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Extreme values of the Dedekind zeta function

by

U. BALAKRISHNAN (Bombay)

In [3] Montgomery has obtained the following inequalities for the Riemann zeta function:

For $1/2<\sigma_0<1$, $T\geqslant T_0(\sigma_0)$ and for a given real θ, there is a t_0 such that $T^{(\sigma_0-1/2)/3}\leqslant t_0\leqslant T$ and

$$\operatorname{Re} e^{-i\theta}\log\zeta(s_0)\geqslant \tfrac{1}{20}(\sigma_0-\tfrac{1}{2})^{1/2}(\log T)^{1-\sigma_0}(\log\log T)^{-\sigma_0},$$

where $s_0=\sigma_0+it_0$ and this with $\theta=0,\ \pi,\ \pm\pi/2$ gives for $1/2<\sigma<1$,

$$\log|\zeta(s)|=\Omega_{\pm}\{(\log t)^{1-\sigma}(\log\log t)^{-\sigma}\}$$

and

$$\arg\zeta(s)=\Omega_{\pm}\{(\log t)^{1-\sigma}(\log\log t)^{-\sigma}\},$$

as $t\to\infty$. Also under the assumption of Riemann hypothesis it is proved there that

$$|\zeta(\tfrac{1}{2}+it)|=\Omega\left(\exp\{\tfrac{1}{20}(\log t/\log\log t)^{1/2}\}\right)$$

and

$$S(t)=\Omega_{\pm}\left((\log t/\log\log t)^{1/2}\right)$$

where

$$S(t)=\pi^{-1}\arg\zeta(\tfrac{1}{2}+it).$$

We obtain below the analogue of these results for the case of Dedekind zeta function ζ_K of a quadratic field K over Q. Also we have the analogue in the case of ζ_{K_1} of a field K_1 with $[K_1:Q]=k>2$, but the result in this case is less precise and restricted to the region $\sigma>1-k^{-1}$ (cf. (25)). We explain some notation before the statements of theorems.

$[x]$, $\|x\|$ as usual denote the integral part and the distance from nearest integer of x respectively. K is a quadratic extension of the field Q of rationals and d denotes the discriminant of K. Also $s=\sigma+it$, $w=u+iv$ with σ, t, u, v real. $C(\alpha)$ would mean a constant depending on α which may be different for its appearence in different places. The constants involved in $\ll$, $\gg$, O are absolute unless stated the dependence explicitly. $f(x) = \Omega(g(x))$ would mean $\overline{\lim\limits_{x\to\infty}}\left|\frac{f(x)}{g(x)}\right| > 0$ and $f(x) = \Omega_{\pm}(g(x))$ would mean (only when $f(x)$ and $g(x)$ are real) $\overline{\lim\limits_{x\to\infty}}\frac{f(x)}{g(x)} > 0$ and $\underline{\lim\limits_{x\to\infty}}\frac{f(x)}{g(x)} < 0$. We state below the results.

THEOREM 1. *Let K be a quadratic extension of Q with discriminant d. Also let $1/2 < \sigma_0 < 1$, $0 < \varepsilon < 1/4$. Then for*

$$T \geqslant T_0(\sigma, \varepsilon) + |d|^{\frac{12}{1-\varepsilon}\frac{1-\sigma_0}{2\sigma_0-1}+4\varepsilon}$$

and any real θ, there is a t_0 with

$$T^{\frac{2}{3(1-\varepsilon)}\frac{2\sigma_0-1}{3-2\sigma_0}} - \log^2 T \leqslant t_0 \leqslant T$$

satisfying

$$\operatorname{Re} e^{-i\theta} \log \zeta_K(\sigma_0 + it_0) \geqslant \lambda (\log T)^{1-\sigma_0} (\log\log T)^{-\sigma_0}$$

with

$$\lambda = \left(\frac{\sinh((1-\sigma_0)/2)}{1-\sigma_0}\right)^2 \left(\operatorname{cosech} 1 \cdot \frac{1-\varepsilon}{6\log 6}\frac{2\sigma_0-1}{3-2\sigma_0}\right)^{1-\sigma_0} - \varepsilon.$$

With the special values $\theta = 0, \pi, \pm\pi/2$ Theorem 1 gives the

COROLLARY. *Let σ be fixed, $1/2 < \sigma < 1$. Then as $t \to \infty$*

$$\log|\zeta_K(\sigma+it)| = \Omega_{\pm}\{(\log t)^{1-\sigma}(\log\log t)^{-\sigma}\}$$

and

$$\arg \zeta_K(\sigma+it) = \Omega_{\pm}\{(\log t)^{1-\sigma}(\log\log t)^{-\sigma}\}.$$

THEOREM 2. *Suppose the Riemann hypothesis for $\zeta_K(s)$ is assumed. That is, we assume, $\zeta_K(\sigma+it) = 0$, $|t| > 0 \Rightarrow \sigma = 1/2$. Then we have as $t \to \infty$*

$$|\zeta_K(\tfrac{1}{2}+it)| = \Omega\left(\exp\left\{\frac{2}{17}\left(\frac{\log t}{\log\log t}\right)^{1/2}\right\}\right)$$

and

$$S_K(t) = \Omega_{\pm}\left\{\left(\frac{\log t}{\log\log t}\right)^{1/2}\right\}$$

where

$$S_K(t) = \pi^{-1}\arg\zeta_K(\tfrac{1}{2}+it).$$

We need the following lemmas for the proof of the theorems.

LEMMA 1. *Let $\theta_1, \theta_2, \ldots, \theta_M$ be arbitrary real numbers and suppose that $0 < \delta < 1/2$. There are at least $[\delta^M(R+1)]$ integers r such that $1 \leqslant r \leqslant R$ and $\|r\theta_m\| \leqslant \delta$ for $1 \leqslant m \leqslant M$.*

This is proved (Lemma 2) in Montgomery [3].

LEMMA 2. *Let $N_K(\sigma_0, T)$ denote the number of zeros of $\zeta_K(s)$ in the rectangle $\sigma_0 \leqslant \sigma \leqslant 1$, $0 \leqslant t \leqslant T$. Then for $\sigma_0 > 1/2$,*

$$N_K(\sigma_0, T) \leqslant \left(c(\varepsilon)|d|^{1/2+\varepsilon}T\right)^{4(1-\sigma_0)/(3-2\sigma_0)}(\log T)^{78}$$

for $T \geqslant T_1(\sigma_0) + \exp\{(200\log|d|)^{1/2}\}$.

Proof. The method of proof is the same as that we use for the case of Riemann zeta function. (See for example (6).) We have for $\sigma > 1$,

$$\zeta_K(s) = \prod_{\mathfrak{p}}(1-N(\mathfrak{p})^{-s})^{-1} = \sum_{n=1}^{\infty} a(n)n^{-s}, \quad a(n) = \sum_{N(\mathfrak{a})=n} 1 \leqslant d(n) \tag{1}$$

where the product is running over the prime ideals $\mathfrak{p}$ of K. Also

$$\zeta_K^{-1}(s) = \sum_{\mathfrak{a}} \frac{\mu(\mathfrak{a})}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty}\frac{b(n)}{n^s}, \quad |b(n)| \leqslant d(n).$$

Consider $M_x(s) = \sum_{n\leqslant x} b(n)n^{-s}$ and

$$\varphi(s) = \zeta_K(s)M_x(s) - 1 = \sum_{n>x} g(n)n^{-s}, \quad |g(n)| \leqslant d^3(n)$$

since $g(n) = \sum_{\substack{d|n\\ d\leqslant x}} b(d)a(n/d)$ and $g(n) = c(n)$ for $n \leqslant x$ where

$$1 = \zeta_K(s)\zeta_K^{-1}(s) = \sum c(n)n^{-s}.$$

Now let $x = T(\log T)^{-1}$ and we have

$$\int_T^{2T} |\varphi(\tfrac{1}{2}+it)|\,dt \leqslant T + \left(\int_T^{2T} |\zeta_K(\tfrac{1}{2}+it)|^2\,dt \int_T^{2T} |M_x(\tfrac{1}{2}+it)|^2\,dt\right)^{1/2} \leqslant c(\varepsilon)|d|^{1/2+\varepsilon}T\log^3 T, \tag{2}$$

using the result

$$\int_T^{2T} |\zeta_K(\tfrac{1}{2}+it)|^2\,dt \leqslant c(\varepsilon)|d|^{1+\varepsilon}T\log^2 T$$

from the theorem in Hinz [2]. (We used the estimate

$$\int_T^{2T} |M_x(\tfrac{1}{2}+it)|^2\,dt \ll T\sum_{n\leqslant x} b^2(n)n^{-1} \ll T(\log x)^4,$$

where the first $\ll$ follows from Theorem 6.1 of Montgomery [4].) Also with

$\delta = (\log T)^{-1}$ we have

$$\int_T^{2T} |\varphi(1+\delta+it)|^2\,dt = \int_T^{2T} \Big|\sum_{n>x} g(n)\,n^{-1-\delta-it}\Big|^2\,dt$$

$$\ll T\sum_{n>x} g^2(n)\,n^{-2-2\delta}+$$

$$+\sum_{\substack{m,n>x\\ m\neq n}} \big(g(m)(mn)^{-(1+\delta)/2}|\log(m/n)|^{-1/2}\big)\big(g(n)(mn)^{-(1+\delta)/2}|\log(m/n)|^{-1/2}\big)$$

$$\ll Tx^{-1}\sum_{n=1}^{\infty} g^2(n)\,n^{-1-2\delta}+\sum_{\substack{m,n>x\\ m\neq n}} g^2(m)(mn)^{-1-\delta}|\log(m/n)|^{-1},$$

using Hölder's inequality for the last sum. Now $g^2(n) \leqslant d^6(n) \leqslant d_{64}(n)$ and we obtain

$$(3)\quad \int_T^{2T} |\varphi(1+\delta+it)|^2\,dt \ll Tx^{-1}\sum_{n=1}^{\infty} d_{64}(n)\,n^{-1-2\delta}+\sum_{m=1}^{\infty} d_{64}(m)\log m\cdot m^{-1-\delta}$$

$$\ll Tx^{-1}\zeta^{64}(1+2\delta)+\big|(\zeta^{64}(1+\delta))'\big| \ll (\log T)^{65}$$

where we have denoted $\left(\frac{d}{ds}f^k(s)\right)_{s=\alpha}$ by $(f^k(\alpha))'$.

We now consider the zeros $\varrho = \beta+i\gamma$ of $\zeta(s)$ with $\sigma_0 \leqslant \beta \leqslant 1$, $T \leqslant \gamma \leqslant 2T$ where $\sigma_0 > 1/2$. Then for $T \geqslant T(\sigma_0)$, σ_0 is $> 1/2+(\log T)^{-1}$ and so $\beta > 1/2+(\log T)^{-1}$. Denote by R the boundary of the rectangle $1/2-\beta \leqslant u \leqslant 1+\delta-\beta$, $-\log^2 T \leqslant v \leqslant \log^2 T$ in the w $(=u+iv)$ plane. Observe

$$(4)\qquad \frac{1}{2\pi i}\int_R \varphi(\varrho+w)\,y^w\,\Gamma(w)\,dw = -1$$

as $w=0$ is inside R and $\varphi(\varrho) = -1$. On the horizontal sides of R, $|v| = \log^2 T$ and the integral is bounded by $\exp(-\frac{1}{10}\log^2 T)$ provided we choose

$$(5)\qquad |d|^{10},\ y < \exp(\tfrac{1}{100}\log^2 T),$$

since for $1/2 \leqslant \beta+u \leqslant 1+\delta$, $\gamma+v \ll\, \gg T$

$$|\varphi(\varrho+w)| \leqslant |\zeta_K(\varrho+w)\,M_x(\varrho+w)|+1 \ll |d|\,T^2$$

as we can take the bound

$$|\zeta_K(\sigma+it)| \leqslant |dt|,\qquad \sigma \geqslant 1/2,\qquad |t| \geqslant 100$$

from Theorem 4 of Rademacher [5]. Hence we get from (4) that

$$(6)\quad 1 \ll y^{1/2-\beta}|\tfrac12-\beta|^{-1}\int_{\gamma-\log^2 T}^{\gamma+\log^2 T} |\varphi(\tfrac12+it)|\,dt$$

$$+\,y^{1+\delta-\beta}|1+\delta-\beta|^{-1}\int_{\gamma-\log^2 T}^{\gamma+\log^2 T} |\varphi(1+\delta+it)|\,dt.$$

Let

$$M_1 = M_1(\varrho) = \int_{\gamma-\log^2 T}^{\gamma+\log^2 T} |\varphi(\tfrac12+it)|\,dt+1/T$$

and

$$M_2 = M_2(\varrho) = \int_{\gamma-\log^2 T}^{\gamma+\log^2 T} |\varphi(1+\delta+it)|\,dt+1/T$$

and let $y = (M_1/M_2)^{2/(1+2\delta)}$. (Observe $M_2 \geqslant 1/T$ and $M_1 \ll |d|\,T\log^3 T$ by (2) and so y satisfies (5).) As $|\frac12-\beta|$, $|1+\delta-\beta| \geqslant (\log T)^{-1}$, (6) gives

$$(7)\qquad 1 \ll \log T\big(M_1^{(1+\delta-\beta)}M_2^{(\beta-1/2)}\big)^{2/(1+2\delta)}.$$

Now divide the region $1 \geqslant \sigma \geqslant \sigma_0$, $T \leqslant t \leqslant 2T$ into rectangles of height $2\log^2 T$ leaving possibly a rectangle of height less than $2\log^2 T$ at the top. Take one set of alternate rectangles and pick a zero each from these rectangles whenever available. Now $\sum_{\varrho}$ will mean summation over the zeros thus chosen. Now we have

$$(8)\quad \sum_{\varrho}\Big(\int_{\gamma-\log^2 T}^{\gamma+\log^2 T} |\varphi(\tfrac12+it)|\,dt+T^{-1}\Big) \leqslant \int_T^{2T} |\varphi(\tfrac12+it)|\,dt+T^{-1}\,T(\log T)^{-2}$$

$$\leqslant C(\varepsilon)\,|d|^{1/2+\varepsilon}\,T\log^3 T,$$

using (2) and

$$(9)\quad \sum_{\varrho}\Big(\int_{\gamma-\log^2 T}^{\gamma+\log^2 T} |\varphi(1+\delta+it)|\,dt+T^{-1}\Big)^2 \ll \sum_{\varrho}\Big(\int_{\gamma-\log^2 T}^{\gamma+\log^2 T}\Big)^2+\sum_{\varrho} T^{-2}$$

$$\ll 2\log T\int_T^{2T} |\varphi(1+\delta+it)|^2\,dt+O(T^{-1})$$

$$\ll (\log T)^{66},$$

using (3). Let M_1' and M_2' be two fixed quantities, to be chosen later. Denote by N_1 the number of zeros from the chosen set for which $M_1 \geqslant M_1'$ and by N_2 those for which $M_2 \geqslant M_2'$. From (8) and (9) we get

$$N_1 M_1' \leqslant C(\varepsilon)\,|d|^{1/2+\varepsilon}\,T\log^3 T \quad\text{and}\quad N_2 M_2'^2 \leqslant (\log T)^{66}$$

so that

$$N_1 \leqslant M_1'^{-1} C(\varepsilon)|d|^{1/2+\varepsilon} T \log^3 T \quad \text{and} \quad N_2 \leqslant M_2'^{-2} (\log T)^{66}.$$

We would choose M_1' and M_2' satisfying

$$M_1' = M_2'^2 C(\varepsilon)|d|^{1/2+\varepsilon} T$$

so that

$$N_1 + N_2 \leqslant 2M_2'^{-2} (\log T)^{66}. \tag{10}$$

We observe from (7) that if we choose

$$M_2' = (\log T)^{-2} \left(C(\varepsilon)|d|^{1/2+\varepsilon} T\right)^{-(1-\sigma_0+\delta)/(3/2-\sigma_0+2\delta)}$$

we would have included every zero from the chosen set either in N_1 or in N_2, since

$$\frac{1-\sigma_0+\delta}{3/2-\sigma_0+2\delta} \geqslant \frac{1-\beta+\delta}{3/2-\beta+2\delta}$$

for all $\beta \geqslant \sigma_0$.

With this value of M_2' we obtain from (10) that

$$N_1 + N_2 \leqslant 2(\log T)^{70} \left(C(\varepsilon)|d|^{1/2+\varepsilon} T\right)^{2(1-\sigma_0+\delta)/(3/2-\sigma_0+2\delta)}.$$

The same estimate holds for a corresponding set of zeros from the other set of alternate rectangles and each rectangle contains less than $2\log^2 T(\log T + \log|d|) \leqslant \log^5 T$ zeros (using (5)) (this estimate holds for the left out rectangle at the top also) and hence

$$N_K(\sigma_0, T, 2T) \leqslant 2(\log T)^{75} \left(C(\varepsilon)|d|^{1/2+\varepsilon} T\right)^{2(1-\sigma_0+\delta)/(3/2-\sigma_0+2\delta)}.$$

Now

$$\frac{1-\sigma_0+\delta}{3/2-\sigma_0+2\delta} = 2\frac{1-\sigma_0}{3-2\sigma_0} + O(\delta)$$

and we have chosen $\delta = (\log T)^{-1}$. Hence for $1/2 < \sigma_0 \leqslant 1$,

$$N_K(\sigma_0, T, 2T) \leqslant \left(C(\varepsilon)|d|^{1/2+\varepsilon} T\right)^{4(1-\sigma_0)/(3-2\sigma_0)} (\log T)^{76}.$$

We can now divide the interval $(0, T)$ into, $[T, \frac{1}{2}T]$, $[\frac{1}{2}T, (\frac{1}{2})^2 T], \ldots$ $\ldots, [T/2^{r-1}, T/2^r]$ with $r \leqslant 2\log T$ for $T/2^r \geqslant T_0(\sigma_0)$. Also for $(0, T/2^r]$ the number of zeros is $< C(\sigma_0) < (\log T)^{76}$ if $T > T_1(\sigma_0)$ and the lemma is now proved.

Remark 1. Here we have obtained explicitly the dependence of the bound for $N_K(\sigma_0, T)$ on d. Heath-Brown [1] has obtained the bound in Lemma 2 with $C(\sigma_0, K)$ in place of our 78. But he has a smaller exponent for T when $3/4 < \sigma \leqslant 1-\varepsilon$.

Remark 2. Let K_1 be an extension of the field Q with $[K_1:Q] = k$. Then we have for $\sigma > 1-k^{-1}$, $T \geqslant T_0(\sigma, K)$

$$N_{K_1}(\sigma, T) \leqslant T^{2(1-\sigma)/(1-k^{-1}-\sigma)} (\log T)^C$$

with $C = C(K_1)$ for $k \geqslant 2$.

LEMMA 3. *We have for* $1/2 \leqslant \sigma < 1$, $a > 1$,

$$\sum_{a^{-1}x < p \leqslant ax} p^{-\sigma}\left(b - |\log(p/x)|\right)$$
$$= x^{1-\sigma}(\log x)^{-1}\left\{2(b-\log a)\frac{\sinh((1-\sigma)\log a)}{1-\sigma} + \right.$$
$$\left. + \left(\frac{2\sinh(\frac{1}{2}(1-\sigma)\log a)}{1-\sigma}\right)^2\right\} + O\left(x^{1-\sigma}(\log x)^{-2}\right).$$

Proof. This follows from the prime number theorem in the form

$$\vartheta(x) = x + O\left(x(\log x)^{-3}\right).$$

In fact we have

$$\sum_{b_1 x < p \leqslant ax} p^{-\sigma}\log p = \vartheta(u)u^{-\sigma}\Big|_{b_1 x}^{ax} + \sigma\int_{b_1 x}^{ax} \vartheta(u)u^{-\sigma-1}\,du$$
$$= x^{1-\sigma}(a^{1-\sigma} - b_1^{1-\sigma})(1-\sigma)^{-1} + O\left(x^{1-\sigma}(\log x)^{-3}\right).$$

Also

$$\sum_{b_1 x < p \leqslant ax} p^{-\sigma} = \vartheta(u)u^{-\sigma}(\log u)^{-1}\Big|_{b_1 x}^{ax} + \int_{b_1 x}^{ax} \left(u^{-1-\sigma}(\log u)^{-2} + \right.$$
$$\left. + \sigma u^{-1-\sigma}(\log x)^{-1}\right)\vartheta(u)\,du$$
$$= x^{1-\sigma}(\log x)^{-1}\{(a^{1-\sigma} - b_1^{1-\sigma})(1-\sigma)^{-1} -$$
$$-(\log x)^{-1}(a^{1-\sigma}\log a - b_1^{1-\sigma}\log b_1)(1-\sigma)^{-1} +$$
$$+(\log x)^{-1}(a^{1-\sigma} - b_1^{1-\sigma})(1-\sigma)^{-2}\} + O\left(x^{1-\sigma}(\log x)^{-3}\right).$$

Now the lemma follows by splitting the sum there to $a^{-1}x < p \leqslant x$ and $x < p \leqslant ax$ and applying the above estimations.

LEMMA 4. *For* $-1 \leqslant \sigma \leqslant 2$ *we have*

$$\log\zeta_K(s) = \sum_{|t-\gamma| \leqslant 1} \log(s-\varrho) + O(\log|dt|)$$

with $-\pi < \operatorname{Im}\log(s-\varrho) \leqslant \pi$ *for any* $t \neq$ *the ordinate of a zero of* $\zeta_K(s)$. (Here the summation is over those $\varrho = \beta + i\gamma$ with $|t-\gamma| \leqslant 1$.)

Proof. The proof is exactly similar to the proof of the corresponding result in the case of Riemann zeta function. (See Theorem 9.6 (B) Titchmarsh [7].)

Now we start with the proof of the theorems. We have from (1)

$$\log\zeta_K(s) = \sum_{\mathfrak{p},m} m^{-1} N(\mathfrak{p})^{-ms} = \sum_{n=2}^{\infty} r(n)\, n^{-s}, \qquad \sigma > 1, \tag{11}$$

where $r(n) = 0$ for any n having at least two distinct prime factors and $r(n) \leqslant 2$ for all n, since there can be at most two distinct prime ideals dividing a rational prime p in a quadratic extension K. Now for $\varepsilon > 0$, $0 < a < 1-\varepsilon$ we have by Lemma 2

$$N_K(\sigma_0, T) \leqslant \tfrac{1}{2} T^{1-\{a(2\sigma_0-1)/(3-2\sigma_0)\}} \tag{12}$$

whenever

$$T \geqslant T_0(\sigma_0, \varepsilon) + |d|^{\frac{1}{2}(4-4\sigma_0+\varepsilon(2\sigma_0-1))/\{(1-a-\varepsilon)(2\sigma_0-1)\}}. \tag{13}$$

Let us divide the rectangle $\sigma_0 \leqslant \sigma \leqslant 1$, $\frac{1}{2} T^{\eta} \leqslant t \leqslant T$ ($\eta = \eta(\sigma_0)$ to be chosen later) into smaller rectangles of height T^{η}. The number of rectangles formed is $R = [T^{1-\eta} - 1/2]$, leaving possibly a rectangle of height less than T^{η} at the top. Consider the integers $1 \leqslant r \leqslant R$. With $\delta = 1/6$ Lemma 1 inform us that there are $[6^{-M}(R+1)]$ of these integers $1 \leqslant r \leqslant R$ with the property that (our $\theta_m = (2\pi)^{-1} T^{\eta} \log p_m$)

$$\|(2\pi)^{-1} r T^{\eta} \log p_m\| \leqslant 1/6 \quad \text{for} \quad 1 \leqslant m \leqslant M. \tag{14}$$

We choose η and M subject to

$$[6^{-M}(R+1)] \geqslant [6^{-M} T^{1-\eta}] > \tfrac{1}{2} T^{1-\{a(2\sigma_0-1)/(3-2\sigma_0)\}}. \tag{15}$$

Hence we have constructed more rectangles than the number of zeros (see (12)) of $\zeta_K(s)$ in $\sigma_0 \leqslant \sigma \leqslant 1$, $0 \leqslant t \leqslant T$ and hence there is at least one rectangle in which $\zeta_K(s)$ is zero free and satisfying (14). Now (15) is satisfied if $\eta = b(2\sigma_0-1)/(3-2\sigma_0)$, $0 < b < a$ and

$$M = \left[(a-b)\frac{2\sigma_0-1}{3-2\sigma_0}\frac{\log T}{\log 6}\right]. \tag{16}$$

Now we have a rectangle $\sigma_0 \leqslant \sigma \leqslant 1$, $r_1 T^{\eta} - \frac{1}{2} T^{\eta} < t < r_1 T^{\eta} + \frac{1}{2} T^{\eta}$ for an r_1 with $1 \leqslant r_1 \leqslant R$, free from zeros of $\zeta_K(s)$ and we can consider $\log\zeta_K(s)$ in this rectangle. Let $t_0 = r_1 T^{\eta}$, $s_0 = \sigma_0 + it_0$ and

$$I = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} \log\zeta_K(s_0+s)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2 e^{\beta s}\, ds$$
$$= \sum \frac{r(n)}{n^{s_0}}\frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2 e^{(\beta-\log n)s}\, ds,$$

using the expansion of $\log\zeta_K(s)$ from (11). Now move the line of integration in

the last integral to $\sigma = 0$ and we obtain

$$I = \sum \frac{r(n)}{n^{s_0}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2\sin\alpha t}{t}\right)^2 e^{(\beta-\log n)it}\, dt \tag{17}$$
$$= \sum \frac{r(n)}{n^{s_0}} \max(0,\ 2|\alpha| - |\beta - \log n|)$$

using the fact that

$$\int_{-\infty}^{\infty}\left(\frac{2\sin\alpha t}{t}\right)^2 \cos\lambda t\, dt = 2\pi\max(0,\ 2|\alpha|-|\lambda|).$$

Now let $\tau = \log^2 T$. We have

$$\int_{1\pm i\tau}^{1\pm i\infty} \log\zeta_K(s_0+s)\left(\frac{e^{\alpha s}-e^{-\alpha s}}{s}\right)^2 e^{\beta s}\, ds \ll e^{2|\alpha|+|\beta|}\tau^{-1}. \tag{18}$$

Hence with the error in (18) we can write the integral I from $1-i\tau$ to $1+i\tau$. Now move the line of integration to $\sigma = 0$. On the horizontal sides the integral is

$$\int_0^1 \log\zeta_K(s_0+\sigma\pm i\tau)\left(\frac{e^{\alpha(\sigma\pm i\tau)}-e^{-\alpha(\sigma\pm i\tau)}}{\sigma\pm i\tau}\right)^2 e^{\beta(\sigma\pm i\tau)}\, d\sigma \tag{19}$$
$$\ll e^{2|\alpha|+|\beta|}(\log t_0 + \log|d|)\tau^{-2},$$

using Lemma 4. Now from (17), (18) and (19) we get

$$\int_{-\tau}^{\tau} \log\zeta_K(s_0+it)\left(\frac{e^{i\alpha t}-e^{-i\alpha t}}{it}\right)^2 e^{i\beta t}\, dt \tag{20}$$
$$= 2\pi\sum r(n)\, n^{-s_0}\max(0,\ 2|\alpha|-|\beta-\log n|) + O\big(e^{2|\alpha|+|\beta|}(\log T)^{-2}\big).$$

We let $\alpha = 1/2$. Give the values $\beta = -\log x$, 0, $\log x$ and multiply the resulting equations from (20) respectively by $\frac{1}{2}e^{-i\theta}$, 1, $\frac{1}{2}e^{i\theta}$ and add (corresponding to the first two values of β the $\sum$ on right side is $O(1)$) to obtain

$$\int_{-\tau}^{\tau} \log\zeta_K(s_0+it)\left(\frac{2\sin(t/2)}{t}\right)^2 \left(\tfrac{1}{2}e^{-i(\theta+\log x)} + 1 + \tfrac{1}{2}e^{i(\theta+\log x)}\right) dt$$
$$= 2\pi e^{i\theta}\sum_{|\log(n/x)|\leqslant 1} r(n)\, n^{-s_0}\big(1-|\log(n/x)|\big) + O\big(1 + x(\log T)^{-2}\big).$$

Thus we have

$$\operatorname{Re} e^{-i\theta}\int_{-\tau}^{\tau} \log\zeta_K(s_0+it)\left(\frac{\sin(t/2)}{t}\right)^2 \big(1+\cos(\theta+t\log x)\big)\, dt \tag{21}$$
$$= \tfrac{1}{2}\pi\sum_{|\log(n/x)|\leqslant 1} r(n)\, n^{-\sigma_0}\cos(t_0\log n)\big(1-|\log(n/x)|\big) + O\big(1+x(\log T)^{-2}\big).$$

Now

$$\int_{-\tau}^{\tau}\left(\frac{\sin(t/2)}{t}\right)^2(1+\cos(\theta+t\log x))\,dt \leqslant 2\int_{-\infty}^{\infty}\left(\frac{\sin(t/2)}{t}\right)^2 dt = \pi.$$

Hence (21) implies that there is a t_1 with $-\tau \leqslant t_1 \leqslant \tau$ for which

(22) $\operatorname{Re} e^{-i\theta}\log\zeta_K(s_0+it_1)$

$$\geqslant \tfrac{1}{2}\sum_{|\log(n/x)|\leqslant 1} r(n)n^{-\sigma_0}\cos(t_0\log n)\big(1-|\log(n/x)|\big)+O\big(1+x(\log T)^{-2}\big).$$

Let p be a rational prime such that there is a prime ideal $\mathfrak{p}$ in K with $N(\mathfrak{p}) = p$. We have for $n = p^2, p^3, \ldots$

$$\sum_{|\log(n/x)|\leqslant 1} r(n)n^{-\sigma_0} \leqslant 2\sum_{p\leqslant 2x^{1/2}}\ \sum_{\substack{l\geqslant 2\\ e^{-1}x\leqslant p^l\leqslant ex}} p^{-l\sigma_0}$$

$$\leqslant 2x^{-\sigma_0}\sum_{p\leqslant 2x^{1/2}}\left(\left[\frac{\log x+1}{\log p}\right]-\left[\frac{\log x-1}{\log p}\right]+1\right)$$

$$\leqslant 14x^{-\sigma_0}\cdot 2x^{1/2}/\log x = O\big((\log x)^{-1}\big) \quad \text{for } \sigma_0 \geqslant 1/2.$$

On the other hand if $\mathfrak{p}$ is such that $N(\mathfrak{p}) = p^2$, a similar bound holds for summation over $n = p^2, p^4, p^6, \ldots$ for $e^{-1}x \leqslant n \leqslant ex$. Hence we can write (22) in the form, for any real θ, we have a t_1 with $-\tau \leqslant t_1 \leqslant \tau$ satisfying

(23) $\operatorname{Re} e^{-i\theta}\log\zeta_K(s_0+it_1)$

$$\geqslant \tfrac{1}{2}\sum_{e^{-1}x\leqslant p\leqslant ex} r(p)p^{-\sigma_0}\cos(t_0\log p)\big(1-|\log(p/x)|\big)+O\big(1+x(\log T)^{-2}\big).$$

Now using (14) we see that $\cos(t_0\log p) \geqslant 1/2$ for all p in the above summation if we choose x subject to the number of primes M in $(e^{-1}x, ex)$ satisfying (16). Now (16) is satisfied if

$$x \sim \frac{a-b}{2\sinh 1}\,\frac{2\sigma_0-1}{3-2\sigma_0}\,\frac{\log T}{\log 6}\log\log T, \tag{24}$$

and with this choice of x the error term in (23) is $O(1)$. Thus (23) reads

$$\operatorname{Re} e^{-i\theta}\log\zeta_K(s_0+it_1) \geqslant \tfrac{1}{4}\sum_{e^{-1}x\leqslant p\leqslant ex} r(p)p^{-\sigma_0}\big(1-|\log(p/x)|\big)+O(1).$$

Now we observe that the prime ideals $\mathfrak{p}$ with $N(\mathfrak{p}) = p \leqslant X$ is asymptotic to $X/\log X$, that is $\sum_{p\leqslant X} r(p) \sim X/\log X$ which is the same density as for rational primes. Also we have $\vartheta_K(x) = \sum_{N\mathfrak{p}\leqslant x}\log N\mathfrak{p} = x+O\big(x(\log x)^{-3}\big)$. (The number of primes dividing d cannot exceed $\log|d|/\log\log|d|$ which is less than $\log T/\log\log T$ by the choice of large T according to (13) and so would not affect the density of prime ideals $\mathfrak{p}$ for which $N(\mathfrak{p}) = p$.) Hence using Lemma 3 we obtain

$$\operatorname{Re} e^{-i\theta}\log\zeta_K(s_0+it_1) \geqslant \frac{1}{4}\left\{\left(\frac{2\sinh\big((1-\sigma_0)/2\big)}{1-\sigma_0}\right)^2-\varepsilon\right\}x^{1-\sigma_0}/\log x$$

and using the value of x from (24) and choosing $a = 2b = 2(1-\varepsilon)/3$ we get Theorem 1.

Towards the proof of Theorem 2 we make the following observations. We assume Riemann hypothesis for $\zeta_K(s)$ and so no zero of $\zeta_K(s)$ lie on $\sigma > 1/2$, $|t| > 0$. Hence in the relation (15) we need η and M to satisfy

$$6^{-M}T^{1-\eta} \geqslant 1.$$

Hence we can choose $M = [(1-\eta)\log T/\log 6]$ and the choice of x in (24) becomes

$$x \sim \tfrac{1}{2}(1-\eta)(\operatorname{cosech} 1)\log T\log\log T/\log 6.$$

Also we can take $T^\eta = \log^3 T$ that is $\eta = 3\log\log T/\log T$. By proceeding exactly as in the case of Theorem 1, we get the relevant constant as

$$4(\sinh^2(1/4))\big(2(\sinh 1)\log 6\big)^{-1/2}-\eta$$

for any $\eta > 0$ provided $T \geqslant T_0(\eta)$ and we have proved Theorem 2.

Remark. We have in the case $[K_1:Q] = k > 2$ that for $1-k^{-1} < \sigma < 1$, $T \geqslant T_0(\sigma_0, K)$ and any real θ, there is a t_0 such that $0 < t_0 \leqslant T$ satisfying

$$\operatorname{Re} e^{-i\theta}\log\zeta_{K_1}(\sigma_0+it_0) \geqslant C(K_1, \sigma_0)(\log T)^{1-\sigma_0}(\log\log T)^{-\sigma_0}. \tag{25}$$

We have an analogue of Theorem 2 also, if we assume $\zeta_{K_1}(s)$ does not have zeros in the region $\sigma > 1-k^{-1}$, $|t| > 0$, but with an implicit constant again.

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