\sigma)/2}\log T \tag{2.15}$$

for any $T^{1/2+\varepsilon} < H \leqslant T$, $\varepsilon>0$ and any $\sigma\in[\frac{1}{2}, 1]$. This result is particularly good when σ is very close to $\frac{1}{2}$. However, our subsequent applications of this lemma do not require the full strength of Selberg's result.

LEMMA 6. *Let* $3\leqslant X\leqslant T^{1/8}$ *and* $\frac{1}{2}\leqslant\sigma\leqslant 1$. *For any non-negative number* ν, *we have*

$$\sum_{\substack{\beta>\sigma\\ T<\gamma\leqslant 2T}} (\beta-\sigma)^{\nu}X^{(\beta-\sigma)} \ll T^{1+(1/2-\sigma)/4}(\log T)^{1-\nu}c^{\nu}\Gamma(\nu+1).$$

Proof. Let $\delta_\nu=1$ or 0 according as ν equals to zero or not. Clearly,

$$\sum_{\substack{\beta>\sigma\\ T<\gamma\leqslant 2T}} (\beta-\sigma)^{\nu}X^{(\beta-\sigma)} = \sum_{\substack{\beta>\sigma\\ T<\gamma\leqslant 2T}} \left\{\int_0^{\beta-\sigma} d(u^{\nu}X^{u}) + \delta_\nu\right\}$$

$$= \int_0^{1-\sigma} \sum_{\substack{\beta>\sigma+u\\ T<\gamma\leqslant 2T}} (\nu u^{\nu-1}X^{u} + u^{\nu}X^{u}\log X)\,du$$

$$+\delta_\nu\{N(\sigma, 2T) - N(\sigma, T)\}$$

$$= \int_0^{1-\sigma} \{N(\sigma+u, 2T) - N(\sigma+u, T)\} \{\nu u^{\nu-1} X^u + u^\nu X^u \log X\} du$$
$$+ \delta_\nu \{N(\sigma, 2T) - N(\sigma, T)\}.$$

Using the estimate (2.15), this is

$$\ll T^{1+(1/2-\sigma)/4} \log T \{ \int_0^{1-\sigma} \nu u^{\nu-1} (XT^{-1/4})^u du + \log X \int_0^{1-\sigma} u^\nu (XT^{-1/4})^u du\}$$
$$+ \delta_\nu T^{1+(1/2-\sigma)/4} \log T$$
$$\ll T^{1+(1/2-\sigma)/4} (\log T)^{1-\nu} c^\nu \Gamma(\nu+1).$$

3. Proof of Theorem 2. We shall prove that

(3.1) $$\sup_{t\in[T,2T]} S(\sigma, t) \geqslant c(\log T/\log\log T)^{1/3}$$
$$\text{for } \tfrac{1}{2} \leqslant \sigma \leqslant \tfrac{1}{2} + (\log\log T/\log T)^{1/3},$$
$$\geqslant c((\sigma - \tfrac{1}{2}) \log T/\log\log T)^{1/2}$$
$$\text{for } \tfrac{1}{2} + (\log\log T/\log T)^{1/3} \leqslant \sigma \leqslant \tfrac{1}{2} + (\log\log T)^{-1}.$$

Throughout this section, we assume $\sigma \in [\frac{1}{2}, \frac{1}{2} + (\log\log T)^{-1}]$, $t \in [T, 2T]$ and

(3.2) $$\tau = 2 \log\log T.$$

Let $V(z) = (z/2)^{-2} \sin^2(\tau z/2)$. Clearly, it satisfies the growth condition (2.10) and its Fourier transform is given by

$$\hat{V}(v) = 2\pi \max(0, \tau - |2\pi v|).$$

With this particular choice of $V(z)$, the imaginary part of equation (2.11) becomes

(3.3) $$\int_{-\infty}^{\infty} \operatorname{Im} \log \zeta(\sigma + i(t+u)) V(u) du$$
$$= 2\pi \operatorname{Im} \sum_{n\leqslant e^\tau} \frac{\Lambda(n)}{\log n} (\tau - \log n) n^{-\sigma-it} + 2\pi \sum_{\beta>\sigma} \int_0^{\beta-\sigma} \operatorname{Im} V(\gamma - t - i\alpha) d\alpha + O(T^{-1}).$$

For the Dirichlet polynomial on the right side, the contribution of those terms corresponding to $n \neq$ prime is

$$\ll \sum_{\substack{p^r \leqslant e^\tau \\ r\geqslant 2}} \tau p^{-r\sigma} \ll \tau \sum_{p\leqslant e^{\tau/2}} p^{-2\sigma} \ll \tau \log \tau.$$

Thus, we can rewrite (3.3) as

(3.4) $$\int_{-\infty}^{\infty} S\left(\sigma, t + \frac{2u}{\tau}\right) u^{-2} \sin^2 u\, du = W(t) + R(t) + O(\log\tau),$$

where

$$W(t) = \operatorname{Im} \sum_{p\leqslant e^\tau} (1 - \tau^{-1} \log p) p^{-\sigma-it}$$

and

(3.5) $$R(t) = \tau \sum_{\beta>\sigma} \int_0^{\beta-\sigma} \operatorname{Im} \left(\frac{\sin(\tau(\gamma - t - i\alpha)/2)}{\tau(\gamma - t - i\alpha)/2} \right)^2 d\alpha.$$

It is easy to show by standard methods that

$$S(\sigma, t) \ll \log|t| \quad \text{as} \quad |t| \to \infty.$$

This implies

$$\int_{|u|>\log T} S\left(\sigma, t + \frac{2u}{\tau}\right) u^{-2} \sin^2 u\, du \ll \int_{\log T}^{\infty} (\log T + \log u) u^{-2} du \ll 1.$$

On the other hand,

$$\int_{|u|\leqslant\log T} S\left(\sigma, t + \frac{2u}{\tau}\right) u^{-2} \sin^2 u\, du \leqslant \sup_{t\in[T/2,3T]} S(\sigma, t).$$

Therefore, we deduced from (3.4) that

(3.6) $$\sup_{t\in[T/2,3T]} S(\sigma, t) \geqslant W(t) + R(t) + O(\log\tau) \quad \text{for all } t \in [T, 2T].$$

Let k be a large positive integer satisfying

(3.7) $$10k \log k \leqslant e^{2\tau/3} \quad \text{and} \quad e^{k\tau} \leqslant T^{1/16}.$$

Our $W(t)$ is of the form $\operatorname{Im} \sum_{p\leqslant e^\tau} a_p p^{-it}$ and we see easily that, for the bove choice of k, Lemma 3 is applicable with $A = 3$. Thus,

$$\{W(t)\}^{2k} dt \geqslant T(ck)^k \Big\{ \sum_{2k\log k < p \leqslant e^{3\tau/4}} p^{-2} \left(1 - \frac{\log p}{\tau}\right)^2 \Big\}^k - O\{(ck)^k (\sum_{p\leqslant e^\tau} p^{1-2\sigma})^k\}.$$

ince $0 \leqslant \sigma - 1/2 \leqslant (\log\log T)^{-1} = 2/\tau$, the sum in the main term, by the rime number theorem, is

$$\geqslant \sum_{e^{2\tau/3}<p\leqslant e^{3\tau/4}} p^{-2\sigma} \geqslant e^{-3}\int_{4\tau(\sigma-1/2)/3}^{3\tau(\sigma-1/2)/2} \frac{dw}{w} - O(\tau^{-1}) = e^{-3}\log\frac{9}{8} - O(\tau^{-1}) \geqslant c.$$

The sum in the remainder term is simply $\leqslant e^{\tau}$. Hence,

$$\int_T^{2T} \{W(t)\}^{2k}\, dt \geqslant T(ck)^k.$$

On the other hand, by (2.5) and (3.7),

$$\int_T^{2T} \{W(t)\}^{2k+1}\, dt \ll k^k \Big(\sum_{p\leqslant e^{\tau}} p^{1-2\sigma}\Big)^{k+1/2} \leqslant k^k e^{2k\tau} \leqslant T^{1/8} k^k.$$

In view of Lemma 4, what remains to be shown is that, for some suitable k and a sufficiently small c_1

$$\int_T^{2T} |R(t)|^{2k+1}\, dt \leqslant T(c_1 k)^k. \tag{3.8}$$

We would then have

$$\sup_{t\in[T,2T]} S(\sigma, t) \geqslant c\sqrt{k}.$$

If RH is true, then $R(t) = 0$, and (3.7) allows us to take

$$k = [c_2 \log T/\log\log T]$$

for some sufficiently small c_2. This gives

$$\sup_{t\in[T,2T]} S(\sigma, t) \geqslant (c\log T/\log\log T)^{1/2}$$

for $\sigma\in[1/2,\ 1/2+(\log\log T)^{-1}]$.

Without assuming the truth of RH, we shall use Lemma 6 to estimate the integral $\int_T^{2T} |R(t)|^{2k+1}\, dt$.

First of all, we can show that for any real x, y,

$$\operatorname{Im}\left(\frac{\sin(x+iy)}{x+iy}\right)^2 \ll |y|\, e^{2|y|}/(1+x^2+y^2).$$

Moreover, $e^y/(1+x^2+y^2)$ is an increasing function of y.

Define for $t\in[T, 2T]$,

$$\theta_t(\sigma) = \max_{\substack{\sigma<\beta\\ |\gamma-t|\leqslant\log^2 T}} (\beta-\sigma) \quad\text{and}\quad \theta_t = \theta_t(1/2) = \max_{\substack{\beta>1/2\\ |\gamma-t|\leqslant\log^2 T}} (\beta-1/2).$$

By (3.2),

$$0 \leqslant \tau\theta_t - \tau\theta_t(\sigma) \leqslant \tau(\sigma-1/2) \leqslant 2. \tag{3.9}$$

It follows from (3.5) that

$$\begin{aligned}
R(t) &\ll \tau \sum_{\beta>\sigma} \int_0^{\beta-\sigma} \tau\alpha e^{\tau\alpha}\{1+(\tau/2)^2(\gamma-t)^2+(\tau\alpha/2)^2\}^{-1}\, d\alpha\\
&\leqslant \sum_{\beta>\sigma} (\beta-\sigma)^2 e^{\tau(\beta-\sigma)}\{(2/\tau)^2+(\gamma-t)^2+(\beta-\sigma)^2\}^{-1}\\
&\leqslant \theta_t^2(\sigma) \sum_{\substack{\beta>\sigma\\ |\gamma-t|\leqslant\log^2 T}} e^{\tau(\beta-\sigma)}\{(2/\tau)^2+(\gamma-t)^2+(\beta-\sigma)^2\}^{-1}\\
&\qquad + e^{\tau/2} \sum_{|\gamma-t|>\log^2 T} (\gamma-t)^{-2}\\
&\leqslant \theta_t^2(\sigma)\, e^{\tau\theta_t} \sum_{\substack{\beta>1/2\\ |\gamma-t|\leqslant\log^2 T}} \{(2/\tau)^2+(\gamma-t)^2+\theta_t^2\}^{-1}\\
&\qquad + \log T \sum_{|\gamma-t|>\log^2 T} (\gamma-t)^{-2}.
\end{aligned}$$

The sum $\sum_{|\gamma-t|>\log^2 T} (\gamma-t)^2 \ll (\log T)^{-1}$ because $N(t+1)-N(t) \leqslant c\log t$.

To estimate the first sum, we quote a result of Selberg from [6], § 7, namely,

$$\sum_{\substack{\beta>1/2\\ |\gamma-t|\leqslant\log^2 T}} \{(2/\tau)^2+(\gamma-t)^2+\theta_t^2\}^{-1} \ll \tau\log T.$$

Thus,

$$R(t) \ll \theta_t^2(\sigma)\, e^{\tau\theta_t}\tau\log T + 1 \ll \tau\theta_t^2(\sigma)\, e^{\tau\theta_t(\sigma)}\log T + 1,$$

by (3.9).

By definition of $\theta_t(\sigma)$,

$$\begin{aligned}
\int_T^{2T} \{\theta_t^2(\sigma)\, e^{\tau\theta_t(\sigma)}\}^{2k+1}\, dt &\leqslant e^{\tau/2}\log^2 T \sum_{\substack{\beta>\sigma\\ T/2<\gamma\leqslant 3T}} (\beta-\sigma)^{4k+2} e^{2k\tau(\beta-\sigma)}\\
&\ll T^{1+(1/2-\sigma)/4}(ck)^{4k}(\log T)^{-4k+1} e^{\tau/2}\log^2 T,
\end{aligned}$$

by Lemma 6 with $X = e^{2k\tau}$. Therefore

$$\begin{aligned}
\int_T^{2T} |R(t)|^{2k+1}\, dt &\ll T^{1-(\sigma-1/2)/4}(ck)^{4k}(\tau\log T)^{2k+1}(\log T)^{-4k+2}\\
&= T^{1-(\sigma-1/2)/4}(c\tau k^2/\log T)^{2k+1}\log^4 T.
\end{aligned}$$

It remains to specify k so that, in addition to (3.7), the inequality

$$\sqrt{k} > (c\tau k^2/\log T)(T^{-(\sigma-1/2)/4}\log^4 T)^{1/(2k+1)}$$

is true for some large constant c.

For $1/2 \leqslant \sigma \leqslant 1/2+(\log\log T/\log T)^{1/3}$, we take

$$k = [c'(\log T/\log\log T)^{2/3}], \quad \text{where } c' \text{ is small.}$$

For $1/2+(\log\log T/\log T)^{1/3} \leqslant \sigma \leqslant 1/2+(\log\log T)^{-1}$, we take

$$k = [(\sigma-1/2)\log T/(16\log\log T)].$$

It is easy to verify that the preceding choice of k satisfies all the requirements. This establishes the inequality (3.8) and hence proves (3.1).

4. Proof of Theorem 4. Assume

$$a > 1/2, \quad T^a < H \leqslant T \quad \text{and} \quad \varepsilon = (a-1/2)/20.$$

LEMMA 7. *For any positive integer k, we have*

$$\int_T^{T+H} \Big|S(t)-\pi^{-1}\operatorname{Im}\sum_{p\leqslant T^{\varepsilon/k}} p^{-1/2-it}\Big|^{2k} dt = O\big(H(ck)^{2k}\big).$$

Proof. This is equation (5.3) of [6], but with the error term made uniform in k. It is slightly better than a corresponding result of Ghosh ([3], Lemma 5), which has $H(ck)^{4k}$ on the right side instead. The improvement here came from a sharpening of his Lemma 2. See [8], Lemma 5.2.

Let $0 < h < 1$, $x = T^{\varepsilon/k}$,

$$Q(t) = S(t)-\pi^{-1}\operatorname{Im}\sum_{p\leqslant x} p^{-1/2-it}$$

and

$$P(t) = \pi^{-1}\operatorname{Im}\sum_{p\leqslant x} p^{-1/2-it}(p^{-ih}-1).$$

By (2.4),

$$\text{(4.1)} \quad \int_T^{T+H} \{P(t)\}^{2k} dt = 2^{-2k}\binom{2k}{k} H \sum_{\mathbf{p}\leqslant x} |a_{\mathbf{p}}|^2 P(\mathbf{p}) + O\big(k^k\big(\sum_{p\leqslant x} p|a_p|^2\big)^k\big),$$

where $a_p = \pi^{-1}p^{-1/2}(p^{-ih}-1)$ for $p \leqslant x$. Using the prime number theorem, we can show that

$$\sum_{p\leqslant x} |a_p|^2 = 2\pi^{-2}\{\sum_{p\leqslant x} p^{-1} - \sum_{p\leqslant x} p^{-1}\cos(h\log p)\}$$
$$= 2\pi^{-2}\log(2+h\log T)+O(\log k).$$

The sum in the main term of (4.1) is equal to

$$k!\sum_{p\leqslant x}|a_p|^2 - O\{k!\textstyle\sum'|a_{\mathbf{p}}|^2\}$$

($\sum'$ denotes the summation over those $\mathbf{p}$ whose entries are not all distinct)

$$= k!\big(\sum_{p\leqslant x}|a_p|^2\big)^k - O\{k!\,k^2\big(\sum_{p\leqslant x}|a_p|^4\big)\big(\sum_{p\leqslant x}|a_p|^2\big)^{k-2}\}.$$

Thus,

$$\text{(4.2)} \quad \int_T^{T+H} \{P(t)\}^{2k} dt = \frac{(2k)!}{2^k\pi^{2k}k!} H\log^k(2+h\log T)$$
$$+O\{H(ck)^k\log k\{\log^{k-1}(2+h\log T)+(\log k)^{k-1}\}\}$$

and

$$\text{(4.3)} \quad \int_T^{T+H} \{P(t)\}^{2k} dt \ll H(ck)^k\{\log^k(2+h\log T)+\log^k k\}.$$

By Lemma 7,

$$\int_T^{T+H} \{Q(t)\}^{2k} dt \ll H(ck)^{2k}.$$

If we write $U(t) = Q(t+h)-Q(t)$, then

$$\text{(4.4)} \quad \int_T^{T+H} \{U(t)\}^{2k} dt \ll H(ck)^{2k}.$$

Now,

$$\int_T^{T+H} \{S(t+h)-S(t)\}^{2k} dt = \int_T^{T+H} \{Q(t+h)-Q(t)+P(t)\}^{2k} dt$$
$$= \int_T^{T+H} \{P(t)\}^{2k} dt$$
$$+O\{c^k\int_T^{T+H} |P(t)|^{2k-1}|U(t)|\,dt + c^k\int_T^{T+H} \{U(t)\}^{2k} dt\}.$$

By Cauchy–Schwarz inequality, (4.3) and (4.4), the first integral in the O-term is

$$\ll \{\int_T^{T+H} \{P(t)\}^{2k} dt\}^{(2k-1)/2k}\{\int_T^{T+H} \{U(t)\}^{2k} dt\}^{1/2k}$$
$$\ll H(ck)^k\{\{\log(2+h\log T)\}^{k-1/2}+\{\log k\}^{k-1/2}\}.$$

Thus, in view of (4.2) and (1.6), we have

$$\int_T^{T+H} \{S(t+h)-S(t)\}^{2k}\,dt$$
$$= A_k H \log^k(2+h\log T)+O\left\{H(ck)^k\left\{k^k+(\log(2+h\log T))^{k-1/2}\right\}\right\}.$$

This proves Theorem 4.

5. Proof of Theorem 3. We shall prove that, as $T\to\infty$,

$$\sup_{t\in[T,2T]}\{S(t+h)-S(t)\}\geqslant c(h\log T)^{1/3}$$

for any $h\in[(\log T)^{-1}, (\log\log T)^{-1}]$.

The idea of the proof is very much the same as that of Theorem 2. The estimation of $\int_T^{2T}|R(t)|^{2k+1}\,dt$ is slightly more complicated and makes use of Theorem 4. In fact, as we shall point out later, it is actually the proof of

$$\inf_{t\in[T,2T]}\{S(t+h)-S(t)\}\leqslant -c(h\log T)^{1/3}$$

that needs Theorem 4.

First of all, we assume that

$$B(\log T)^{-1}<h\leqslant(\log\log T)^{-1}$$

for some sufficiently large absolute constant B. This is no real restriction, because if

$$(\log T)^{-1}\leqslant h\leqslant B(\log T)^{-1}\quad\text{and}\quad S(t+Bh)-S(t)\geqslant c(Bh\log T)^{1/3},$$

then there exists $n\leqslant B$ such that

$$S(t+nh)-S(t+(n-1)h)\geqslant\frac{c}{B}(Bh\log T)^{1/3}.$$

For this section, we let

(5.1) $$\tau = 3/(2h).$$

In Lemma 5, we take $\sigma=1/2$ and

$$V(z)=(z/2)^{-2}\sin^2(\tau z/2).$$

Replace t by $t+h/2$ and $t-h/2$ successively, the imaginary part of the difference of the two equations arising from (3.3) can be written as

$$\int_{-\infty}^{\infty}\{S(t+h/2+2u/\tau)-S(t-h/2+2u/\tau)\}\,\tau^{-2}\,V(2u/\tau)\,du$$

$$=-2\sum_{n\leqslant e^\tau}\frac{\Lambda(n)}{\log n}\left(1-\frac{\log n}{\tau}\right)\sin\left(\tfrac{1}{2}h\log n\right)n^{-1/2}\cos(t\log n)$$
$$+\tau^{-1}\sum_{\beta>1/2}\int_0^{\beta-1/2}\operatorname{Im}\{V(\gamma-t-h/2-i\alpha)-V(\gamma-t+h/2-i\alpha)\}\,d\alpha+O(T^{-1}).$$

Following the same type of arguments that lead to (3.6), we deduced that

$$\sup_{t\in[T/2,3T]}\{S(t+h)-S(t)\}\geqslant W(t)+R(t)-O(1)\quad\text{for all } t\in[T,2T],$$

where

$$W(t)=-2\operatorname{Re}\sum_{p\leqslant e^\tau}\sin\left(\tfrac{1}{2}h\log p\right)\left(1-\frac{\log p}{\tau}\right)p^{-1/2-it}$$

and

(5.2) $$R(t)=\tau^{-1}\sum_{\beta>1/2}\int_0^{\beta-1/2}\operatorname{Im}\{V(\gamma-t-h/2-i\alpha)-V(\gamma-t+h/2-i\alpha)\}\,d\alpha.$$

Let k be a large positive integer satisfying

(5.3) $$10k\log k\leqslant e^{2\tau/3}\quad\text{and}\quad e^{k\tau}\leqslant T^{1/48}.$$

Our $W(t)$ is of the form $\operatorname{Re}\sum_{p\leqslant e^\tau}a_p p^{-it}$ with

$$a_p=-2p^{-1/2}\sin\left(\tfrac{1}{2}h\log p\right)(1-\tau^{-1}\log p).$$

For k satisfying (5.3), Lemma 3 is applicable to $W(t)$ with $A=4$. Thus,

$$\int_T^{2T}\{W(t)\}^{2k}\,dt\geqslant T(ck)^k\Big(\sum_{2k\log k<p\leqslant e^{3\tau/4}}a_p^2\Big)^k-O\Big\{k^k\Big(\sum_{p\leqslant e^\tau}pa_p^2\Big)^k\Big\}$$
$$\geqslant T(ck)^k\sum_{e^{2\tau/3}<p\leqslant e^{3\tau/4}}h^2p^{-1}\log^2 p-O(k^k e^{\tau k})$$
$$\geqslant T(ck)^k,$$

by (5.1). Also, by (2.5), we have

$$\int_T^{2T}\{W(t)\}^{2k+1}\,dt\ll k^k\Big(\sum_{p\leqslant e^\tau}pa_p^2\Big)^{k+1/2}\ll T^{1/24}k^k.$$

At this point, if RH is true, then $R(t)=0$ and we would have

$$\sup_{t\in[T/2,3T]}\{S(t+h)-S(t)\}\geqslant\sup_{t\in[T,2T]}W(t)-O(1)\geqslant c\sqrt{k},$$

by Lemma 4. Choosing

$$k = \left[\frac{\log T}{50\tau}\right] = \left[\frac{h}{75}\log T\right]$$

(which satisfies (5.3)), this gives the unpublished result of Selberg.

Without assuming RH, we shall prove that

$$\int_T^{2T} |R(t)|^{2k+1}\,dt \ll T\left(\frac{ck^2}{h\log T}\right)^{2k+1}, \tag{5.4}$$

provided

$$(h\log T)^{1/2} < k \leqslant h\log T. \tag{5.5}$$

When B is sufficiently large, the choice

$$k = \left[\left(\frac{h}{B}\log T\right)^{2/3}\right]$$

will satisfy (5.3), (5.5) as well as the inequality

$$ck^2/(h\log T) < \sqrt{k}$$

for some large constant c. We would then have, by Lemma 4,

$$\sup_{t\in[T/2,3T]}\{S(t+h)-S(t)\} \geqslant \sup_{t\in[T,2T]}\{W(t)+R(t)\}-O(t)$$
$$\geqslant c\sqrt{k} \geqslant c(h\log T)^{1/3}.$$

So, it remains to prove (5.4).

Let

$$f(z) = z^{-2}\sin^2 z.$$

For any real numbers u, v, δ, $\delta>0$, we have

$$\operatorname{Im}\{f(u+\delta+iv)-f(u-\delta+iv)\} = \int_{u-\delta}^{u+\delta}\operatorname{Im} f'(x+iv)\,dx.$$

Since $f'(x)$ is real,

$$|\operatorname{Im} f'(x+iv)| \leqslant |f'(x+iv)-f'(x)| \leqslant \int_0^{|v|}|f''(x+iy)|\,dy.$$

It is easy to show that

$$f''(x+iy) = f''(z) = z^{-4}\{(2z^2-3)\cos 2z - 4z\sin 2z + 3\} \ll e^{2|y|}/(1+x^2+y^2).$$

Hence,

$$\operatorname{Im}\{f(u+\delta+iv)-f(u-\delta+iv)\} \ll \int_{u-\delta}^{u+\delta}\int_0^{|v|} e^{2y}(1+x^2+y^2)^{-1}\,dy\,dx$$
$$\ll |v|\,e^{2|v|}\int_{u-\delta}^{u+\delta}(1+x^2+v^2)^{-1}\,dx$$
$$\ll \delta|v|\,e^{2|v|}(1+u^2+v^2)^{-1},$$

if $\delta \leqslant 1/2$. Thus, in view of (5.1) and the fact $V(z) = \tau^2 f(\tau z/2)$, we deduced from (5.2) that

$$R(t) \ll \tau\sum_{\beta>1/2}\int_0^{\beta-1/2}\alpha h\tau^2 e^{\tau\alpha}\{1+(\tau(\gamma-t)/2)^2+(\tau\alpha/2)^2\}^{-1}\,d\alpha$$
$$\ll \sum_{\beta>1/2}(\beta-1/2)^2 e^{\tau(\beta-1/2)}\{(2/\tau)^2+(\gamma-t)^2+(\beta-1/2)^2\}^{-1}$$
$$\ll \sum_{\substack{\beta>1/2\\ |\gamma-t|\leqslant T/2}} + \sum_{\substack{\beta>1/2\\ |\gamma-t|>T/2}}(\beta-1/2)^2 e^{\tau(\beta-1/2)}\{h^2+(\gamma-t)^2\}^{-1}.$$

The second sum, estimated crudely, is

$$\ll \sum_{|\gamma-t|>T/2} e^{\tau/2}(\gamma-t)^{-2} \ll T^{1/2}\sum_{|\gamma-t|>T/2}(\gamma-t)^{-2}.$$

By a familiar argument,

$$\sum_{|\gamma-t|>T/2}(\gamma-t)^{-2} \ll T^{-1}\log T.$$

Thus,

$$R(t) \ll 1 + \sum_{\substack{\beta>1/2\\ |\gamma-t|\leqslant T/2}}(\beta-1/2)^2 e^{\tau(\beta-1/2)}\{h^2+(\gamma-t)^2\}^{-1}. \tag{5.6}$$

Let

$$R_1(t) = \sum_{\substack{\beta>1/2\\ |\gamma-t|\leqslant T/2}}\{h^2+(\gamma-t)^2\}^{-1}.$$

Lemma 8. *For any positive integer ν, $\nu \leqslant 10h\log T$, we have*

$$\int_T^{2T}|R_1(t)|^{\nu}\,dt \ll T(ch^{-1}\log T)^{\nu}. \tag{5.7}$$

Proof. By Cauchy–Schwarz inequality,

$$\int_T^{2T}|R_1(t)|^{\nu}\,dt \leqslant T^{1/2}\Big\{\int_T^{2T}|R_1(t)|^{2\nu}\,dt\Big\}^{1/2}.$$

So, we may assume ν is even. From its definition,

$$R_1(t) \leqslant \sum_{n=0}^{[1/\sqrt{h}]} \sum_{nh \leqslant |\gamma-t| < (n+1)h} \{h^2+(\gamma-t)^2\}^{-1} + \\ + \sum_{\sqrt{h} < |\gamma-t| \leqslant 1} h^{-1} + \sum_{1 < |\gamma-t| \leqslant T/2} (\gamma-t)^{-2}.$$

Since both

$$\sum_{|\gamma-t| \leqslant 1} 1 \quad \text{and} \quad \sum_{1 < |\gamma-t|} (\gamma-t)^{-2} \quad \ll \log t \ll \log T,$$

$$(5.8) \quad R_1(t) \ll \sum_{n=0}^{[1/\sqrt{h}]} (n^2+1)^{-1} h^{-2} \{N(t+(n+1)h) - N(t+nh) + N(t-nh) \\ - N(t-(n+1)h)\} + h^{-1} \log T.$$

For any $u \in [T, 2T]$, (1.1) implies

$$(5.9) \quad 0 \leqslant N(u+h) - N(u) \ll h \log T + S(u+h) - S(u).$$

If we want to prove

$$\sup_{t \in [T,2T]} \{S(t+h) - S(t)\} \geqslant c(h \log T)^{1/3},$$

we certainly may assume

$$S(u+h) - S(u) \leqslant c(h \log T) \quad \text{for all } u \in [T, 2T].$$

It then follows from (5.8) and (5.9) that

$$R_1(t) \ll \sum_{n=0}^{[1/\sqrt{h}]} (n^2+1)^{-1} h^{-2} (h \log T) + h^{-1} \log T \ll h^{-1} \log T.$$

This leads to (5.7).

However, for the proof that

$$\inf_{u \in [T,2T]} \{S(u+h) - S(u)\} \leqslant -c(h \log T)^{1/3},$$

the above argument does not work, in particular, we cannot assume that

$$\sup_{t \in [T,2T]} \{S(t+h) - S(t)\} \leqslant c(h \log T).$$

As an alternative, we use Theorem 4 which says that, on the average, $S(u+h) - S(u)$ is small.

From (5.8) and (5.9), we have

$$R_1(t) \ll h^{-1} \log T + h^{-2} \sum_{n \leqslant 1/\sqrt{h}} n^{-2} \{|S(t+nh) - S(t+(n-1)h)| \\ + |S(t-(n-1)h) - S(t-nh)|\}.$$

Thus, by Cauchy–Schwarz inequality and Theorem 4, notice that ν is even,

$$\int_T^{2T} |R_1(t)|^\nu dt \ll T(ch^{-1} \log T)^\nu + (ch^{-2})^\nu \Big(\sum_{n=1}^{\infty} n^{-2}\Big)^{\nu-1} \\ \times \Big\{ \sum_{n \leqslant 1/\sqrt{h}} n^{-2} \Big(\int_T^{2T} |S(t+nh) - S(t+(n-1)h)|^\nu dt \\ + \int_T^{2T} |S(t-(n-1)h) - S(t-nh)|^\nu dt \Big) \Big\} \\ \ll T(ch^{-1} \log T)^\nu + T(ch^{-2})^\nu \{\nu^\nu + \nu^{\nu/2} (\log(h \log T))^{\nu/2}\} \\ \ll T(ch^{-1} \log T)^\nu.$$

This proves the lemma.

If $R_2(T)$ is the sum on the right of (5.6), we have, by Cauchy–Schwarz inequality and the above lemma,

$$\int_T^{2T} |R_2(t)|^{2k+1} dt \\ \leqslant \int_T^{2T} |R_1(t)|^{2k+1/2} \Big\{ \sum_{\substack{\beta > 1/2 \\ |\gamma-t| \leqslant T/2}} (\beta-\tfrac{1}{2})^{4(2k+1)} e^{2(2k+1)\tau(\beta-1/2)} \{h^2+(\gamma-t)^2\}^{-1} \Big\}^{1/2} dt \\ \leqslant \Big\{ \int_T^{2T} |R_1(t)|^{4k+1} dt \Big\}^{1/2} \Big\{ \int_T^{2T} \sum_{\substack{\beta > 1/2 \\ |\gamma-t| \leqslant T/2}} (\beta-\tfrac{1}{2})^{4(2k+1)} e^{2(2k+1)\tau(\beta-1/2)} \\ \times \{h^2+(\gamma-t)^2\}^{-1} dt \Big\}^{1/2} \\ \leqslant \{T(ch^{-1} \log T)^{4k+1}\}^{1/2} \Big\{ \sum_{\substack{\beta > 1/2 \\ T/2 < \gamma \leqslant 3T}} h^{-1} (\beta-\tfrac{1}{2})^{4(2k+1)} e^{2(2k+1)\tau(\beta-1/2)} \Big\}^{1/2},$$

provided

$$k < h \log T.$$

To estimate the sum over the zeros of $\zeta(s)$, we use Lemma 6 with $X = e^{2\tau(2k+1)}$ (which is $\leqslant T^{1/8}$ by (5.3)) and $\sigma = 1/2$. This yields

$$\int_T^{2T} |R_2(t)|^{2k+1} dt \leqslant \{T(ch^{-1} \log T)^{4k+1} T(\log T)^{-8k-3} h^{-1} k^{8k}\}^{1/2} \\ \leqslant T\Big(\frac{ck^2}{h \log T}\Big)^{2k+1}.$$

In view of (5.6), this proves (5.4) and hence completes the proof of Theorem 3.

6. Ω_+ result for $S_1(t)$. We shall prove that

$$(6.1) \quad S_1(t) = \Omega_+\big((\log t)^{1/2} (\log \log t)^{-9/4}\big).$$

Let $t \in [T, 2T]$ and

(6.2) $$\tau = \log\log T.$$

We choose in Lemma 5

$$V(z) = \tau \exp(-\tau^2 z^2).$$

Its Fourier transform is given by

$$\hat{V}(v) = \sqrt{\pi} \exp(-\tau^2 \pi^2 v^2).$$

With this $V(z)$, the real part of equation (2.11) is

(6.3) $$\int_{-\infty}^{\infty} \log|\zeta(\sigma + i(t+u/\tau))|\, e^{-u^2}\, du$$

$$= \sqrt{\pi}\, \mathrm{Re} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log n} \exp(-\tfrac{1}{4}\tau^{-2}\log^2 n)\, n^{-\sigma - it}$$

$$+ 2\pi\, \mathrm{Re} \sum_{\sigma<\beta} \int_0^{\beta-\sigma} \exp\{-\tau^2(\gamma - t - i\alpha)^2\}\, d\alpha + O(T^{-1}).$$

It is not difficult to show that (see, for instance, (1.9) of [6])

(6.4) $$S_1(t) = \pi^{-1} \int_{1/2}^{2} \log|\zeta(\sigma+it)|\, d\sigma + O(1).$$

Thus, after integrating (6.3) with respect to σ over the interval $[\frac{1}{2}, 2]$, we have

(6.5) $$\int_{-\infty}^{\infty} S_1\left(t + \frac{u}{\tau}\right) e^{-u^2}\, du$$

$$= \frac{1}{\sqrt{\pi}} \mathrm{Re} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{\log^2 n} \exp(-\tfrac{1}{4}\tau^{-2}\log^2 n)\, n^{-1/2-it}$$

$$+ 2\tau\, \mathrm{Re} \sum_{\beta>1/2} \int_0^{\beta-1/2} (\beta - 1/2 - \alpha) \exp\{-\tau^2(\gamma - t - i\alpha)^2\}\, d\alpha + O(1)$$

$$= \frac{1}{\sqrt{\pi}} \mathrm{Re} \sum_{p \leqslant e^{2\tau^2}} (\log p)^{-1} \exp(-\tfrac{1}{4}\tau^{-2}\log^2 p)\, p^{-1/2-it}$$

$$+ 2\tau \sum_{\beta>1/2} \int_0^{\beta-1/2} (\beta - 1/2 - \alpha) \exp\{\tau^2(\alpha^2 - (\gamma - t)^2)\} \cos\{2\alpha\tau^2(\gamma - t)\}\, d\alpha + O(1).$$

Let

(6.6) $$\lambda = \frac{1}{10}(\log\log T)^{-3/2}$$

and divide the second sum on the right side into three parts, namely,

$$R_1(t) = \sum_{\substack{1/2<\beta\leqslant 1/2+\lambda \\ |\gamma-t|\leqslant \pi(8\lambda\tau^2)^{-1}}}, \quad R_2(t) = \sum_{\substack{1/2<\beta\leqslant 1/2+\lambda \\ |\gamma-t|> \pi(8\lambda\tau^2)^{-1}}}, \quad R_3(t) = \sum_{\beta>1/2+\lambda}.$$

In $R_1(t)$, each term is non-negative because $|2\alpha\tau^2(\gamma - t)| \leqslant \pi/4$. Hence

(6.7) $$R_1(t) \geqslant 0.$$

Plainly,

$$R_2(t) \ll \sum_{\substack{1/2<\beta\leqslant 1/2+\lambda \\ |\gamma-t|> \pi(8\lambda\tau^2)^{-1}}} (\beta - \tfrac{1}{2})^2 \exp\{-\tau^2(\gamma-t)^2 + \tau^2(\beta-\tfrac{1}{2})^2\}$$

$$\ll \lambda^2 e^{\tau^2\lambda^2} \sum_{|\gamma-t|>\pi(8\lambda\tau^2)^{-1}} e^{-\tau^2(\gamma-t)^2}.$$

We further divide this sum into

$$\sum_{|\gamma-t|>1} \quad \text{and} \quad \sum_{\pi(8\lambda\tau^2)^{-1}<|\gamma-t|\leqslant 1}.$$

The first one, by a familiar argument using the fact that $N(t+1) - N(t) \leqslant c\log t$, is $\ll e^{-\tau^2}\log T \ll 1$. The second one is $\ll \exp\{-\tau^2\pi^2(8\lambda\tau^2)^{-2}\} \times \log T \ll 1$. Hence

(6.8) $$R_2(t) \ll (\log\log T)^{-3}.$$

Finally,

(6.9) $$|R_3(t)| \leqslant \sum_{\substack{1/2+\lambda<\beta \\ |\gamma-t|\leqslant 1}} (\beta-\tfrac{1}{2})^2 \exp\{\tau^2(\beta-\tfrac{1}{2})^2 - \tau^2(\gamma-t)^2\}$$

$$+ \sum_{\substack{1/2+\lambda<\beta \\ |\gamma-t|>1}} \exp\{-\tfrac{3}{4}\tau^2(\gamma-t)^2\}$$

$$\leqslant \sum_{\substack{1/2+\lambda<\beta \\ |\gamma-t|\leqslant 1}} (\beta-\tfrac{1}{2})^2 \exp\{\tau^2(\beta-\tfrac{1}{2})^2 - \tau^2(\gamma-t)^2\} + e^{-\tau^2/2}.$$

It is well known that (a direct consequence of (2.13) and (6.4)) $S_1(t) = O(\log t)$. Using the now familiar argument, we showed that

$$\sup_{t\in[T/2, 3T]} S_1(t) \geqslant \int_{-\infty}^{\infty} S_1(t + u/\tau)\, e^{-u^2}\, du + O(1).$$

Thus, collecting the estimates (6.7), (6.8) and (6.9) into equation (6.5), we have

$$(6.10)\qquad \sup_{t\in[T/2,3T]} S_1(t) \geqslant W(t)+\tau\{R_1(t)+R_2(t)+R_3(t)\}-O(1)$$

$$\geqslant W(t)+R(t)-O(1) \quad \text{for all } t\in[T, 2T],$$

where

$$W(t) = \operatorname{Re}\sum_{p\leqslant e^{2\tau^2}} \{\sqrt{\pi p}(\log p)\exp(-\tfrac{1}{4}\tau^{-2}\log^2 p)\}^{-1} p^{-it}$$

and

$$|R(t)| \leqslant \tau \sum_{\substack{1/2+\lambda<\beta\\ |\gamma-t|\leqslant 1}} (\beta-\tfrac{1}{2})^2 \exp\{\tau^2(\beta-\tfrac{1}{2})^2-\tau^2(\gamma-t)^2\}.$$

Let

$$(6.11)\qquad k = \left[\frac{\log T}{240}(\log\log T)^{-5/2}\right].$$

Using Lemmas 2 and 3 as in the previous sections, we can show that

$$(6.12)\qquad \int_T^{2T} \{W(t)\}^{2k+1}\,dt \ll T^{1/2}(ck)^k$$

and

$$(6.13)\qquad \int_T^{2T} \{W(t)\}^{2k}\,dt$$

$$\geqslant T(ck)^k\{\sum_{2k\log k<p\leqslant e^{3\tau^2/2}} (p\log^2 p)^{-1}\exp(-\tfrac{1}{2}\tau^{-2}\log^2 p)\}^k - O(T^{1/2}(ck)^k)$$

$$\geqslant T\{ck(\log\log T)^{-2}\}^k.$$

Furthermore, by Cauchy–Schwarz inequality and Lemma 6,

$$\int_T^{2T} |R(t)|^{2k+1}\,dt$$

$$\leqslant \tau^{2k+1}\int_T^{2T}\{\sum_{\substack{1/2+\lambda<\beta\\ |\gamma-t|\leqslant 1}} (\beta-\tfrac{1}{2})^{4k+2}\exp\{(2k+1)\tau^2(\beta-\tfrac{1}{2})^2-\tau^2(\gamma-t)^2\}\}$$

$$\times\{\sum_{|\gamma-t|\leqslant 1} e^{-\tau^2(\gamma-t)^2}\}^{2k}\,dt$$

$$\leqslant (c\tau\log T)^{2k}\sum_{\substack{1/2+\lambda<\beta\\ T/2<\gamma\leqslant 3T}} (\beta-\tfrac{1}{2})^{4k+2}\exp\{(2k+1)\tau^2(\beta-\tfrac{1}{2})^2\}$$

$$\leqslant (c\tau\log T)^{2k}T^{1-\lambda/4}(\log T)^{-4k-1}k^{4k}.$$

In view of (6.2), (6.6) and (6.11), we have

$$\int_T^{2T} |R(t)|^{2k+1}\,dt \leqslant Tc^k.$$

With this and the estimates (6.12) and (6.13), Lemma 4 implies that

$$\sup_{t\in[T,2T]}\{W(t)+R(t)\} \geqslant c\sqrt{k}/\log\log T \geqslant c(\log T)^{1/2}(\log\log T)^{-9/4}.$$

This together with (6.10) proves (6.1).

Acknowledgement. The content of this paper is a portion of the author's Ph. D. thesis, written under the direction of Professor Atle Selberg at Princeton University. The author acknowledges with gratitude many valuable discussions with Professor Selberg.

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Received on 8.3.1985