On powers of the theta-function greater than the eighth

by

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1. Introduction. The powers of the classical theta-function

\[ \Theta(\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau} = 1 + 2 \sum_{n=1}^{\infty} e^{\pi i n^2 \tau}, \quad \text{Im} \tau > 0, \]

have been the object of a good deal of attention since early in the century, both because of their connection with the problem of the representation of integers by sums of squares (integral powers) and because of their intrinsic interest as modular forms (arbitrary real powers). A powerful approach to the study of \( \Theta^s(\tau) \) when \( s > 4 \) (and even for an interval of values \( s \leq 4 \), but this is another, more complex, matter) is to compare it with the related Eisenstein series

\[ \Psi_s(\tau) = \frac{1}{2} \sum_{M=1}^{\infty} \frac{\vartheta_s(M)(c\tau + d)^{-s/2}}{\left| \begin{array}{cc} c & d \\ a & b \end{array} \right|}, \]

defined for \( \tau \in \mathcal{H} \), the upper half-plane. (The notation in (2) will be explained in § 2, below.) Once one has shown that \( \Psi_s \) is a modular form with formal properties precisely the same as those of \( \Theta^s \), standard results from the theory of modular forms yield a formula for \( r_s(n) \), the coefficients in the expansion of \( \Theta^s \):

\[ \Theta^s(\tau) = 1 + \sum_{n=1}^{\infty} r_s(n) e^{\pi i n^2 \tau}, \quad \tau \in \mathcal{H}. \]

The formula is exact when \( 4 < s \leq 8 \) and asymptotic for \( s > 8 \). The major technical difficulty in this approach is to show that \( \Psi_s \) is indeed a modular form for arbitrary \( s > 4 \). (The situation is much simpler for \( s \in \mathbb{Z}, s \geq 5 \).) This has been carried out in detail in [1] and [7], pp. 238–243. Petersson [6] has computed the Fourier expansions at the parabolic cusps of the functions \( \Psi_s \), \( s > 4 \), and indeed of all Eisenstein series of real weight greater than 2 on general finitely generated Fuchsian groups of the first kind.

As suggested above, when \( 4 < s \leq 8 \) \( \Theta^s = \Psi_s \), but this is no longer the
case for any \( s > 8 \). Thus for \( s > 8 \) it is of interest to determine the precise relationship between the two functions. Toward this end it is necessary to introduce a modular function \( F \) invariant with respect to \( \Gamma_0 \), the modular subgroup on which both \( \Theta' \) and \( \Psi'_s \) are modular forms. \( \Gamma_0 \) is of index 3 in the full modular group \( \Gamma(1) \) and it has a fundamental region with two cusps: \( \infty \) and \( -1 \). Putting

\[
g(t) = \frac{\Theta(t+1)}{\Theta(t)}, \quad f(t) = 1 - g(t),
\]

we define \( F(t) = f(t)g(t) \) and observe (5) that \( F \) is not only invariant with respect to \( \Gamma_0 \) for \( F(Mt) = F(t) \) for \( M \in \Gamma_0 \), but is in fact a \textit{Hauptmodul} for \( \Gamma_0 \); that is to say, \( F \) has precisely one zero of order 1 and one pole of order 1 (measured in the appropriate local variable) in a fundamental region for \( \Gamma_0 \), so that every modular function with respect to \( \Gamma_0 \) is a rational function in \( F \). In fact it can be shown from properties of \( \Theta \) that \( F \) has a zero of order 1 at the cusp \( \infty \), a pole of order 1 at the cusp \( -1 \) and no other zeros or poles in a fundamental region.

This circumstance makes the function \( F \) especially well suited for comparison with

\[
\Phi_s(t) = \Psi'_s(t)/\Theta'(t), \quad s > 4,
\]
a modular function on \( \Gamma_0 \). The expansions of \( \Theta' \) and \( \Psi'_s \) at the two cusps \( \infty \) and \( -1 \) show that the only possible pole of \( \Phi_s \) in a fundamental region is at the cusp \( -1 \). From these properties of \( \Phi_s \) and those of \( F \) mentioned above it follows that \( \Phi_s \) is a polynomial in \( F \) of degree equal to the order of the pole of \( \Phi_s \) at \( -1 \).

In a letter I received some time ago, Pierre Barrucand raised the question of determining precisely the coefficients of this polynomial, in particular when it is of degree 1 (\( 8 < s < 8 \)). Calculation of these coefficients in effect determines the precise relationship between \( \Theta' \) and \( \Psi'_s \); it is the purpose of this article to carry out this determination when \( 8 < s < 16 \). We prove two theorems.

**Theorem 1.** For \( 8 < s < 16 \) and \( \tau \in \mathcal{H} \),

\[
\Phi_s(t) = 1 + c(s)F(t),
\]

where \( c(s) \) is a constant depending only upon \( s \). That is,

\[
\Psi'_s(t) = \Theta'(t)\left[1 + c(s)F(t)\right].
\]

The statement of Theorem 2, which gives the explicit expression for \( c(s) \), requires further notation. The statement is to be found in Section 5.

In principle, the method of proof of these theorems applies to all \( s > 8 \), but the details are more complex when \( s > 16 \). I am indebted to M. Barrucand for his stimulating and extensive correspondence.

**2. The modular forms.** That \( \Theta(t) \) has no zeros in \( \mathcal{H} \) and is a modular form of weight 1/2 on \( \Gamma_0 \), with multiplier system \( v_\theta \), related to the Jacobi symbol \( (n; \ell) \equiv 1 \), are familiar facts, consequences of the connection of \( \Theta(t) \) with the Dedekind function \( \eta(t) \) (1) of Chapter 3:

\[
\eta(t) = \eta^2\left(\frac{t+1}{2}\right)/\eta(t+1), \quad \tau \in \mathcal{H}.
\]

\( \eta(t) \) is a modular form of weight 1/2 on the full modular group \( \Gamma(1) \). More explicitly, \( \eta(t) \) satisfies (2) of Chapter 3:

\[
\eta(t+1) = e^{\pi i t} \eta(t), \quad \eta(-1/t) = e^{-\pi i t} \tau^{1/2} \eta(t),
\]

for \( \tau \in \mathcal{H} \). (Here and elsewhere in this article we adopt the convention that \( \tau = |\tau| \exp(ik \arg \tau) \) for complex \( \tau \) and arbitrary real \( k \).) The following functional equations for \( \Theta(t) \) result immediately from (5) and (6):

\[
\Theta(Mt) = v_\theta(M)\left(c(t) + 1\right)^{1/2} \Theta(t), \quad \tau \in \mathcal{H},
\]

for all \( M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0 \). As a consequence of (8) \( v_\theta \) satisfies a “consistency condition” in weight 1/2; we shall comment further about this in Section 4.

In the definition (2) of \( \Psi'_s \), we have written \( v_\theta \) for \( v_\Psi \), the multiplier system of weight \( s/2 \) connected with \( \Theta(t) \). The summation in (2) is to be taken over all \( M = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \Gamma_0 \) with distinct lower rows; that is, over all pairs of rational integers \( c, d \) with \( (c, d) = 1 \) and \( c+d \) odd. For \( s > 4 \) the series converges absolutely and this guarantees that analytic difficulties do not arise in showing that, like \( \Theta' \), \( \Psi'_s \) is an entire modular form of weight \( s/2 \) and multiplier system \( v_\Psi \) on \( \Gamma_0 \). Nevertheless, when \( s \) is not an integer the proof contains formal difficulties; these have been worked out in detail in [1] and [7]. These same problems are evident here in our derivation (5) of the expansion, in explicit form, of \( \Psi'_s \) at the cusp \( \infty \).

It is not too hard to prove from (5) and (6) that the functions \( f(t), g(t) \) defined by (4) satisfy the relations

\[
f(t+2) = f(t), \quad g(t+2) = g(t)
\]

\[
f(-1/t) = g(t), \quad g(-1/t) = f(t).
\]

Since \( \Gamma_0 \) is generated by the two transformations \( S^2 \): \( \tau \rightarrow \tau+2 \) and \( T \): \( \tau \rightarrow -1/\tau \), it follows that the function \( F(t) = f(t)/g(t) \) is invariant with respect to \( \Gamma_0 \). Furthermore, it can be shown from the definitions (4) and the expansion (1) of \( \Theta(t) \) at the cusp \( \infty \) that

\[
\frac{1}{16} e^{-\pi i} f(t) \rightarrow 1, \quad g(t) \rightarrow 1
\]
as \( y = \operatorname{Im} \tau \to +\infty \). Since \( F(\tau + 2) = F(\tau) \), this implies that the expansion of \( F(\zeta) \) at \( \infty \) has the form
\[
F(\zeta) = 16e^{2\pi i} + \text{higher powers of } e^{2\pi i}.
\]

To determine the behavior of \( F(\zeta) \) at the cusp \(-1\) consider the function \( F(-1-1/\tau) \) at \( \infty \). From (4),
\[
g(\tau + 1) = \frac{\Theta^4(\tau)}{\Theta^4(\tau + 1)} = \frac{1}{g(\tau)} = 1 + f(\tau)/g(\tau)
\]
and
\[
f(\tau + 1) = 1 - g(\tau + 1) = -f(\tau)/g(\tau).
\]

Hence,
\[
F\left(-1 - \frac{1}{\tau}\right) = F\left(1 - \frac{1}{\tau}\right) = f\left(1 - \frac{1}{\tau}\right)g\left(1 - \frac{1}{\tau}\right)
\]
\[
= \frac{f(-1/\tau)}{g(-1/\tau)}\left\{1 + f(-1/\tau)g(-1/\tau)\right\} = -\frac{g(\tau)}{f(\tau)}\left\{1 + \frac{g(\tau)}{f(\tau)}\right\} = \frac{G(\tau)}{f^2},
\]
so that
\[
G(\tau + 1) = -\frac{g(\tau + 1)}{f(\tau + 1)^2} = -\frac{1}{g(\tau)}\frac{g(\tau)^2}{f(\tau)^2} = -\frac{g(\tau)}{f(\tau)^2} = G(\tau).
\]

On the other hand, by (9) and the periodicity of \( f \) and \( g \), \( G(\tau) \) has an expansion of the form
\[
G(\tau) = \frac{1}{16\tau^2}e^{-2\pi i\tau} + \text{nonnegative powers of } e^{2\pi i\tau} + \text{positive powers of } e^{2\pi i\tau/2}
\]
\[
= \frac{1}{16\tau^2}e^{-2\pi i\tau} + \text{nonnegative powers of } e^{-2\pi i\tau},
\]
This, in turn, assumes the form
\[
G(\tau) = \frac{1}{16\tau^2}e^{-2\pi i\tau} + \text{nonnegative powers of } e^{2\pi i\tau},
\]

since \( G(\tau + 1) = G(\tau) \) implies that only even powers of \( e^{2\pi i\tau} \) appear. Thus,
\[
F(\tau) = G\left(-\frac{1}{\tau + 1}\right)
\]
\[
= \frac{1}{16\tau^2}e^{-2\pi i(-1/\tau + 1)} + \text{nonnegative powers of } e^{2\pi i(-1/\tau + 1)},
\]
the expansion of \( F \) at \(-1\).
Choose $c(s) = -16^2 \beta(s)$; then $H(\tau) \equiv \Phi_2(\tau) = 1 + c(s) F(\tau)$ is bounded as $\tau \rightarrow -1$, vertically, so that $H(\tau)$, a bounded modular function, is constant. Since $H(\tau) \rightarrow 0$ as $\tau \rightarrow i \infty$, $H(\tau) \equiv 0$ and Theorem 1 follows.

4. Multiplier systems. Theorem 2, to be stated in Section 5 below, presents an explicit formula for $c(s)$. The determination of $c(s)$, while straightforward in principle, is considerably more involved than the simple proof of Theorem 1. Since $c(s) = -16^2 \beta(s)$ we may compute the constant $\beta(s)$ occurring in (15) and this, in turn, amounts to calculation of $d_0$, for $8 < s < 16$ and $d_1$ for $s = 16$. We require a result of H. Petersson on multiplier systems connected with arbitrary real weight.

Let $\Gamma$ be a discrete group of $2 \times 2$ matrices with real entries and determinant 1 (i.e. $\Gamma$ acts on $\mathcal{H}$) and $k$ a real number. We say a complex-valued function $v$ on $\Gamma$ is a multiplier system of weight $k$ on $\Gamma$ provided $|v(M)| = 1$ for $M$ in $\Gamma$ and

$$v(M_1 M_2)(c_1 \tau + d_1)^k = v(M_1) v(M_2)(c_1 M_2 \tau + d_1)^k(c_2 \tau + d_2)^k,$$

for all $M_1, M_2 \in \Gamma$, where $M_3 = M_1 M_2$ and $M_4 = \begin{pmatrix} c_{i1} & d_{i1} \\ c_{i2} & d_{i2} \end{pmatrix}$, $1 \leq i \leq 3$. The identity (16) is called the "consistency condition (in weight $k$)" for $v$.

With $X = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$, $Y = \begin{pmatrix} y_1 & x_2 \\ y_2 & y_1 \end{pmatrix}$, $XY = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$ and $k$ a real number, define

$$\sigma_k(X, Y) = \sigma(X, Y) = \frac{\left(\begin{array}{ll} x_1 & x_2 \\ y_1 & y_2 \end{array}\right) \tau + \left(\begin{array}{ll} y_1 & x_2 \\ y_2 & y_1 \end{array}\right)}{(c_1 \tau + d_1)^k}.$$

Then (16) can be rewritten as

$$v(M_1 M_2) = v(M_1) v(M_2).$$

It is worth noting that when $k$ is an integer $\sigma(X, Y) = 1$, so that the consistency condition reduces to $v(M_1 M_2) = v(M_1) v(M_2)$. For all real $k$ $|\sigma(X, Y)| = 1$. The result of Petersson is

**Lemma 3** ([5], p. 379). For $A$ a real $2 \times 2$ matrix and $v$ a multiplier system of weight $k$ on the discrete group $\Gamma$, let $v^*$ be the function defined on $A^{-1} \Gamma A$ by

$$v^*(M) = v(AMA^{-1}) \frac{\sigma(AMA^{-1}, A)}{\sigma(A, M)}, \quad M \in A^{-1} \Gamma A.$$

Then $v^*$ is a multiplier system of weight $k$ on the group $A^{-1} \Gamma A$.

In the proof of Lemma 3 we shall make use of the following result, which is also required in the derivation of Theorem 2 ([5], p. 378).

**Lemma 4.** If $X, Y, Z$ are real $2 \times 2$ matrices of determinant $> 0$, then

$$\sigma(X, Y Z) = \frac{\sigma(X, Y) \sigma(X Y, Z)}{\sigma(Y, Z)}.$$

5. We can now state and prove

**Theorem 2.** Let $A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$. Then the constant $c(s)$ is given by the formula

$$c(s) = \frac{-16^2 \left(2\pi s \right)^{d_2}}{2^2 \Gamma(s/2)} \left(\frac{s}{8-1}\right)^{d_2} \times \sum_{d=1}^{d_0} d^{-d_2} \sum_{0 < l < d} \frac{(s-1)^{l-1}}{\sigma_l(A^{-1} M_{d_l}, d)} e^{-2\pi(s-l)l}/d,$$

for $8 < s \leq 16$.

In (20) $\sum$ indicates that the summation is taken over integers $\delta$ such that $(\delta, d) = 1$. Also $M_{d_l-1} = \begin{pmatrix} \delta & 0 \\ d_l & \delta \end{pmatrix} M_A$, where $M \in \Gamma$; thus $v^*(A^{-1} M_{d_l-1})$ is defined. When $s$ is a multiple of 8, then $v_{s+1} = 1$ and $s = 1$, so that (20) simplifies considerably when $s = 16$.

**Proof of Theorem 2.** The proof requires Lemmas 3 and 4 and the Lipschitz summation formula [4]:

$$\sum_{n=0}^{\infty} \frac{e^{2\pi i m n}}{(\tau - m)^2} = \frac{e^{-\pi i/(\tau - m)^2}}{\Gamma(\tau)} \sum_{n+x>0} (n+x)^{-1} e^{2\pi i(n+x)\tau},$$

for $\tau \in \mathcal{H}$, $\lambda > 1$ and $0 \leq x < 1$. We note that the calculations we carry out here are valid as long as $s > 4$.

From the definition (2) of $\Psi_{s}$ and the consistency conditions (16) applied to $v = v_s (k = s/2)$, we find that

$$\Psi_s(\tau) = 1 + \sum_{c=1}^{\infty} \sum_{d=1}^{\infty} \Psi_s M_{c,d}(c \tau + d)^{-s/2},$$

where $\sum_{c,d}$ indicates that the summation is to be taken over $c, d \in Z$ such that $(c, d) = 1$ and $c + d = 0$. Such pairs of integers form precisely the set of distinct lower rows of elements of $\Gamma_0$. Thus, $M_{c,d} = \begin{pmatrix} c & d \\ \tau & \delta \end{pmatrix} \in \Gamma_0$, with an appropriate choice of upper row. (That the choice does not affect the value of $v_s(M_{c,d})$ follows from the consistency conditions for $v_s$ and the fact that
Let $A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, so that $A \in \Gamma(1)$ and $A(\infty) = -1$. We have

$$
\Psi_\sigma(At) = \tau^{n/2} \left\{ \tau^{-n/2} + \sum_{c \in \mathbb{Z}} \sum_{d\geq 0} \frac{v_\phi(M_{cd})}{\sigma(A, A^{-1} M_{cd}) (d - c)^{n/2}} \right\}.
$$

Since

$$\sigma(A, A^{-1} M_{cd}) = (cA + d)^{n/2}(c)^{n/2}/[(d - c)^{n/2}(d - c)^{n/2}],$$

we may rewrite the above expression as

$$
\Psi_\sigma(At) = \tau^{n/2} \left\{ \tau^{-n/2} + \sum_{c \in \mathbb{Z}} \sum_{d\geq 0} \frac{v_\phi(M_{cd})}{\sigma(A, A^{-1} M_{cd}) (d - c)^{n/2}} \right\}.
$$

From the definition of $v_\phi$,

$$v_\phi(M_{cd}) \sigma(A, A^{-1} M_{cd}) = v_\phi(A^{-1} M_{cd} A) \sigma(A, A^{-1} M_{cd} A),$$

so that replacing $d - c$ by $d$, we have

$$\Psi_\sigma(At) = \tau^{n/2} \left\{ \tau^{-n/2} + \sum_{c \in \mathbb{Z}} \sum_{d\geq 0} \frac{v_\phi(A^{-1} M_{cd} + c) A)}{\sigma(A, A^{-1} M_{cd} + c) (d - c)^{n/2}} \right\},$$

where the summation indicates summation over $d \in \mathbb{Z}$ such that $(c, d) = 1$ and $d$ is odd. A calculation shows that $M_{cd} + c = M_{d, -c}$, so that

$$\Psi_\sigma(At) = \tau^{n/2} \left\{ \tau^{-n/2} + \sum_{c \in \mathbb{Z}} \sum_{d\geq 0} \frac{v_\phi(A^{-1} M_{d, -c})}{\sigma(A, A^{-1} M_{d, -c}) (d - c)^{n/2}} \right\}.$$

We note that $A^{-1} M_{d, -c} = A^{-1} M_{d, -c + c} A = A^{-1} M_{d, -c} A = A^{-1} M_{d, -c} A$. We want to observe that the summand (22) is unchanged if we replace $(c, d)$ by $(-c, -d)$, suppose $A^{-1} M_{d, -c} A$ has denominator $d - c$, and write $A^{-1} M_{d, -c} = M_{d, -c}$. Then, the definition of $\sigma$ we have

$$v_\phi(A^{-1} M_{d, -c}) (d - c)^{n/2} = (A^{-1} M_{d, -c} A + d)^{n/2}. $$

and so that the summand in (22) is equal to

$$\Psi_\sigma(At) = \tau^{n/2} \left\{ \tau^{-n/2} + \sum_{c \in \mathbb{Z}} \sum_{d\geq 0} \frac{v_\phi(A^{-1} M_{d, -c})}{\sigma(A, A^{-1} M_{d, -c}) (d - c)^{n/2}} \right\}.$$ 

(Here we have used $A = \begin{pmatrix} -1 & * \\ 1 & 0 \end{pmatrix}$.) But by the consistency conditions (16) for $v_\phi$, the expression (23) remains invariant if we replace $(d, c)$ by $(-d, -c)$, since this replaces $(d', -c')$ by $(-d', c')$. Thus (22) becomes

$$\Psi_\sigma(At) = \tau^{n/2} \left\{ \tau^{-n/2} + \sum_{c \in \mathbb{Z}} \sum_{d\geq 0} \frac{v_\phi(A^{-1} M_{d, -c})}{\sigma(A, A^{-1} M_{d, -c}) (d - c)^{n/2}} \right\}.$$ 

But we claim that

$$v_\phi(A^{-1} M_{1, 0}) \sigma(A, A^{-1} M_{1, 0}) \tau^{n/2} = \tau^{n/2},$$

from which follows

$$\Psi_\sigma(At) = \tau^{n/2} \sum_{d\geq 0} \sum_{c \in \mathbb{Z}} \frac{v_\phi(A^{-1} M_{d, -c})}{\sigma(A, A^{-1} M_{d, -c}) (d - c)^{n/2}},$$

where the interchange of the double sum is justified by absolute convergence. The claim that $v_\phi(A^{-1} M_{1, 0}) \sigma(A, A^{-1} M_{1, 0}) = 1$ is immediate from the definitions of $v_\phi$ and $\sigma$.

Let $U \tau = \tau + 1$. In the inner sum of (25) we put $c = \delta + dm$, where $m \in \mathbb{Z}$ and $\delta$ runs through a reduced residue system modulo $d$. Then $M_{d, -\delta} = M_{d, -\delta} U^{-m}$ and $A^{-1} M_{d, -\delta} = A^{-1} M_{d, -\delta} U^m \in U^{-1} \Gamma_0 A = \Gamma_0(2)$, since $U \in \Gamma_0(2)$. We shall use the consistency condition (16) for $v_\phi$ in the form

$$v_\phi(M_1, M_2) = \sigma(M_1, M_2) v_\phi(M_1) v_\phi(M_2),$$

for $M_1, M_2 \in \Gamma_0(2)$. With $M_1 = A^{-1} M_{d, -\delta}$, $\delta = U^{-m}$, this gives

$$v_\phi(A^{-1} M_{d, -\delta}) = \sigma(A^{-1} M_{d, -\delta} U^{-m}) v_\phi(A^{-1} M_{d, -\delta}) v_\phi(U^{-m}).$$

On the other hand, Lemma 4 with $X = A$, $Y = A^{-1} M_{d, -\delta}$, $Z = U^{-m}$ states that

$$\sigma(A, A^{-1} M_{d, -\delta} U^{-m}) = \frac{\sigma(A, A^{-1} M_{d, -\delta} U^{-m})}{\sigma(A^{-1} M_{d, -\delta} U^{-m})},$$

But by the definition (17) $\sigma(M_{d, -\delta}, U^{-m}) = 1$, so that

$$v_\phi(A^{-1} M_{d, -\delta}) \sigma(A, A^{-1} M_{d, -\delta} U^{-m}) = v_\phi(A^{-1} M_{d, -\delta}) v_\phi(U^{-m}) v_\phi(A^{-1} M_{d, -\delta}).$$

From these considerations the expression (25) for $\Psi_\sigma(At)$ takes the form

$$\Psi_\sigma(At) = \tau^{n/2} \sum_{d\geq 0} \sum_{c \in \mathbb{Z}} \frac{v_\phi(A^{-1} M_{d, -c})}{\sigma(A, A^{-1} M_{d, -c}) (d - \delta - dm)^{n/2}},$$

where $\sum_{c \in \mathbb{Z}}$ indicates that $(\delta, d) = 1$. Define $\kappa$ by $v_\phi(U) = e^{2\pi i \kappa}$, $0 \leq \kappa < 1$. (This is the same constant $\kappa$ that appears in (14).) By the consistency condition (16) for $v_\phi$ we have $v_\phi(U^{-m}) = e^{2\pi i \kappa m}$, for $m \in \mathbb{Z}$. Thus (25) becomes

$$\Psi_\sigma(At) = \tau^{n/2} \sum_{d\geq 0} \sum_{c \in \mathbb{Z}} \frac{v_\phi(A^{-1} M_{d, -c})}{\sigma(A, A^{-1} M_{d, -c}) (d - \delta - dm)^{n/2}}.$$

Rewrite the inner sum as

$$\tau^{n/2} \sum_{m=0}^{\infty} \frac{e^{2\pi i \kappa m}}{(\tau - \delta/d - m)^{n/2}},$$

for $M_1, M_2 \in \Gamma_0(2)$.
apply the Lipschitz summation formula (21) with \( \tau \) replaced by \( \tau - \delta/d \) and (26) takes the form

\[
\Psi_s(A \tau) = \frac{e^{-\pi i s/4} (2\pi)^{s/2}}{\Gamma(s/2)} \sum_{d \geq 0} d^{-s/2} \sum_{0 < \delta < d} \frac{\nu_s(A^{-1} M_{d,-\delta})}{\sigma(A, A^{-1} M_{d,-\delta})} \times \sum_{n+\kappa > 0} (n+\kappa)^{s/2-1} e^{2\pi i (n+\kappa) \tau / (\tau - \delta/d)}
\]

Here we have defined

\[
a_n = a_n(s) = \frac{e^{-\pi i s/4} (2\pi)^{s/2}}{\Gamma(s/2)} \sum_{d \geq 0} d^{-s/2} \sum_{0 < \delta < d} \frac{\nu_s(A^{-1} M_{d,-\delta})}{\sigma(A, A^{-1} M_{d,-\delta})} e^{2\pi i (n+\kappa) \tau / (\tau - \delta/d)}
\]

Replace \( \tau \) by \( A^{-1} \tau = (-\tau + 1)^{-1} \) in (27); since \( \arg((-\tau + 1)^{-1}) = \pi - \arg(\tau + 1) \), we have

\[\frac{1}{-\tau + 1} \frac{1}{\pi - \arg(\tau + 1)} = \frac{1}{\pi - \arg(\tau - 1)} \frac{1}{\pi - \arg(\tau + 1)} = \frac{1}{\pi - \arg(\tau - 1)} \frac{1}{\pi - \arg(\tau + 1)} \]

so that (27) becomes

\[
\Psi_s(\tau) = e^{\pi i /2} (\tau + 1)^{-s/2} \sum_{n+\kappa > 0} a_n(s) e^{2\pi i (n+\kappa) \tau / (\tau + 1)}
\]

The expansion (29) with \( a_n(s) \) given by (28) is nothing more than an explicit form of the expansion (14) for \( \Psi_s(\tau) \). Comparison shows that \( d_n = e^{\pi i /2} a_n \).

Recall that \( \kappa = s/8 - 1 \) for \( 8 < s < 16 \) and \( \kappa = 0 \) for \( s = 16 \) and further that

\[
c(s) = -16^2 \beta(s) = \begin{cases} -16^2 e^{-\pi i /4} 2^{-s} \, d_0, & 8 < s < 16, \\ -16^2 e^{-\pi i /4} 2^{-s} \, d_1, & s = 16 \end{cases}
\]

by (15) and (13). The formula (20) follows since \( d_n = e^{\pi i /2} a_n \); the proof of Theorem 2 is complete.

6. Proof of Lemmas 3 and 4. We begin with the proof of Lemma 4. Let \( X, Y \) have lower rows \( x_1 \tau + x_2, y_1 \tau + y_2 \), respectively, as in (17). Denote the lower rows of \( XY, YZ \) and \( X'YZ \) by \( x_1' \tau + x_2', y_1' \tau + y_2' \) and \( x_1' \tau + x_2', y_1' \tau + y_2' \), respectively. From the definition (17) it follows that

\[
\sigma(Y, Z) = \frac{(y_1 Z + y_2)h(x_1 \tau + x_2)}{(y_1' \tau + y_2')h(x_1' \tau + x_2')}
\]

and

\[
\sigma(X, Y) = \frac{(x_1 Y \tau + x_2)h(y_1 \tau + y_2)}{(x_1' \tau + y_2')h(x_1' \tau + x_2')}
\]

Next observe that since \( \sigma(X, Y) \) is analytic in \( \mathcal{H} \) and \( \|\sigma(X, Y)\| = 1 \), \( \sigma(X, Y) \) is independent of \( \tau \) in \( \mathcal{H} \). Hence,

\[
\sigma(X, Y) = \frac{(x_1 Y \tau + x_2)h(y_1 Z \tau + y_2)}{(x_1' \tau + x_2')h(x_1' \tau + x_2')}
\]

To verify (19) we simply form the products \( \sigma(X, Y) \sigma(Y, Z) \) and \( \sigma(X, Y) \sigma(X, Z) \), using the expressions (30a) and (30d). In both instances we obtain

\[
(x_1 Y \tau + x_2)h(y_1 Z \tau + y_2)h(x_1' \tau + x_2')h(x_1' \tau + x_2')
\]

The identity (19) follows.

We turn now to the proof of Lemma 3. We must show that for \( M_1, M_2 \in \mathcal{A}^{-1} \Gamma A, M_3 = M_1 M_2 \), the relation (16) holds with \( \nu \) replaced by \( \nu^s \). This amounts to showing that

\[
\nu^s(M_1 M_2) = \sigma(M_1, M_2) \sigma(M_1 M_2 A^{-1}, A)
\]

for \( M_1, M_2 \in \mathcal{A}^{-1} \Gamma A \). By definition of \( \nu^s \),

\[
\nu^s(M_1 M_2) = \nu(AM_1 M_2 A^{-1}) \frac{\sigma(AM_1 M_2 A^{-1}, A)}{\sigma(A, M_1 M_2)}
\]

But by the consistency condition (16) for \( \nu \) on \( \Gamma \),

\[
\nu(AM_1 M_2 A^{-1}) = \nu(AM_1 A^{-1} AM_2 A^{-1})
\]

Thus by the definition (18) of \( \nu^s \) the right-hand side of (31) becomes

\[
\nu^s(M_1) \nu^s(M_2) \frac{\sigma(M_1 A^{-1}, A)\sigma(M_2 A^{-1}, A)}{\sigma(M_1 A^{-1}, A)\sigma(M_2 A^{-1}, A)}
\]

Now by Lemma 4,

\[
\sigma(A, M_1 M_2) = \frac{\sigma(A, M_1) \sigma(M_1 M_2)}{\sigma(M_1, M_2)}
\]

so that (31) becomes

\[
\nu^s(M_1 M_2) = \sigma(M_1, M_2) \nu^s(M_1) \nu^s(M_2) \times \left\{ \frac{\sigma(A, M_2) \sigma(M_1 A^{-1}, A) \sigma(M_1 M_2 A^{-1}, A) \sigma(M_1, M_2)}{\sigma(M_1, M_2)} \right\}
\]
The proof will be complete if we show that the expression in braces is \( \equiv 1 \). By Lemma 4, with \( X = AM_1, Y = M_2 A^{-1} \) and \( Z = A \),

\[
\sigma(AM_1, M_2 A^{-1}, A) = \frac{\sigma(M_2 A^{-1}, A) \sigma(M_1, M_2)}{\sigma(AM_1, M_2 A^{-1})},
\]

and again, with \( X = AM_1 A^{-1}, Y = A \) and \( Z = M_2 A^{-1} \), it follows that

\[
\sigma(AM_1 A^{-1}, AM_2 A^{-1}) = \frac{\sigma(M_1 A^{-1}, A) \sigma(M_1, M_2 A^{-1})}{\sigma(A, M_2 A^{-1})}.
\]

Hence \( \{ \} = \frac{\sigma(A, M_2) \sigma(M_2 A^{-1}, A)}{\sigma(A, M_1 A^{-1}) \sigma(AM_2 A^{-1}, A)} \), after the obvious cancellations are made. But Lemma 4, once again, with \( X = Z = A, Y = V_2 A^{-1} \), implies that

\[
\sigma(AM_2 A^{-1}, A) = \frac{\sigma(A, M_3) \sigma(M_2 A^{-1}, A)}{\sigma(A, M_2 A^{-1})}.
\]

This completes the proof.

7. Concluding remarks. We can make use of the expression (28) for \( a_n(s) \) to show that \( \sim \) in contradistinction to the situation when \( s \leq 8 \) for \( s > 8 \) the equation \( \psi \varphi(r) = \Theta^2(r) \) never occurs. Since the zero of \( \Theta^2(r) \) at the cusp \( -1 \) has exact order \( s/8 \), it is sufficient to prove that \( a_n(s) \neq 0 \) for \( 8 \leq s \leq 8 \). Thus the order of the zero of \( \psi \varphi(r) \) at \(-1\) is at most 1, while for \( \Theta^2(r) \) the order of the zero at \(-1\) is \( > 1 \) for \( s > 8 \).

In fact, we shall show simply from (28) that \( a_n(s) \neq 0 \) for all \( n \) and \( s \geq 6 \).

By the triangle inequality,

\[
|a_n(s)| \geq \frac{(2n)^{\gamma_2}}{\Gamma(s/2)} (n + \lambda)^{\gamma_2 - 1} \left( 1 - \sum_{d = 3}^{\infty} \sum_{d' = 3}^{\infty} d^{-\gamma_2} \left( \sum_{0 < \epsilon < d} \epsilon \right) \right).
\]

But,

\[
\sum_{d = 3}^{\infty} \sum_{d' = 3}^{\infty} d^{-\gamma_2} \left( \sum_{0 < \epsilon < d} \epsilon \right) \leq \sum_{d = 3}^{\infty} d^{-\gamma_2 - 1} \varphi(d) < \sum_{d = 3}^{\infty} \sum_{d' = 3}^{\infty} d^{-\gamma_2 - 1} < \zeta(s/2 - 1) - 1.
\]

Hence by (32),

\[
|a_n(s)| \geq \frac{(2n)^{\gamma_2}}{\Gamma(s/2)} (n + \lambda)^{\gamma_2 - 1} \left( 2 - \zeta(s/2 - 1) \right) - 1.
\]

as long as \( s > 6 \). This proof is to be found as well in [8].

References