integers 'under' $\lambda = \sum_{k=0}^{\infty} \lambda_k 2^k$. Here we have shown that whenever $\lambda \in Z_2$ is irrational then the language $\mathscr{L}(\lambda)$ under λ is not recognisable by a finite automaton.

Our present theorem can of course be proved without appeal to the theory of finite automata. Indeed the referee points out that λ may be approximated by a rational a/b:

$$|\lambda - a/b|_p < p^{-2k} \quad \text{with} \quad |a|, |b| < p^k.$$

Then

$$(1+X)^{\lambda b} \equiv (1+X)^a \ (\text{mod } X^{p^{2k}})$$

and given a polynomial $P(X, Y)$ with zero $Y = (1+X)^\lambda$ one readily shows that, with k sufficiently large, the polynomial one constructs to vanish at $(1+X)^{\lambda b}$ also vanishes at $(1+X)^a$. Of course this is a significantly less elegant argument than the one we present.

References

[1] J.-P. Allouche, *Somme des chiffres et transcendance*, Bull. Soc. Math. France 110 (1982), pp. 279–285 (see lemma at p. 281).

[2] A. Blanchard et M. Mendès France, *Symmètrie et transcendance*, Bull. Sci. Math. (2) 106 (1982), pp. 325–335.

[3] G. Christol, T. Kamae, M. Mendès France et G. Rauzy, *Suites algèbriques, automates et substitutions*, Bull. Soc. Math. France 108 (1980), pp. 401–419.

[4] F. M. Dekking, M. Mendès France and A. J. van der Poorten, *FOLDS!* The Mathematical Intelligencer 4 (1982), pp. 130–138; II *Symmetry disturbed*, ibid., pp. 173–181; III *More Morphisms*, ibid., pp. 190–195.

[5] S. Eilenberg, *Automata, Languages and Machines*, Vol. A, Academic Press, 1974, (see proposition 5.1, p. 24).

[6] H. Furstenberg, *Algebraic functions over finite fields*, J. Algebra 7 (1967), pp. 271–277.

[7] N. Koblitz, *P-adic analysis; a short course on recent work*, Cambridge U.P.; LMS Lecture Notes 46, 1980 (note p. 15).

[8] J. H. Loxton and A. J. van der Poorten, *Arithmetic properties of the solutions of a class of functional equations*, J. Reine Angew. Math. 330 (1982), pp. 159–172.

[9] M. Mendès France, A. J. van der Poorten, *Automata and p-adic numbers*, in: *Théorie des nombres, Colloque de Luminy, 1983*, (Unpublished manuscript).

U.E.R. DE MATHÉMATIQUES ET D'INFORMATIQUE
UNIVERSITÉ DE BORDEAUX I
TALENCE, FRANCE

SCHOOL OF MATHEMATICS AND PHYSICS
MACQUARIE UNIVERSITY
NORTH RYDE, NEW SOUTH WALES, AUSTRALIA

Received on 17.7.1984
and in revised form on 21.2.1985

ACTA ARITHMETICA
XLVI (1986)

An iteration problem involving the divisor function*

by

CLAUDIA SPIRO (Buffalo, N.Y.)

1. Introduction. Let $d(n)$ denote the number of positive integers dividing the positive integer n. For every integer $I \geqslant 3$, recursively define an integer sequence $S(I) = \{a_k\}_{k=0}^{\infty}$ by

$$a_0 = I; \quad a_{k+1} = a_k + (-1)^k d(a_k) \quad \text{for } k \geqslant 0. \tag{1}$$

For example, the sequence $S(275)$ is

$$275, \overline{281, 279, 285, 277, 279, 273}, \tag{2}$$

where the bar indicates that the sequence becomes periodic immediately after the term 275, and that one full period consists, in order, of the terms 281, 279, 285, 277, 279, 273. We will show that every sequence contains only integers exceeding 2 (see Proposition 1). If a sequence $S(I)$ is eventually periodic, we call the period a *cycle*. If there are n terms in the cycle, we say that it is an *n-cycle*, or a *cycle of length n*. Thus, the cycle in (2) has length 6.

At the West Coast Number Theory Conference, held in Los Angeles in December of 1977, we stated the following conjectures about these sequences:

CONJECTURE 1. *For every I, the sequence $S(I)$ is eventually periodic.*

CONJECTURE 2. *For each n, there are infinitely many $2n$-cycles.*

CONJECTURE 3. *The series $\sum 1/n$, summed over all positive integers which are elements of at least one cycle, diverges.*

CONJECTURE 4. *Every element of a given cycle has the same parity.*

In this paper, we establish that there are infinitely many 2-cycles (see Theorem 1). This is Conjecture 2 for $n = 1$. In addition, we show (see Theorem 3) that Conjecture 2 follows from the Prime k-tuples Conjecture.

2. Elementary properties of the sequences $S(I)$. Throughout this paper, m and n will denote positive integers, and p will denote a prime. Proposition 1,

* This investigation was supported in part by a Research Development Fund Award from the State University of New York Research Foundation.

below, implies that the sequences $S(I)$ are well-defined (i.e. that none of the a_k are negative or 0).

PROPOSITION 1. *If $I \geqslant 3$, then $a_k \geqslant 3$ for all k.*

Proof. This result is true by inspection, for $3 \leqslant I \leqslant 8$. If $I \geqslant 9$, and $a_k \geqslant 9$ for all k, then the assertion is clear. If $I \geqslant 9$ but there is a k with $a_k < 9$, let m be minimal subject to $a_m < 9$. By (1), m is odd, and $a_m = a_{m-1} - d(a_{m-1})$. But since $a_{m-1} \geqslant 9$, and since $d(n) \leqslant 2\sqrt{n}$ for all n, we have

$$a_m \geqslant a_{m-1} - 2\sqrt{a_{m-1}} \geqslant 3.$$

The proposition now follows from its truth for I with $3 \leqslant I \leqslant 8$. ∎

The next proposition is an approximation to Conjecture 4.

PROPOSITION 2. *If n is fixed, and the numbers comprising a $2n$-cycle are sufficiently large (perhaps depending upon n), then these numbers have the same parity.*

Proof. Let a be the minimal number in a $2n$-cycle, and let b be the maximal number in it. Since $d(m)$ is odd if and only if m is a perfect square, it suffices to show that the cycle cannot have any squares in it if a is sufficiently large. By (1) and the well-known estimate $d(m) = o(\sqrt{m})$ as $m \to \infty$, we have $a < b < a + \sqrt{a}$ if a is sufficiently large. Hence, the cycle contains at most one square. But if it contained exactly one square, then when we started on a, and ran through the $2n$ terms of the cycle, we would find exactly one parity change, contradicting the fact that $a \equiv a \pmod 2$. We further remark that after the sequence hit the square and changed parity, it could not hit the same square again without first undergoing a second parity change. Hence, the cycle contains no squares, and the result is established. ∎

Conjecture 4 asserts that we can delete the condition from Proposition 2 that the terms of our $2n$-cycle be sufficiently large.

The cycles with minimal term less than 100 are

$\overline{3, 5}$, $\overline{5, 7}$, $\overline{6, 10}$, $\overline{10, 14}$, $\overline{11, 13}$, $\overline{12, 18}$, $\overline{17, 19}$, $\overline{22, 26}$, $\overline{29, 31}$, $\overline{34, 38}$, $\overline{35, 39}$, $\overline{41, 43}$, $\overline{44, 50}$, $\overline{51, 55}$, $\overline{58, 62}$, $\overline{59, 61}$, $\overline{60, 72}$, $\overline{65, 69}$, $\overline{70, 78}$, $\overline{71, 73}$, $\overline{72, 84}$, $\overline{82, 86}$, $\overline{84, 96}$, $\overline{87, 91}$, $\overline{91, 95}$, $\overline{92, 98}$, $\overline{93, 97}$, $\overline{95, 99}$, $\overline{96, 108}$.

3. Statement and proof of the main theorem. By definition, I is the smallest integer in a 2-cycle if and only if $d(I + d(I)) = d(I)$. Accordingly, we will be able to deduce the existence of infinitely many 2-cycles from the following theorem.

THEOREM 1. *For $x \geqslant 3$,*

$$\#\{n \leqslant x:\ d(n+d(n)) = d(n)\} \gg x(\log x)^{-7}.$$

Our method of proof depends on the construction of a certain polynomial with integer coefficients of the form $f(m) = \prod_{h=1}^{7} (b_h m + c_h)$. We will then apply results derived from sieve methods to show that for arbitrarily large positive integers m, there are two factors $b_h m + c_h$ and $b_j m + c_j$ (with the subscripts h, j possibly depending on m), such that both factors are squarefree, and have the same number of prime divisors. Thus, $d(b_h m + c_h)$ will equal $d(b_j m + c_j)$. We will then multiply the $b_h m + c_h$ and $b_j m + c_j$ by appropriate quantities, to obtain our integers n with $d(n + d(n)) = d(n)$. This method was recently exployed by Heath-Brown [3], to establish that

$$\#\{n \leqslant x:\ d(n) = d(n+1)\} \gg x(\log x)^{-7}.$$

It is necessary to note that for that result, it was harder to determine the coefficients of the appropriate polynomial, and that his procedure for choosing those coefficients is very different from the procedure we use here. Again apart from choosing the polynomial, this method was first employed by the author (see Theorem 4.3.3 of [4]), to show that $d(n) = d(n+5040)$ infinitely often.

The next lemma is central to the determination of the coefficients of our $f(m)$.

LEMMA 1. *For each $j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ and each $k \in \{75, 125, 375\}$, there exists a positive integer $l < 10^8$ such that*

(i) $l/d(l) = 2^j k$,

(ii) *no prime exceeding* 11 *divides* l.

Proof of lemma. We present the integers l in the following table.

	$k = 75$	$k = 125$	$k = 375$
$j = 0$	$2^2 \cdot 3^3 \cdot 5^2$	5^4	$3^2 \cdot 5^4$
$j = 1$	$2^4 \cdot 3 \cdot 5^3$	$2^2 \cdot 3 \cdot 5^4$	$2^3 \cdot 3^2 \cdot 5^4$
$j = 2$	$2^5 \cdot 3^3 \cdot 5^2$	$2^2 \cdot 3^2 \cdot 5^4$	$2^6 \cdot 3 \cdot 5^3 \cdot 7$
$j = 3$	$2^4 \cdot 3 \cdot 5^4$	$2^5 \cdot 3 \cdot 5^4$	$2^6 \cdot 3^2 \cdot 5^3 \cdot 7$
$j = 4$	$2^4 \cdot 3^2 \cdot 5^4$	$2^7 \cdot 5^4$	$2^6 \cdot 3 \cdot 5^4 \cdot 7$
$j = 5$	$2^9 \cdot 3 \cdot 5^3$	$2^6 \cdot 5^4 \cdot 7$	$2^6 \cdot 3^2 \cdot 5^4 \cdot 7$
$j = 6$	$2^9 \cdot 3^2 \cdot 5^3$	$2^{11} \cdot 3 \cdot 5^3$	$2^8 \cdot 3^3 \cdot 5^4$
$j = 7$	$2^8 \cdot 3^5 \cdot 5^2$	$2^{10} \cdot 5^3 \cdot 11$	$2^{10} \cdot 3^2 \cdot 5^3 \cdot 11$
$j = 8$	$2^9 \cdot 3^2 \cdot 5^4$	$2^{11} \cdot 3 \cdot 5^4$	$2^{10} \cdot 3 \cdot 5^4 \cdot 11$

By inspection, these values of l satisfy (ii). The purely numerical verification of (i) for each l is left to the reader. ∎

We remark that these numbers were computed by hand, by examining multiples of small powers of 2. The reason for choosing $\{75, 125, 375\}$ as our set of possible values of k will be clearer after the proof of the theorem, and we will discuss it at that point.

Proof of theorem. First, observe that

(3) $$C_h \doteq 210h-11 \quad \text{is prime, for } h = 1, \ldots, 7.$$

(In fact, $210h-11$ is prime for $0 \leqslant h \leqslant 10$, although we will not need that information here.) Let

(4) $$p_1 = 13, \quad p_2 = 17, \quad p_3 = 19, \quad p_4 = 23, \quad p_5 = 29, \quad p_6 = 31, \quad p_7 = 37.$$

By the Chinese Remainder Theorem, there is a solution $s > 0$ of the simultaneous system of congruences in m

(5) $$11!\,C_h m + 1 \equiv p_h^{157499} \pmod{p_h^{157500}}, \quad h = 1, \ldots, 7.$$

Here, the exponent $157499 = 210\cdot 2\cdot 3\cdot 5^3 - 1$ is chosen to enable us to apply Lemma 1 at a later stage (see (19) and (23)–(29), below). Set

(6) $$q_h = [11!\,C_h s + 1]\, p_h^{-157499}, \quad h = 1, \ldots, 7.$$

Then each q_h is a positive integer satisfying

(7) $$(q_h, 11!\,p_h) = 1.$$

Put

(8) $$A = 11! \prod_{h=1}^{7} p_h^{157500}.$$

We claim that

(9) $$(C_h A, q_h) = 1, \quad h = 1, \ldots, 7.$$

In view of (6) and (7), it is enough to show that if $1 \leqslant j \leqslant 7$, and $j \neq h$, then $p_j \nmid q_h$. By (4),

$$C_j[11!\,C_h s+1] - C_h[11!\,C_j s+1] = 210(j-h) \not\equiv 0 \pmod{p_j}.$$

Since s is a solution of the simultaneous system (5), p_j divides $11!\,C_j s+1$. Hence, p_j cannot divide $11!\,C_h s+1$. Therefore, (6) yields $p_j \nmid q_h$, as asserted.

Consider the polynomial with integer coefficients

(10) $$f(m) = \prod_{h=1}^{7} (p_h^{-157499} C_h A m + q_h).$$

We will show that there is no prime p such that $p \mid f(m)$ for all m. Indeed, it follows from (4), (8) and (9) that $f(m)$ is never divisible by any prime less than 38. If $p \geqslant 38$, then (9) implies the existence of at most one solution of the congruence

$$p_h^{-157499} C_h A m + q_h \equiv 0 \pmod{p}.$$

Hence, there are at least $p-7 > 0$ solutions of

$$f(m) \not\equiv 0 \pmod{p},$$

and our assertion follows.

We state the sieve result which we will use to regulate the number of prime divisors of the factors $p_h^{-157499} Am + q_h$ of $f(m)$, as the next lemma.

LEMMA 2. *Let α_i, β_i $(i = 1, 2, \ldots, 7)$ be integers satisfying*

(11) $$\Big(\prod_{i=1}^{7} \alpha_i\Big) \prod_{1 \leqslant u < v \leqslant 7} (\alpha_u \beta_v - \alpha_v \beta_u) \neq 0.$$

Assume also that for every prime p, there is an m for which

(12) $$g(m) \doteq \prod_{i=1}^{7} (\alpha_i m + \beta_i) \not\equiv 0 \pmod{p}.$$

Then for any integer $N > 0$, we have

$$\#\{m \leqslant x:\ (g(m), N) = 1,\ \mu(g(m)) \neq 0,\ \omega(g(m)) \leqslant 27\} \gg x(\log x)^{-7},$$

where μ denotes the Möbius function, and $\omega(m)$ is the number of distinct primes dividing m.

This lemma follows from Theorem 10.5 of Halberstam and Richert's book [2], as improved by Xie [5]. We have stated the special case of these results where $g(m)$ has 7 linear factors. The general statement is given as Lemma 2 of [3].

We apply the lemma with $f(m)$ in place of $g(m)$ (so that $\alpha_i = p_i^{-157499} C_i A$ and $\beta_i = q_i$), and with $N = 1500!$. We have already shown that the second condition of this lemma holds. Clearly, since there is no fixed prime p dividing any of the factors $\alpha_i m + \beta_i$, we cannot have $\alpha_u \beta_v = \alpha_v \beta_u$ unless $\alpha_u = \pm\alpha_v$ and $\beta_u = \pm\beta_v$. By inspection, we do not have $\beta_u = \pm\beta_v$, here, for $u \neq v$. Therefore,

(13) $$\#\{m \leqslant x:\ (f(m), 1500!) = 1,\ \mu(f(m)) \neq 0,\ \omega(f(m)) \leqslant 27\} \gg x(\log x)^{-7}.$$

Let $M > 0$ be any integer satisfying

(14) $$\mu(f(M)) \neq 0, \quad \omega(f(M)) \leqslant 27, \quad (f(M), 1500!) = 1.$$

We claim that there are h, j with $1 \leqslant h < j \leqslant 7$, possibly depending upon M, such that

(15) $$\omega(p_h^{-157499} C_h AM + q_h) = \omega(p_j^{-157499} C_j AM + q_j).$$

Indeed, if there were no h, j satisfying (15), then we would have

$$27 \geqslant \omega(f(n)) = \sum_{i=1}^{7} \omega(p_i^{-157499} C_i AM + q_i) \geqslant \sum_{i=1}^{7} i = 28.$$

Furthermore, this argument shows that we can choose h, j so that

$$\omega(p_h^{-157499}C_hAM+q_h)=\omega(p_j^{-157}C_jAM+q_j)=t, \tag{16}$$

with

$$1\leqslant t\leqslant 6. \tag{17}$$

Since $\mu(f(m))\neq 0$, (16) yields

$$d(p_h^{-157499}C_hAM+q_h)=d(p_j^{-157499}C_jAM+q_j)=2^t. \tag{18}$$

From (4), (18), and the last equation in (14), it follows that

$$d(C_hAM+p_h^{157499}q_h)=d(C_jAM+p_j^{157499}q_j)=2^{t+1}\cdot 210\cdot 3\cdot 5^3. \tag{19}$$

From (3) and the final equation in (14), we can conclude that

$$d\big(C_j(C_hAM+p_h^{157499}q_h)\big)=d\big(C_h(C_jAM+p_j^{157499}q_j)\big)=2^{t+2}\cdot 210\cdot 3\cdot 5^3. \tag{20}$$

Furthermore, it follows from (6) that

$$C_j(C_hAM+p_h^{157499}q_h)-C_h(C_jAM+p_j^{157499}q_j)=C_j-C_h. \tag{21}$$

Redefine h and j if necessary, so that $C_j>C_h$, and set

$$r=C_h(C_jAM+p_j^{157499}q_j). \tag{22}$$

Hence, (20), (21), and (22) yield

$$d(r)=d(r+C_j-C_h)=2^{t+2}\cdot 210\cdot 3\cdot 5^3. \tag{23}$$

We will verify that there is a positive integer l, bounded independently of r, such that $n=rl$ satisfies $d(n+d(n))=d(n)$. The idea is to arrange that

$$(r+C_j-C_h)l-rl=(C_j-C_h)l=2^{t+2}\cdot 210\cdot 3\cdot 5^3\,d(l) \tag{24}$$

(the first equality being, of course, trivial). By (3),

$$C_j-C_h=210u, \tag{25}$$

where

$$1\leqslant u\leqslant 6. \tag{26}$$

Substituting (25) into (23) yields

$$d(r)=d(r+210u)=2^{t+2}\cdot 210\cdot 3\cdot 5^3. \tag{27}$$

Furthermore, equation (3) and the last condition of (14) imply that

$$(11!,\,r)=(11!,\,r+210u)=1. \tag{28}$$

By Lemma 1, there is a positive integer $l<10^8$, none of whose prime factors exceed 11, such that

$$l/d(l)=2^{t+2}\cdot 3\cdot 5^3/u, \tag{29}$$

for all integers t and u respectively satisfying (17) and (26). Let $n=rl$. Then (23), (25), (27), and (29) imply that

$$d(n)=d\big(n+(C_j-C_h)l\big)=210ul=(C_j-C_h)l. \tag{30}$$

Therefore, $d(n)=d(n+d(n))$. Since $n=rl$, $l<10^8$, and there are only finitely many choices for h and j such that (15) holds, we can deduce the theorem from (21), (13), and our choice of M. ■

At this point, we will indicate why we chose the ranges of j and k as we did in the statement of Lemma 1. As seen from the sentence containing (29), we never needed the lemma for $j=0$. However, if we replace 157499 by $78749=210\cdot 3\cdot 5^3-1$ in (5), then Lemma 1 permits us to consider a polynomial with 8 linear factors in place of the $f(m)$ which we chose. If we make this change, then we are required to restate Lemma 2 for polynomials with 8 linear factors. Halberstam and Richert [2] (see Theorem 10.5, and Table 3 on p. 285) proved results containing Lemma 2 with 7 replaced by 8 in (11), (12), and the final inequality, and with 27 replaced by 34 in that inequality. Since $\sum_{i=1}^{8} i=36>34$ (compare with the inequality preceding (16)), this result is sufficient to obtain the weaker estimate

$$\#\{n\leqslant x:\ d(n+d(n))=d(n)\}\gg x(\log x)^{-8}.$$

Hence, Xie's improvements are unnecessary, if we merely desire to show that infinitely many 2-cycles exist.

Secondly (cf. the sentence containing (29), and the derivation of (30)), we needed to choose a set of the form $\{3B,\,5B,\,15B\}$ for the range of k, where B is a positive integer. The choice $\{75,\,125,\,375\}=\{3\cdot 5^2,\,5^3,\,3\cdot 5^3\}$, with $B=25$, was motivated by the relationships

$$3=3^2/d(3^2),\quad 3\cdot 2^{-1}=3/d(3),\quad 5^3=5^4/d(5^4),\quad 5^3\cdot 2^{-2}=5^3/d(5^3),$$

and the fact that $(3,\,5)=1$.

The limit of the current method of proof is the replacement of 7 by 2 in Theorem 1 (cf. Theorem 10.5 of [2]). Making this improvement by these methods amounts to proving the Twin-Prime Conjecture, or establishing a related result of equal difficulty. On the other hand, any pair of twin primes is an example of a cycle. We believe that the number of cycles with minimal term not exceeding x is substantially greater than the number of twin primes p, $p+2$ with $p\leqslant x$. Bearing in mind that $\sum 1/p$, summed over primes such that $p+2$ is prime, converges [1], we have stated our belief about the number of cycles in the form of Conjecture 3.

For completeness, and because the argument is much shorter than Heath-Brown's proof that $d(n) = d(n+1)$ infinitely often, we will present Theorem 2, below. By using 7 linear factors instead of the 8 that we used in Theorem 4.33 of [4], we have reduced the constant 5040 to 2520 with no extra work.

THEOREM 2. $\# \{n < x\colon d(n) = d(n+2520)\} \gg x(\log x)^{-7}$.

Proof. Let

$$S = \{11, 17, 23, 29, 41, 47, 53\},$$

and put

$$f(m) = \prod_{s \in S} (sm+1).$$

Then Lemma 2 implies that

(31) $\# \{m \leqslant x\colon \mu(f(m)) \neq 0,\ \omega(f(m)) \leqslant 27,\ (f(m), 60!) = 1\} \gg x(\log x)^{-7}$.

Let M be any positive integer such that

$$\mu(f(M)) \neq 0, \quad \omega(f(M)) \leqslant 27, \quad (f(M), 60!) = 1.$$

By the same justification that was used to establish the sentence containing (15) in the proof of the last theorem, there exist $\gamma, \delta \in S$ such that $\gamma < \delta$, and

$$\omega(\gamma M+1) = \omega(\delta M+1).$$

Since the primes γ, δ are in S, $(f(M), 60!) = 1$, and $\mu(f(M)) \neq 0$, it follows that $\delta - \gamma | 2520$, and that

$$d\left(\frac{2520}{\delta-\gamma}\right) d(\delta(\gamma M+1)) = d\left(\frac{2520}{\delta-\gamma}\right) d(\gamma(\delta M+1)).$$

We can now conclude from the multiplicativity of d that

$$d(N) = d(N+2520) \quad \text{for} \quad N = \frac{2520}{\delta-\gamma} \cdot \gamma \cdot (\delta M+1).$$

The theorem is now an immediate consequence of (31). ∎

4. A conditional proof of Conjecture 2. In this section, we prove that Conjecture 2 follows from the Prime k-tuples Conjecture, which we state here for convenience.

THE PRIME k-TUPLES CONJECTURE. *Let $\{\alpha_i\}_{i=1}^{k}$, $\{\beta_i\}_{i=1}^{k}$ be integer sequences with no α_i equal to 0. Suppose that for every prime p, there is an n with*

$$\prod_{i=1}^{k} (\alpha_i n+\beta_i) \not\equiv 0 \pmod{p}.$$

Then there are infinitely many n such that $\alpha_i n+\beta_i$ is prime for all i.

For related conjectures, see pp. 1–2 of [2].

THEOREM 3. *The Prime k-Tuples Conjecture implies Conjecture 2.*

Let N be any fixed positive integer, and set $P = \prod_{p \leqslant 2N+3} p$, where the product runs over primes. Consider the polynomial

$$h(m) = \prod_{k=0}^{N-1} (m+2kP)(m+(2k+3)P).$$

Choose primes $p_0 < p_1 < \ldots < p_{2n-1}$ exceeding $2N+4$. Then by the Chinese Remainder Theorem, there is a unique solution S to the system of $2N$ congruences in m

(32) $$m+2kP \equiv p_{2k}^{3P/2-1} \pmod{p_{2k}^{3P/2}}, \quad 0 \leqslant k \leqslant N-1;$$

(33) $$m+(2k+3)P \equiv p_{2k+1}^{P/2-1} \pmod{p_{2k+1}^{P/2}}, \quad 0 \leqslant k \leqslant N-2;$$

(34) $$m+(2N+1)P \equiv p_{2N-1}^{(N+1/2)P-1} \pmod{p_{2N-1}^{(N+1/2)P}};$$

(35) $$m \equiv 1 \pmod{P}.$$

Put

(36) $$E = P \prod_{k=0}^{2N-1} p_k^{2NP}.$$

Define the linear functions $L_k(m)$, $0 \leqslant k \leqslant 2N-1$, by

(37) $$L_{2k}(m) = p_{2k}^{1-3P/2}(Em+S+2kP) \doteqdot \lambda_{2k} m+\mu_{2k}, \quad 0 \leqslant k \leqslant N-1;$$

(38) $$L_{2k+1}(m) = p_{2k}^{1-P/2}(Em+S+(2k+3)P) \doteqdot \lambda_{2k+1} m+\mu_{2k+1}, \quad 0 \leqslant k \leqslant N-2;$$

(39) $$L_{2N-1}(m) = p_{2N-1}^{1-(N+1/2)P}(Em+S+(2N+1)P) \doteqdot \lambda_{2N-1} m+\mu_{2N-1}.$$

From (32)–(34), (36)–(39), and the definition of P, we can conclude that λ_k and μ_k are integers for $0 \leqslant k \leqslant 2N-1$, and that

(40) $$p_k \nmid L_k(m) \quad \text{for all } k, m.$$

We claim that

(41) $$(\lambda_k, \mu_k) = 1, \quad 0 \leqslant k \leqslant 2N-1.$$

Since E/λ_k is a power of the prime p_k, (41) will follow from (40) if we show that

(42) $$p \nmid (E, \mu_k) \quad \text{for all } p \neq p_k.$$

If $p \leqslant 2N+3$, then (37), (38), (39), and the fact that S satisfies (35) imply that $p \nmid \mu_k$. We remark that (36) now gives

(43) $$(L_k(m), P) = 1 \quad \text{for all } k, m.$$

If $p = p_j$ for some j, then (32), (33), and (34) yield $p \nmid S$. But then, we can conclude from (36)–(40) and the inequality $p_j > 2N+4$ that $p \nmid \mu_k$. We remark that (36) and (43) now yield

$$(L_k(m), E) = 1 \quad \text{for all } k, m, \tag{44}$$

and that (42) now follows from (36).

Let

$$J(m) \doteqdot \prod_{k=0}^{2N-1} L_k(m).$$

We assert that for every prime p, there is an m such that $p \nmid J(m)$. For $p \leqslant 2N+3$, this assertion is an immediate consequence of (43). If $p > 2N+3$, then by (41), there is at most one solution to the congruence $L_k(m) \equiv 0 \pmod{p}$ for each k. Accordingly, there are at least $p-(2N-1) > 0$ solutions to $J(m) \not\equiv 0 \pmod{p}$.

Therefore, if we assume the validity of the Prime k-tuples Conjecture, then there are infinitely many m such that

$$L_k(m) \text{ is prime for all } k.$$

For any such m, it follows from (37) and (44) that

$$d(Em+S+2kP) = 3P, \quad 0 \leqslant k \leqslant N-1.$$

Similarly, we can conclude from (38), (39), and (44) that

$$d(Em+S+(2k+3)P) = P, \quad 0 \leqslant k \leqslant N-2;$$

$$d(Em+S+(2N+1)P) = (2N+1)P.$$

Consequently, when $I = Em+S$, the sequence $S(I) = \{a_i\}_{i=0}^{\infty}$ referred to in the introduction is given by

$$a_i = Em+S+2kP \quad \text{if} \quad i \equiv 2k \pmod{2N}, \quad 0 \leqslant k \leqslant N-1;$$
$$a_i = Em+S+(2k+3)P \quad \text{if} \quad i \equiv 2k+1 \pmod{2N}, \quad 0 \leqslant k \leqslant N-1.$$

The theorem follows.

References

[1] V. Brun, *Le série* $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\frac{1}{29}+\frac{1}{31}+\frac{1}{41}+\frac{1}{43}+\frac{1}{59}+\frac{1}{61}+\ldots$ *où les dénominateurs sont "nombres premiers jumeaux" est convergente où finie*, Bull. Sci. Math. (2) 43 (1919), pp. 124–128.

[2] H. Halberstam and H.-E. Richert, *Sieve Methods*, L.M.S. monographs, no. 4, Academic Press, London, New York 1974, xiv+364 pp.

[3] D. R. Heath-Brown, *The divisor function at consecutive integers*, Mathematica 31 (1984), pp. 141–149.

[4] C. A. Spiro, *The frequency with which an integral-valued, prime-independent, multiplicative or additive function of n divides a polynomial function of n*, Ph.D. Dissertation, University of Illinois at Urbana/Champaign, 1981, viii+179 pp.

[5] X. Xie, *On the k-twin primes problem*, Acta Math. Sinica 26 (1983), pp. 378–384.

STATE UNIVERSITY OF NEW YORK
BUFFALO, NEW YORK

Received on 3.9.1984
and in revised form on 4.3.1985 (1448)