ACTA ARITHMETICA
XLVI (1986)

On powers of the theta-function greater than the eighth

by

MARVIN I. KNOPP (Philadelphia, Penn.)

1. Introduction. The powers of the classical theta-function

$$\Theta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} = 1 + 2\sum_{n=1}^{\infty} e^{\pi i n^2 \tau}, \qquad \operatorname{Im}\tau > 0, \tag{1}$$

have been the object of a good deal of attention since early in the century, both because of their connection with the problem of the representation of integers by sums of squares (integral powers) and because of their intrinsic interest as modular forms (arbitrary real powers). A powerful approach to the study of $\Theta^s(\tau)$ when $s > 4$ (and even for an interval of values $s \leqslant 4$, but this is another, more complex, matter) is to compare it with the related Eisenstein series

$$\Psi_s(\tau) = \frac{1}{2}\sum \bar{v}_s(M)(c\tau+d)^{-s/2}, \tag{2}$$

defined for $\tau \in \mathscr{H}$, the upper half-plane. (The notation in (2) will be explained in § 2, below.) Once one has shown that Ψ_s is a modular form with formal properties precisely the same as those of Θ^s, standard results from the theory of modular forms yield a formula for $r_s(n)$, the coefficients in the expansion of Θ^s:

$$\Theta^s(\tau) = 1 + \sum_{n=1}^{\infty} r_s(n)\, e^{\pi i n \tau}, \qquad \tau \in \mathscr{H}. \tag{3}$$

The formula is exact when $4 < s \leqslant 8$ and asymptotic for $s > 8$. The major technical difficulty in this approach is to show that Ψ_s is indeed a modular form for arbitrary $s > 4$. (The situation is much simpler for $s \in Z$, $s \geqslant 5$.) This has been carried out in detail in [1] and [7], pp. 238–243. Petersson [6] has computed the Fourier expansions at the parabolic cusps of the functions Ψ_s, $s > 4$, and indeed of all Eisenstein series of real weight greater than 2 on general finitely generated Fuchsian groups of the first kind.

As suggested above, when $4 < s \leqslant 8$ $\Theta^s = \Psi_s$, but this is no longer the case for any $s > 8$. Thus for $s > 8$ it is of interest to determine the precise relationship between the two functions. Toward this end it is necessary to introduce a modular function F invariant with respect to Γ_Θ, the modular subgroup on which both Θ^s and Ψ_s are modular forms. Γ_Θ is of index 3 in the full modular group $\Gamma(1)$ and it has a fundamental region with two cusps: $i\infty$ and -1. Putting

$$g(\tau) = \frac{\Theta^4(\tau+1)}{\Theta^4(\tau)}, \quad f(\tau) = 1 - g(\tau), \tag{4}$$

we define $F(\tau) = f(\tau)g(\tau)$ and observe (§ 2) that F is not only invariant with respect to Γ_Θ ($F(M\tau) = F(\tau)$ for $M \in \Gamma_\Theta$), but is in fact a *Hauptmodul* for Γ_Θ; that is to say, F has precisely one zero of order 1 and one pole of order 1 (measured in the appropriate local variable) in a fundamental region for Γ_Θ, so that every modular function with respect to Γ_Θ is a rational function in F. In fact it can be shown from properties of Θ that F has a zero of order 1 at the cusp $i\infty$, a pole of order 1 at the cusp -1 and no other zeros or poles in a fundamental region.

This circumstance makes the function F especially well suited for comparison with

$$\Phi_s(\tau) = \Psi_s(\tau)/\Theta^s(\tau), \quad s > 4,$$

a modular function on Γ_Θ. The expansions of Θ^s and Ψ_s at the two cusps $i\infty$ and -1 show that the only possible pole of Φ_s in a fundamental region is at the cusp -1. From these properties of Φ_s and those of F mentioned above it follows that Φ_s is a polynomial in F of degree equal to the order of the pole of Φ_s at -1.

In a letter I received some time ago, Pierre Barrucand raised the question of determining precisely the coefficients of this polynomial, in particular when it is of degree 1 ($8 < s \leqslant 16$). Calculation of these coefficients in effect determines the precise relationship between Θ^s and Ψ_s; it is the purpose of this article to carry out this determination when $8 < s \leqslant 16$. We prove two theorems.

THEOREM 1. *For* $8 < s \leqslant 16$ *and* $\tau \in \mathscr{H}$,

$$\Phi_s(\tau) = 1 + c(s)F(\tau),$$

where $c(s)$ *is a constant depending only upon* s. *That is,*

$$\Psi_s(\tau) = \Theta^s(\tau)[1 + c(s)F(\tau)].$$

The statement of Theorem 2, which gives the explicit expression for $c(s)$, requires further notation. The statement is to be found in Section 5.

In principle, the method of proof of these theorems applies to all $s > 8$, but the details are more complex when $s > 16$. I am indebted to M. Barrucand for his stimulating and extensive correspondence.

2. The modular forms. That $\Theta(\tau)$ has no zeros in $\mathscr{H}$ and is a modular form of weight 1/2 on Γ_Θ, with multiplier system v_Θ related to the Jacobi symbol ($v_\Theta^8 \equiv 1$), are familiar facts, consequences of the connection of $\Theta(\tau)$ with the Dedekind function $\eta(\tau)$ ([2], Chapter 3):

$$\Theta(\tau) = \eta^2\left(\frac{\tau+1}{2}\right)\Big/\eta(\tau+1), \quad \tau \in \mathscr{H}. \tag{5}$$

$\eta(\tau)$ is a modular form of weight 1/2 on the full modular group $\Gamma(1)$. More explicitly, $\eta(\tau)$ satisfies ([2], Chapter 3)

$$\eta(\tau+1) = e^{\pi i/12}\eta(\tau), \quad \eta(-1/\tau) = e^{-\pi i/4}\tau^{1/2}\eta(\tau), \tag{6}$$

for $\tau \in \mathscr{H}$. (Here and elsewhere in this article we adopt the convention that

$$-\pi \leqslant \arg z < \pi, \quad z^k = |z|^k \exp(ik \arg z) \tag{7}$$

for complex z and arbitrary real k.) The following functional equations for $\Theta(\tau)$ result immediately from (5) and (6):

$$\Theta(M\tau) = v_\Theta(M)(c\tau+d)^{1/2}\Theta(\tau), \quad \tau \in \mathscr{H}, \tag{8}$$

for all $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\Theta$. As a consequence of (8) v_Θ satisfies a "consistency condition" in weight 1/2; we shall comment further about this in Section 4.

In the definition (2) of Ψ_s we have written v_s for v_Θ^s, the multiplier system of weight $s/2$ connected with $\Theta^s(\tau)$. The summation in (2) is to be taken over all $M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\Theta$ with distinct lower rows; that is, over all pairs of rational integers c, d with $(c, d) = 1$ and $c+d$ odd. For $s > 4$ the series converges absolutely and this guarantees that analytic difficulties do not arise in showing that, like Θ^s, Ψ_s is an entire modular form of weight $s/2$ and multiplier system v_s on Γ_Θ. Nevertheless, when s is not an integer the proof contains formal difficulties; these have been worked out in detail in [1] and [7]. These same problems are evident here in our derivation (§ 5) of the expansion, in explicit form, of Ψ_s at the cusp -1.

It is not too hard to prove from (5) and (6) that the functions $f(\tau)$, $g(\tau)$ defined by (4) satisfy the relations

$$f(\tau+2) = f(\tau), \quad g(\tau+2) = g(\tau)$$
$$f(-1/\tau) = g(\tau), \quad g(-1/\tau) = f(\tau).$$

Since Γ_Θ is generated by the two transformations S^2: $\tau \to \tau+2$ and T: $\tau \to -1/\tau$, it follows that the function $F(\tau) = f(\tau)g(\tau)$ is invariant with respect to Γ_Θ. Furthermore, it can be shown from the definitions (4) and the expansion (1) of $\Theta(\tau)$ at the cusp $i\infty$ that

$$\tfrac{1}{16}e^{-\pi i\tau}f(\tau) \to 1, \quad g(\tau) \to 1 \tag{9}$$

as $y = \operatorname{Im}\tau \to +\infty$. Since $F(\tau+2) = F(\tau)$, this implies that the expansion of $F(\tau)$ at $i\infty$ has the form

(10) $$F(\tau) = 16e^{\pi i\tau} + \text{higher powers of } e^{\pi i\tau}.$$

To determine the behavior of $F(\tau)$ at the cusp -1 consider the function $F(-1-1/\tau)$ at $i\infty$. From (4),

$$g(\tau+1) = \frac{\Theta^4(\tau)}{\Theta^4(\tau+1)} = \frac{1}{g(\tau)} = 1 + \frac{f(\tau)}{g(\tau)}$$

and

$$f(\tau+1) = 1 - g(\tau+1) = -f(\tau)/g(\tau).$$

Hence,

$$F\left(-1-\frac{1}{\tau}\right) = F\left(1-\frac{1}{\tau}\right) = f\left(1-\frac{1}{\tau}\right) g\left(1-\frac{1}{\tau}\right)$$
$$= -\frac{f(-1/\tau)}{g(-1/\tau)}\left\{1+\frac{f(-1/\tau)}{g(-1/\tau)}\right\} = -\frac{g(\tau)}{f(\tau)}\left\{1+\frac{g(\tau)}{f(\tau)}\right\} \equiv G(\tau).$$

Now

$$G = -\frac{g}{f}\left(1+\frac{g}{f}\right) = -\frac{g(f+g)}{f^2} = -\frac{g}{f^2},$$

so that

$$G(\tau+1) = -\frac{g(\tau+1)}{f(\tau+1)^2} = -\frac{1}{g(\tau)}\frac{g(\tau)^2}{f(\tau)^2} = -\frac{g(\tau)}{f(\tau)^2} = G(\tau).$$

On the other hand, by (9) and the periodicity of f and g, G has an expansion of the form

$$G(\tau) = -\frac{1 + \text{positive powers of } e^{\pi i\tau}}{16^2(e^{\pi i\tau} + \text{higher powers of } e^{\pi i\tau})^2}$$
$$= -\frac{1}{16^2}e^{-2\pi i\tau} + \alpha e^{-\pi i\tau} + \text{nonnegative powers of } e^{\pi i\tau}.$$

This, in turn, assumes the form

$$G(\tau) = -\frac{1}{16^2}e^{-2\pi i\tau} + \text{nonnegative powers of } e^{2\pi i\tau},$$

since $G(\tau+1) = G(\tau)$ implies that only even powers of $e^{\pi i\tau}$ appear. Thus,

(11) $$F(\tau) = G\left(-\frac{1}{\tau+1}\right)$$
$$= -\frac{1}{16^2}e^{-2\pi i(-1/(\tau+1))} + \text{nonnegative powers of } e^{2\pi i(-1/(\tau+1))},$$

the expansion of F at -1.

3. Proof of Theorem 1. The proof entails comparison of Φ_s and F at the two cusps $i\infty$ and -1 in the fundamental region. The behavior of F at these cusps has been determined in Section 2. We turn to the behavior of Φ_s at $i\infty$. It follows readily from the definitions (1) and (2) that $\Theta(\tau) \to 1$ and $\Psi_s(\tau) \to 1$ as $\tau \to i\infty$. Since we also know that $\Phi_s(\tau+2) = \Phi_s(\tau)$, Φ_s has the expansion at $i\infty$:

$$\Phi_s(\tau) = 1 + \text{positive powers of } e^{2\pi i\tau},$$

valid in $\mathscr{H}$. By (10), then, $\Phi_s(\tau) - [1 + c(s)F(\tau)] \to 0$ as $\tau \to i\infty$, regardless of the choice of $c(s)$.

To determine the behavior of Φ_s at -1 we consider separately Ψ_s and Θ^s at -1. By [2], pp. 46–48, Θ has the expansion at -1:

$$\Theta(\tau) = (\tau+1)^{-1/2}\sum_{n=0}^{\infty} b_n \exp\{2\pi i(n+1/8)(-1/(\tau+1))\}, \qquad b_0 = 2e^{i\pi/4}.$$

From this it follows that for real s

(12) $$\Theta^s(\tau) = (\tau+1)^{-s/2} t^{s/8} \sum_{n=0}^{\infty} c_n t^n, \qquad t = \exp\{-2\pi i/(\tau+1)\},$$

with

(13) $$c_0 = 2^s e^{\pi i s/4} \neq 0.$$

Thus Θ^s has a zero of exact order $s/8$ in the local variable t at the cusp -1. On the other hand, standard results on Eisenstein series ([3], pp. 278–281) show that $\Psi_s(\tau) \to 0$ as τ approaches vertically any parabolic cusp inequivalent to $i\infty$ under Γ_Θ. In specific terms, at -1 Ψ_s has the expansion

(14) $$\Psi_s(\tau) = (\tau+1)^{-s/2} \sum_{n+\varkappa>0} d_n t^{n+\varkappa},$$

where $\varkappa \in [0, 1)$ is determined from the multiplier system v_Θ. In this instance, $\varkappa = s/8 - [s/8]$. (For real x, $[x]$ denotes the largest integer $\leqslant x$.) Note that $\varkappa = s/8 - 1 > 0$ for $8 < s < 16$ and $\varkappa = 0$ for $s = 16$.

It now follows from (12), (13) and (14) that at -1 Φ_s has the expansion

$$\Phi_s(\tau) = \begin{cases} t^{-1}\sum_{n=0}^{\infty} d_n' t^n, & 8 < s < 16, \\ t^{-2}\sum_{n=1}^{\infty} d_n' t^n, & s = 16. \end{cases}$$

For $8 < s \leqslant 16$, then, at -1

$$\Phi_s(\tau) = \beta t^{-1} + \text{nonnegative powers of } t,$$

where the constant $\beta = \beta(s)$ is given by

(15) $$\beta(s) = \begin{cases} d_0' = d_0/c_0, & 8 < s < 16, \\ d_1' = d_1/c_0, & s = 16. \end{cases}$$

Choose $c(s) = -16^2\beta(s)$; then $H(\tau) \equiv \Phi_3(\tau) - [1+c(s)F(\tau)]$ is bounded as $\tau \to -1$, vertically, so that $H(\tau)$, a bounded modular function, is constant. Since $H(\tau) \to 0$ as $\tau \to i\infty$, $H(\tau) \equiv 0$ and Theorem 1 follows.

4. Multiplier systems. Theorem 2, to be stated in Section 5 below, presents an explicit formula for $c(s)$. The determination of $c(s)$, while straightforward in principle, is considerably more involved than the simple proof of Theorem 1. Since $c(s) = -16^2\beta(s)$ we must calculate the constant $\beta(s)$ occurring in (15) and this, in turn, amounts to calculation of d_0 for $8 < s < 16$ and d_1 for $s = 16$. We require a result of H. Petersson on multiplier systems connected with arbitrary real weight.

Let Γ be a discrete group of 2×2 matrices with real entries and determinant 1 (i.e. Γ acts on $\mathscr{H}$) and k a real number. We say a complex-valued function v on Γ is a *multiplier system of weight k on Γ* provided $|v(M)| = 1$ for M in Γ and

(16) $$v(M_1 M_2)(c_3\tau+d_3)^k = v(M_1)v(M_2)(c_1 M_2\tau+d_1)^k(c_2\tau+d_2)^k,$$

for all $M_1, M_2 \in \Gamma$, where $M_3 = M_1 M_2$ and $M_i = \begin{pmatrix} * & * \\ c_i & d_i \end{pmatrix}$, $1 \leqslant i \leqslant 3$. The identity (16) is called the "consistency condition (in weight k)" for v.

With $X = \begin{pmatrix} * & * \\ x_1 & x_2 \end{pmatrix}$, $Y = \begin{pmatrix} * & * \\ y_1 & y_2 \end{pmatrix}$, $XY = \begin{pmatrix} * & * \\ z_1 & z_2 \end{pmatrix}$ and k a real number, define

(17) $$\sigma_k(X, Y) = \sigma(X, Y) = \frac{(x_1 Y\tau+x_2)^k(y_1\tau+y_2)^k}{(z_1\tau+z_2)^k}.$$

Then (16) can be rewritten as

$$v(M_1 M_2) = \sigma(M_1, M_2)v(M_1)v(M_2).$$

It is worth noting that when k is an integer $\sigma(X, Y) \equiv 1$, so that the consistency condition reduces to $v(M_1 M_2) = v(M_1)v(M_2)$. For all real k $|\sigma(X, Y)| \equiv 1$. The result of Petersson is

LEMMA 3 ([5], p. 379). *For A a real 2×2 matrix and v a multiplier system of weight k on the discrete group Γ, let v^* be the function defined on $A^{-1}\Gamma A$ by*

(18) $$v^*(M) = v(AMA^{-1})\frac{\sigma(AMA^{-1}, A)}{\sigma(A, M)}, \quad M \in A^{-1}\Gamma A.$$

Then v^* is a multiplier system of weight k on the group $A^{-1}\Gamma A$.

In the proof of Lemma 3 we shall make use of the following result, which is also required in the derivation of Theorem 2 ([5], p. 378).

LEMMA 4. *If X, Y, Z are real 2×2 matrices of determinant > 0, then*

(19) $$\sigma(X, YZ) = \frac{\sigma(X, Y)\sigma(XY, Z)}{\sigma(Y, Z)}.$$

Remarks. It is clear from the definition (17) that $|\sigma(X, Y)| = 1$. Thus $|v^*(M)| = 1$ follows from $|v(AMA^{-1})| = 1$ and Lemma 3 will follow if we can show that v^* satisfies (16) with Γ replaced by $A^{-1}\Gamma A$. When k is an integer (18) reduces to $v^*(M) = v(AMA^{-1})$ and there is nothing to prove. For arbitrary real k, however, this result is not widely known and we therefore include a proof. In order to avoid diversion from our main purpose – the proof of Theorem 2 – we postpone the proofs of Lemmas 3 and 4 until Section 6.

5. We can now state and prove

THEOREM 2. *Let $A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. Then the constant $c(s)$ is given by the formula*

(20) $$c(s) = \frac{-16^2(2\pi)^{s/2}}{2^s\Gamma(s/2)}\left(\frac{s}{8}-1\right)^{s/2-1} \times \sum_{\substack{d=1\\ d\ \mathrm{odd}}}^{\infty} d^{-s/2} \sum_{0\leqslant\delta<d}{}' \frac{\bar{v}_s^*(A^{-1}M_{d,-\delta})}{\sigma(A, A^{-1}M_{d,-\delta})} e^{-2\pi i(s/8-1)\delta/d},$$

for $8 < s \leqslant 16$.

In (20) $\sum'$ indicates that the summation is taken over integers δ such that $(\delta, d) = 1$. Also $M_{d,-\delta} = \begin{pmatrix} * & * \\ d & -\delta \end{pmatrix} = MA$, where $M \in \Gamma_\Theta$; thus $\bar{v}_s^*(A^{-1}M_{d,-\delta})$ is defined. When s is a multiple of 8, then $\bar{v}_s \equiv 1$ and $\sigma \equiv 1$, so that (20) simplifies considerably when $s = 16$.

Proof of Theorem 2. The proof requires Lemmas 3 and 4 and the *Lipschitz summation formula* [4]:

(21) $$\sum_{m=-\infty}^{\infty} \frac{e^{2im\varkappa}}{(\tau-m)^\lambda} = \frac{e^{-\pi i\lambda/2}(2\pi)^\lambda}{\Gamma(\lambda)} \sum_{n+\varkappa>0} (n+\varkappa)^{\lambda-1} e^{2\pi i(n+\varkappa)\tau},$$

for $\tau \in \mathscr{H}$, $\lambda > 1$ and $0 \leqslant \varkappa < 1$. We note that the calculations we carry out here are valid as long as $s > 4$.

From the definition (2) of Ψ_s and the consistency conditions (16) applied to $v = v_s$ $(k = s/2)$, we find that

$$\Psi_s(\tau) = 1 + \sum_{c=1}^{\infty} \sum_{d=-\infty}^{\infty}{}^* \bar{v}_s(M_{c,d})(c\tau+d)^{-s/2},$$

where $\sum\sum^*$ indicates that the summation is to be taken over $c, d \in Z$ such that $(c, d) = 1$ and $c+d$ is odd. Such pairs of integers form precisely the set of distinct lower rows of elements of Γ_Θ. Thus, $M_{c,d} = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_\Theta$, with an appropriate choice of upper row. (That the choice does not affect the value of $v_s(M_{c,d})$ follows from the consistency conditions for v_s and the fact that

$v_s(S^2)=1$.) Let $A=\begin{pmatrix}-1&-1\\1&0\end{pmatrix}$, so that $A\in\Gamma(1)$ and $A(\infty)=-1$. We have

$$\Psi_s(A\tau)=\tau^{s/2}\left\{\tau^{-s/2}+\sum_{c=1}^{\infty}\sum_{d=-\infty}^{\infty}{}^{*}\frac{v_s(M_{c,d})}{(cA\tau+d)^{s/2}\tau^{s/2}}\right\}.$$

Since

$$\sigma(M_{c,d},A)=(cA\tau+d)^{s/2}(\tau)^{s/2}/[(d-c)\tau-c]^{s/2},$$

we may rewrite the above expression as

$$\Psi_s(A\tau)=\tau^{s/2}\left\{\tau^{-s/2}+\sum_{c=1}^{\infty}\sum_{d=-\infty}^{\infty}{}^{*}\frac{\bar v_s(M_{c,d})}{\sigma(M_{c,d},A)\{(d-c)\tau-c\}^{s/2}}\right\}.$$

From the definition of v_s^*,

$$v_s(M_{c,d})\sigma(M_{c,d},A)=v^*(A^{-1}M_{c,d}A)\sigma(A,A^{-1}M_{c,d}A),$$

so that, replacing $d-c$ by d, we have

$$\Psi_s(A\tau)=\tau^{s/2}\left\{\tau^{-s/2}+\sum_{c=1}^{\infty}\sum_{d=-\infty}^{\infty}{}''\frac{\bar v_s^*(A^{-1}M_{c,d+c}A)}{\sigma(A,A^{-1}M_{c,d+c}A)(d\tau-c)^{s/2}}\right\},$$

where $\sum''$ indicates summation over $d\in Z$ such that $(c,d)=1$ and d is odd. A calculation shows that $M_{c,d+c}A=M_{d,-c}$, so that

$$\Psi_s(A\tau)=\tau^{s/2}\left\{\tau^{-s/2}+\sum_{c=1}^{\infty}\sum_{d=-\infty}^{\infty}{}''\frac{\bar v_s^*(A^{-1}M_{d,-c})}{\sigma(A,A^{-1}M_{d,-c})(d\tau-c)^{s/2}}\right\}. \tag{22}$$

We note that $A^{-1}M_{d,-c}=A^{-1}M_{c,d+c}A\in A^{-1}\Gamma_\Theta A=\Gamma_0(2)$.

We want to observe that the summand in (22) is unchanged if we replace (c,d) by $(-c,-d)$. Suppose $A^{-1}M_{d,-c}\tau$ has denominator $d'\tau-c'$ and write $A^{-1}M_{d,-c}=M_{d',-c'}$. Then, by the definition of σ we have

$$\sigma(A,A^{-1}M_{d,-c})(d\tau-c)^{s/2}=(\gamma A^{-1}M_{d,-c}\tau+\delta)^{s/2}(d'\tau-c')^{s/2},$$

so that the summand in (22) is equal to

$$\bar v_s^*(M_{d',-c'})(A^{-1}M_{d,-c}\tau)^{-s/2}(d'\tau-c')^{-s/2}. \tag{23}$$

(Here we have used $A=\begin{pmatrix}*&*\\1&0\end{pmatrix}$.) But by the consistency conditions (16) for v^*, the expression (23) remains invariant if we replace (d,c) by $(-d,-c)$, since this replaces $(d',-c')$ by $(-d',c')$. Thus (22) becomes

$$\Psi_s(A\tau)=\tau^{s/2}\left\{\tau^{-s/2}+\sum_{\substack{c=-\infty\\c\neq0}}^{\infty}\sum_{d>0}{}''\frac{\bar v_s^*(A^{-1}M_{d,-c})}{\sigma(A,A^{-1}M_{d,-c})(d\tau-c)^{s/2}}\right\}. \tag{24}$$

But we claim that

$$v_s^*(A^{-1}M_{1,0})\sigma(A,A^{-1}M_{1,0})\tau^{s/2}=\tau^{s/2},$$

from which follows

$$\Psi_s(A\tau)=\tau^{s/2}\sum_{\substack{d>0\\d\,\mathrm{odd}}}\sum_{\substack{c=-\infty\\(c,d)=1}}^{\infty}\frac{\bar v_s^*(A^{-1}M_{d,-c})}{\sigma(A,A^{-1}M_{d,-c})(d\tau-c)^{s/2}}, \tag{25}$$

where the interchange of the double sum is justified by absolute convergence. The claim that $v_s^*(A^{-1}M_{1,0})\sigma(A,A^{-1}M_{1,0})=1$ is immediate from the definitions of v_s^* and σ.

Let $U\tau=\tau+1$. In the inner sum of (25) we put $c=\delta+dm$, where $m\in Z$ and δ runs through a reduced residue system modulo d. Then $M_{d,-c}=M_{d,-\delta}U^{-m}$ and $A^{-1}M_{d,-\delta}=A^{-1}M_{d,-c}U^m\in A^{-1}\Gamma_\Theta A=\Gamma_0(2)$, since $U\in\Gamma_0(2)$. We shall use the consistency condition (16) for v_s^* in the form

$$v_s^*(M_1M_2)=\sigma(M_1,M_2)v_s^*(M_1)v_s^*(M_2),$$

for $M_1,M_2\in\Gamma_0(2)$. With $M_1=A^{-1}M_{d,-\delta}$, $M_2=U^{-m}$, this gives

$$\begin{aligned}v_s^*(A^{-1}M_{d,-c})&=v_s^*(A^{-1}M_{d,-\delta}U^{-m})\\&=\sigma(A^{-1}M_{d,-\delta},U^{-m})v_s^*(A^{-1}M_{d,-\delta})v_s^*(U^{-m}).\end{aligned}$$

On the other hand, Lemma 4 with $X=A$, $Y=A^{-1}M_{d,-\delta}$, $Z=U^{-m}$ states that

$$\sigma(A,A^{-1}M_{d,-\delta}U^{-m})=\frac{\sigma(A,A^{-1}M_{d,-\delta})\sigma(M_{d,-\delta},U^{-m})}{\sigma(A^{-1}M_{d,-\delta},U^{-m})}.$$

But by the definition (17) $\sigma(M_{d,-\delta},U^{-m})=1$, so that

$$\begin{aligned}&v_s^*(A^{-1}M_{d,-c})\sigma(A,A^{-1}M_{d,-\delta}U^{-m})\\&\qquad=\sigma(A,A^{-1}M_{d,-\delta})v_s^*(U^{-m})v_s^*(A^{-1}M_{d,-\delta}).\end{aligned}$$

From these considerations the expression (25) for $\Psi_s(A\tau)$ takes the form

$$\Psi_s(A\tau)=\tau^{s/2}\sum_{\substack{d>0\\d\,\mathrm{odd}}}\sum_{0\leqslant\delta<d}{}'\frac{\bar v_s^*(A^{-1}M_{d,-\delta})}{\sigma(A,A^{-1}M_{d,-\delta})}\sum_{m=-\infty}^{\infty}\frac{\bar v_s^*(U^{-m})}{(d\tau-\delta-dm)^{s/2}},$$

where $\sum'$ indicates that $(\delta,d)=1$.

Define $\varkappa$ by $v_s^*(U)=e^{2\pi i\varkappa}$, $0\leqslant\varkappa<1$. (This is the same constant $\varkappa$ that appears in (14).) By the consistency condition (16) for v_s^* we have $v_s^*(U^{-m})=e^{-2\pi im\varkappa}$, for $m\in Z$. Thus (25) becomes

$$\Psi_s(A\tau)=\tau^{s/2}\sum_{\substack{d>0\\d\,\mathrm{odd}}}\sum_{0\leqslant\delta<d}{}'\frac{\bar v_s^*(A^{-1}M_{d,-\delta})}{\sigma(A,A^{-1}M_{d,-\delta})}\sum_{m=-\infty}^{\infty}\frac{e^{2\pi im\varkappa}}{(d\tau-\delta-dm)^{s/2}}. \tag{26}$$

Rewrite the inner sum as

$$d^{-s/2}\sum_{m=-\infty}^{\infty}\frac{e^{2\pi im\varkappa}}{(\tau-\delta/d-m)^{s/2}},$$

apply the Lipschitz summation formula (21) with τ replaced by $\tau-\delta/d$ and (26) takes the form

$$(27)\qquad \Psi_s(A\tau) = \frac{e^{-\pi is/4}(2\pi)^{s/2}}{\Gamma(s/2)}\tau^{s/2}\sum_{\substack{d>0\\ d\ \mathrm{odd}}} d^{-s/2}\sum_{0\leqslant\delta<d}{}' \frac{\bar{v}_s^*(A^{-1}M_{d,-\delta})}{\sigma(A, A^{-1}M_{d,-\delta})}\times$$
$$\times\sum_{n+\varkappa>0}(n+\varkappa)^{s/2-1}e^{2\pi i(n+\varkappa)(\tau-\delta/d)}$$
$$=\tau^{s/2}\sum_{n+\varkappa>0}a_n e^{2\pi i(n+\varkappa)\tau}.$$

Here we have defined

$$(28)\qquad a_n = a_n(s) = \frac{e^{-\pi is/4}(2\pi)^{s/2}}{\Gamma(s/2)}(n+\varkappa)^{s/2-1}\times$$
$$\times\sum_{\substack{d>0\\ d\ \mathrm{odd}}} d^{-s/2}\sum_{0\leqslant\delta<d}{}' \frac{\bar{v}_s^*(A^{-1}M_{d,-\delta})}{\sigma(A, A^{-1}M_{d,-\delta})}e^{-2\pi i(n+\varkappa)\delta/d}.$$

Replace τ by $A^{-1}\tau = -(\tau+1)^{-1}$ in (27); since $\arg\left(-(\tau+1)^{-1}\right) = \pi-\arg(\tau+1)$, we have

$$\left(-(\tau+1)^{-1}\right)^{s/2} = e^{\pi is/2}(\tau+1)^{-s/2},$$

so that (27) becomes

$$(29)\qquad \Psi_s(\tau) = e^{\pi is/2}(\tau+1)^{-s/2}\sum_{n+\varkappa>0}a_n(s)e^{2\pi i(n+\varkappa)\tau}.$$

The expansion (29) with $a_n(s)$ given by (28) is nothing more than an explicit form of the expansion (14) for $\Psi_s(\tau)$. Comparison shows that $d_n = e^{\pi is/2}a_n$. Recall that $\varkappa = s/8-1$ for $8<s<16$ and $\varkappa = 0$ for $s = 16$ and further that

$$c(s) = -16^2\beta(s) = \begin{cases}-16^2 e^{-\pi is/4}2^{-s}d_0, & 8<s<16,\\ -16^2 e^{-\pi is/4}2^{-s}d_1, & s = 16,\end{cases}$$

by (15) and (13). The formula (20) follows since $d_n = e^{\pi is/2}a_n$; the proof of Theorem 2 is complete.

6. Proof of Lemmas 3 and 4. We begin with the proof of Lemma 4. Let X, Y have lower rows $x_1\tau+x_2$, $y_1\tau+y_2$, respectively, as in (17). Denote the lower rows of XY, YZ and XYZ by $x_1'\tau+x_2'$, $y_1'\tau+y_2'$ and $x_1''\tau+x_2''$, respectively. From the definition (17) it follows that

$$(30a)\qquad \sigma(Y, Z) = \frac{(y_1Z\tau+y_2)^k(z_1\tau+z_2)^k}{(y_1'\tau+y_2')^k},$$

$$(30b)\qquad \sigma(X, YZ) = \frac{(x_1YZ\tau+x_2)^k(y_1'\tau+y_2')^k}{(x_1''\tau+x_2'')^k}$$

and

$$(30c)\qquad \sigma(XY, Z) = \frac{(x_1'Z\tau+x_2')^k(z_1\tau+z_2)^k}{(x_1''\tau+x_2'')^k}.$$

Next observe that since $\sigma(X, Y)$ is analytic in $\mathscr{H}$ and $|\sigma(X, Y)|\equiv 1$, $\sigma(X, Y)$ is independent of τ in $\mathscr{H}$. Hence,

$$(30d)\qquad \sigma(X, Y) = \frac{(x_1YZ\tau+x_2)^k(y_1Z\tau+y_2)^k}{(x_1'Z\tau+x_2')^k}.$$

To verify (19) we simply form the products $\sigma(X, YZ)\sigma(Y, Z)$ and $\sigma(X, Y)\sigma(XY, Z)$, using the expressions (30a)–(30d). In both instances we obtain

$$\frac{(x_1YZ\tau+x_2)^k(y_1Z\tau+y_2)^k(z_1\tau+z_2)^k}{(x_1''\tau+x_2'')^k}.$$

The identity (19) follows.

We turn now to the proof of Lemma 3. We must show that for $M_1, M_2\in A^{-1}\Gamma A$, $M_3 = M_1M_2$, the relation (16) holds with v replaced by v^*. This amounts to showing that

$$v^*(M_1M_2) = \sigma(M_1, M_2)v^*(M_1)v^*(M_2),$$

for $M_1, M_2\in A^{-1}\Gamma A$. By definition of v^*,

$$(31)\qquad v^*(M_1M_2) = v(AM_1M_2A^{-1})\frac{\sigma(AM_1M_2A^{-1}, A)}{\sigma(A, M_1M_2)}.$$

But by the consistency condition (16) for v on Γ,

$$v(AM_1M_2A^{-1}) = v(AM_1A^{-1}AM_2A^{-1})$$
$$= \sigma(AM_1A^{-1}, AM_2A^{-1})v(AM_1A^{-1})v(AM_2A^{-1}).$$

Thus by the definition (18) of v^* the right-hand side of (31) becomes

$$v^*(M_1)v^*(M_2)\frac{\sigma(A, M_1)\sigma(A, M_2)}{\sigma(AM_1A^{-1}, A)\sigma(AM_2A^{-1}, A)}\times$$
$$\times\frac{\sigma(AM_1A^{-1}, AM_2A^{-1})\sigma(AM_1M_2A^{-1}, A)}{\sigma(A, M_1M_2)}.$$

Now by Lemma 4,

$$\sigma(A, M_1M_2) = \frac{\sigma(A, M_1)\sigma(AM_1, M_2)}{\sigma(M_1, M_2)},$$

so that (31) becomes

$$v^*(M_1M_2) = \sigma(M_1, M_2)v^*(M_1)v^*(M_2)\times$$
$$\times\left\{\frac{\sigma(A, M_2)\sigma(AM_1A^{-1}, AM_2A^{-1})\sigma(AM_1M_2A^{-1}, A)}{\sigma(AM_1A^{-1}, A)\sigma(AM_2A^{-1}, A)\sigma(AM_1, M_2)}\right\}.$$

The proof will be complete if we show that the expression in braces is $\equiv 1$. By Lemma 4, with $X = AM_1$, $Y = M_2A^{-1}$ and $Z = A$,

$$\sigma(AM_1M_2A^{-1}, A) = \frac{\sigma(M_2A^{-1}, A)\sigma(AM_1, M_2)}{\sigma(AM_1, M_2A^{-1})},$$

and again, with $X = AM_1A^{-1}$, $Y = A$ and $Z = M_2A^{-1}$, it follows that

$$\sigma(AM_1A^{-1}, AM_2A^{-1}) = \frac{\sigma(AM_1A^{-1}, A)\sigma(AM_1, M_2A^{-1})}{\sigma(A, M_2A^{-1})}.$$

Hence $\{\ \} = \dfrac{\sigma(A, M_2)\sigma(M_2A^{-1}, A)}{\sigma(A, M_2A^{-1})\sigma(AM_2A^{-1}, A)}$, after the obvious cancellations are made. But Lemma 4, once again, with $X = Z = A$, $Y = V_2A^{-1}$, implies that

$$\sigma(AM_2A^{-1}, A) = \frac{\sigma(A, M_2)\sigma(M_2A^{-1}, A)}{\sigma(A, M_2A^{-1})}.$$

This completes the proof.

7. Concluding remarks. We can make use of the expression (28) for $a_n(s)$ to show that — in contradistinction to the situation when $s \leqslant 8$ — for $s > 8$ the equation $\Psi_s(\tau) = \Theta^s(\tau)$ never occurs. Since the zero of $\Theta^s(\tau)$ at the cusp -1 has exact order $s/8$, it is sufficient to prove that $a_0(s) \neq 0$ for $8 \nmid s$ and $a_1(s) \neq 0$ for $8 | s$. (Thus the order of the zero of $\Psi_s(\tau)$ at -1 is at most 1, while for $\Theta^s(\tau)$ the order of the zero at -1 is > 1 for $s > 8$.)

In fact, we shall show simply from (28) that $a_n(s) \neq 0$ for all n and $s \geqslant 6$. By the triangle inequality,

$$|a_n(s)| \geqslant \frac{(2\pi)^{s/2}}{\Gamma(s/2)}(n+\varkappa)^{s/2-1}\Big\{1-\Big|\sum_{\substack{d\geqslant 3\\ d\,\text{odd}}} d^{-s/2} \sum_{0\leqslant\delta<d}{}' (\ )\Big|\Big\}. \tag{32}$$

But,

$$\Big|\sum_{\substack{d\geqslant 3\\ d\,\text{odd}}} d^{-s/2} \sum_{0\leqslant\delta<d}{}' (\ )\Big| \leqslant \sum_{\substack{d\geqslant 3\\ d\,\text{odd}}}{}' d^{-s/2}\varphi(d) < \sum_{\substack{d\geqslant 3\\ d\,\text{odd}}} d^{-s/2+1} < \zeta(s/2-1)-1.$$

Hence by (32),

$$|a_n(s)| > \frac{(2\pi)^{s/2}}{\Gamma(s/2)}(n+\varkappa)^{s/2-1}\{2-\zeta(s/2-1)\} > 0,$$

as long as $s \geqslant 6$. This proof is to be found as well in [8].

References

[1] P. T. Bateman and M. Knopp, *On lattice points in spheres of fractional dimensions*, unpublished manuscript.

[2] M. Knopp, *Modular functions in analytic number theory*, Markham Publishing Co., Chicago 1970.

[3] J. Lehner, *Discontinuous groups and automorphic functions*, American Mathematical Society, Providence 1964.

[4] R. Lipschitz, *Untersuchung der Eigenschaften einer Gattug von unendlichen Reihen*, J. Reine Angew. Math. 105 (1889), pp. 127–156.

[5] H. Petersson, *Theorie der automorphen formen beliebiger reeler Dimension und ihre Darstellung durch eine neue Art Poincaréscher Reihen*, Math. Ann. 103 (1930), pp. 369–436.

[6] — *Über die Entwicklungskoeffizienten der ganzen Modulformen und ihre Bedeutung für die Zahlentheorie*, Abh. Math. Sem. Univ. Hamburg 8 (1931), pp. 215–242.

[7] R. Rankin, *Modular forms and functions*, Cambridge University Press, Cambridge 1977.

[8] — *Sums of squares and cusp forms*, Amer. J. Math. 87 (1965), pp. 857–860.

Received on 9.10.1984
and in revised form on 21.1.1985 (1461)