as in the proof of Lemma 4.8 in [1] (so that, the basic intervals $\mathfrak{m}_{a,q}$'s are defined with $1 \leqslant q \leqslant P^{(5/16)}$, $|\alpha - a/q| \leqslant (qQ)^{-1}$). Let the λ_i's be as in Lemma 10.1 (and the f_i's be defined correspondingly). As already established in [1] (by using Weyl's inequality),

$$(11.1) \qquad f(\alpha) \ll P^{1-(1/16)+\delta_0} \quad \text{if} \quad \alpha \in m.$$

Write

$$(11.2) \qquad r_5(N) = \int_{Q^{-1}}^{1+Q^{-1}} f^4(\alpha)\{f(\alpha) f_1(\alpha) \ldots f_8(\alpha)\}^2 e(-N\alpha)\, d\alpha.$$

With α_9 defined by (10.2), we see that (from Lemma 10.1) the contribution to $r_5(N)$ from m is

$$(11.3) \qquad \ll P^{4(1-1/16+\delta_0)}(PP_1 \ldots P_8)^2 P^{-5\alpha_9+\delta_0} \ll P^{-5-\delta_0}(P^4)(PP_1 \ldots P_8)^2$$

since $5\alpha_9 + (4/16) > 5$.

For the treatment of $\mathfrak{m}$, and the transition to the singular series, we make the obvious modifications in [2]. Davenport uses 7 fifth powers in the singular series. (This can however, be simplified by using a larger number of fifth powers.) In place of estimating $(f^7 - g^7)$, and its integral over $\mathfrak{m}$, we estimate $(f^6 f_8 - g^6 g_8)$ and its integral by starting with

$$f^6 f_8 - g^6 g_8 = (f^6 - g^6) f_8 + g^6 (f_8 - g_8),$$

and using (4.30), (4.31), (4.32) in [1]. (Note that $\mathfrak{m}_{a,q}$'s are defined with $q \leqslant P^{1/2}$ in [2], and, this hardly makes any difference.) It would then follow (as in [2]) from (11.3), that $r_5(N) \gg P(P_1 \ldots P_8)^2$, proving that $G(5) \leqslant 22$.

Reference

[2] H. Davenport, *On Waring's problem for fifth and sixth powers*, Amer. J. Math. 64 (1942), pp. 199–207.

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Some problems involving powers of integers

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1. Introduction. There are a number of famous problems which appear to be questions about the distribution of powers of integers. For example, Catalan's conjecture that 8 and 9 are the only powers which differ by 1, and even Fermat's last theorem, have implications of this sort. Many such questions, including Catalan's conjecture, can now be resolved in principle by invoking lower bounds for linear forms in the logarithms of algebraic numbers. (This technique and its applications, which include many powerful results on Diophantine equations and inequalities, are surveyed in [5].) The main point of this paper is a lower bound for simultaneous linear forms in logarithms with some interesting applications to powers and integers which are almost powers.

Consider first the problem of estimating the number of perfect powers in a short interval. J. Turk [7] has shown that the interval $[N, N+N^{1/2}]$ can contain at most $c(\log N)^{1/2}$ powers, for some positive constant c. This is the appropriate interval to examine because any longer interval $[N, N+N^{(1/2)+\varepsilon}]$, with $\varepsilon > 0$, trivially contains $\frac{1}{2}N^{\varepsilon}(1+o(1))$ squares. This question is also discussed in [6], where it arises in the context of exponential Diophantine equations. An appendix to [6] explains how to use linear forms in logarithms, continued fractions and some brute force computation to find all 21 solutions of the inequality $|p^a - q^b| < p^{a/2}$ in positive integers a and b and primes p and q with $p < q < 20$. Probably, the number of prime powers in any interval $[N, N+N^{1/2}]$ is bounded. The computations in [6] give just one example, namely 11^2, 5^3 and 2^7, of three prime powers in such an interval. In this direction, we shall prove:

THEOREM 1. *The interval* $[N, N+N^{1/2}]$ *with* $N \geqslant 16$, *say, contains at most*

$$\exp\left(40(\log\log N \log\log\log N)^{1/2}\right)$$

perfect powers.

Similar questions can be posed about numbers which are almost powers, or have various other assigned multiplicative structures. Such numbers still tend to be sparse. We give the following definition, from a number of plausible alternatives, to illustrate the results that can be obtained. Given $\delta > 0$, let P_δ denote the set of numbers of the shape $z = mx^a$ with a, m and x integers, $a \geqslant 2$ and $m \leqslant \exp(a^{1-\delta})$.

THEOREM 2. *The interval* $[N, N+N^{1/2}]$ *with* $N \geqslant 16$, *say, contains at most*

$$\exp\exp(80\delta^{-1}(\log\log N \log\log\log N)^{1/2})$$

elements of P_δ.

Finally, we consider the Diophantine equation

$$N = \frac{x^a-1}{x-1} = \frac{y^b-1}{y-1},$$

to be solved in integers a, b, x and y with $a > b > 2$ and $y > x > 1$. The only known solutions appear to be

$$31 = \frac{2^5-1}{2-1} = \frac{5^3-1}{5-1}$$

and

$$8191 = \frac{2^{13}-1}{2-1} = \frac{90^3-1}{90-1}.$$

Mąkowski and Schinzel [4] have obtained a number of results showing that the equation has no other solutions when one or more of the variables is restricted. The equation has also appeared, marking a certain exceptional case, in work of Bateman and Stemmler [2] on Waring's problem in number fields. Probably the equation has only finitely many solutions, but in any case, the number of solutions for a given N should be bounded. However, we can only prove that there are relatively few solutions.

THEOREM 3. *Suppose* $\varepsilon > 0$. *For a given integer* N, *the equation* $N = (x^a-1)/(x-1)$ *has at most* $c(\log N)^{(1/2)+\varepsilon}$ *solutions in integers* a *and* x *with* $a > 2$ *and* $x > 1$. (*The positive constant* c *depends only on* ε.)

2. Linear forms in logarithms. At the present time, the strongest results on the problems discussed in Section 1 have been obtained by exploiting lower bounds for linear forms in the logarithms of algebraic numbers. The theorems of Section 1 are applications of a new theorem of this type.

We shall consider the linear forms

$$\Lambda_i = \beta_{i0}+\beta_{i1}\log\alpha_1 + \ldots + \beta_{in}\log\alpha_n \quad (1 \leqslant i \leqslant t),$$

where the α's and β's denote algebraic numbers. We assume that the α's are non-zero and multiplicatively independent, that the matrix (β_{ij}) formed by the β's has rank t, and that the logarithms have their principal values.

Further, we suppose that the height of α_j is at most $A_j (\geqslant 4)$, the height of β_{ij} is at most $B(\geqslant 4)$, and that the field K generated by the α's and β's over the rationals has degree at most d. We set

$$\Omega = \log A_1 \ldots \log A_n.$$

With this notation, we prove:

THEOREM 4.

$$\max_{1\leqslant i\leqslant t} |\Lambda_i| > \exp(-C(\Omega\log\Omega)^{1/t}\log(B\Omega)), \quad \textit{where} \quad C = (16nd)^{200n}.$$

In case $t = 1$, the theorem gives a lower bound for a single linear form which is essentially the principal result proved by A. Baker in [1]. Here, it is unnecessary to assume that the α's are multiplicatively independent. This assumption, or something similar, is needed in our theorem to guarantee that the linear forms are independent. Our proof follows, as closely as possible, the work of Baker in [1] and we shall be content to give only an outline of the argument. No particular significance should be attached to the constant 200 appearing in C and the related constants in Theorems 1 to 3. The value of C given here has been chosen for historical reasons.

A result in the same spirit as Theorem 4 was obtained by Ramachandra (J. Austral. Math. Soc. 10 (1969), pp. 197–203) at an early stage of the development of the theory of linear forms in logarithms. By adapting Ramachandra's comments on his result to the present context, we may perhaps view Theorem 4 as support for the belief that the quantity $\Omega = \log A_1 \ldots \log A_n$ in the current lower bounds for linear forms should be replaced by $\log A_1 + \ldots + \log A_n$.

3. Proof of Theorem 4. To begin, we assume that the linear forms have the special shape

$$\Lambda_i = \beta_{i0}+\beta_{i1}\log\alpha_1 + \ldots + \beta_{is}\log\alpha_s - \log\alpha_{s+i} \quad (1 \leqslant i \leqslant t)$$

with $s+t = n$ $(\geqslant 3)$, that the α's and β's are elements of a field K with degree at most d $(\geqslant 8)$, that the numbers $\beta_{1j}, \ldots, \beta_{tj}$ have a common denominator not exceeding B for each fixed j, and that

$$\max_{1\leqslant i\leqslant t} |\Lambda_i| < \exp(-C(\Omega\log\Omega')^{1/t}\log(B\Omega))$$

with $\Omega' = \log A_1 \ldots \log A_s$ and $C = (nd)^{80n}$. We proceed to show that $\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}$ generate an extension of K of degree less than q^n for some prime q.

We define

$$L = (k\Omega\log\Omega')^{1/t}, \quad h = L_{-1}+1 = [\log(BL)],$$

$$L_j = [k^{-\varepsilon}L/\log A_j] \quad (0 \leqslant j \leqslant n),$$

where $\varepsilon = 1/(3n)$, $k = (nd)^{40n}$ and $A_0 = \Omega'$. We introduce the function

$$\Phi(z_0, \ldots, z_s) = \sum_{(\lambda)} p(\lambda)\big(\Delta(z_0+\lambda_{-1}; h)\big)^{\lambda_0+1} \exp\big((\lambda_{s+1}\beta_{10}+ \ldots +\lambda_n\beta_{t0})z_0\big)\alpha_1^{\gamma_1 z_1} \ldots \alpha_s^{\gamma_s z_s},$$

where the $p(\lambda) = p(\lambda_{-1}, \ldots, \lambda_n)$ are integers to be determined later,

$$\Delta(x; h) = (x+1)\ldots(x+h)/h!,$$

$$\gamma_j = \lambda_j+\lambda_{s+1}\beta_{1j}+ \ldots +\lambda_n\beta_{tj} \quad (1 \leqslant j \leqslant s),$$

and the sum is taken over all integral vectors $\lambda = (\lambda_{-1}, \ldots, \lambda_n)$ with $0 \leqslant \lambda_j \leqslant L_j$ $(-1 \leqslant j \leqslant n)$. Further, for any non-negative integers $m_0, \ldots, m_s$, we denote by $f(z)$ the function

$$f(z) = (\log\alpha_1)^{-m_1}\ldots(\log\alpha_s)^{-m_s}\left(\frac{\partial}{\partial z_0}\right)^{m_0}\ldots\left(\frac{\partial}{\partial z_s}\right)^{m_s}\Phi(z_0, \ldots, z_s),$$

evaluated at $z_0 = \ldots = z_s = z$, and we denote by $g(z)$ the function obtained from $f(z)$ by substituting α_{s+i} for

$$\alpha'_{s+i} = e^{\beta_{i0}}\alpha_1^{\beta_{i1}}\ldots\alpha_s^{\beta_{is}} \quad (1 \leqslant i \leqslant t).$$

Thus

$$g(z) = \sum_{(\lambda)} p(\lambda)\psi(\lambda; z)\alpha_1^{\lambda_1 z}\ldots\alpha_n^{\lambda_n z}\gamma_1^{m_1}\ldots\gamma_s^{m_s},$$

where

$$\psi(\lambda; z) = \left(\frac{d}{dz}+\lambda_{s+1}\beta_{10}+ \ldots +\lambda_n\beta_{t0}\right)^{m_0}\big(\Delta(z+\lambda_{-1}; h)\big)^{\lambda_0+1}.$$

We can choose the integers $p(\lambda)$, not all zero and with absolute values at most e^{8Lh}, so that $g(l) = 0$ for all integers l with $1 \leqslant l \leqslant hk^{(1/2)\varepsilon}$ and all non-negative integers $m_0, \ldots, m_s$ with $m_0+ \ldots +m_s \leqslant L$. Indeed, these requirements amount to a number of linear equations for the unknowns $p(\lambda)$ and we can obtain a non-trivial solution by a version of Siegel's lemma such as that given by Waldschmidt [8], Lemma 1.3.1. The estimates are made as in Lemma 7 of [1]. We derive various inequalities involving the functions $f(z)$ and $g(z)$ as in Lemma 8 of [1]. For any non-negative integers $m_0, \ldots, m_s$ with $m_0+ \ldots +m_s \leqslant L$, we have

$$|f(z)| \leqslant \exp\{12L(h+ndk^{-\varepsilon}|z|)\}.$$

Next, let l and q be positive integers with $l \leqslant hk^{1/2}$ and $q \leqslant k^{\varepsilon}$. Our assumptions imply that

$$|\log\alpha'_{s+i}-\log\alpha_{s+i}| < (B\Omega)^{-kL} \quad (1 \leqslant i \leqslant t),$$

for suitable determinations of the $\log\alpha'_{s+i}$, so we have

$$|f(l/q)-g(l/q)| < (B\Omega)^{-kL/2}.$$

Furthermore, if $g(l/q) \neq 0$, then

$$|f(l/q)| > \exp\{-24Ldq^n(h+ndk^{-\varepsilon}l)\}.$$

These inequalities allow us to perform an inductive extrapolation and interpolation argument as in Lemmas 9 and 10 of [1]. From this, we obtain $g(l/q) = 0$ for all integers l and q with $1 \leqslant l \leqslant hq$, $1 \leqslant q \leqslant k^{\varepsilon}$ and $(l, q) = 1$ and for all non-negative integers $m_0, \ldots, m_s$ with $m_0+ \ldots +m_s \leqslant L/6$.

Let q be any prime with $6 < q \leqslant k^{\varepsilon}$ and suppose that the field $K(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q})$ is an extension of K of degree q^n. The main inductive argument of Section 6 of [1] yields the following proposition. For each non-negative integer J with $q^J \leqslant k\Omega'\log\Omega'$, there are integers $p^{(J)}(\lambda) = p^{(J)}(\lambda_{-1}, \ldots, \lambda_n)$, not all zero and with absolute values at most e^{8Lh}, such that $g^{(J)}(l) = 0$ for all integers l with $1 \leqslant l \leqslant hq^J$, $(l, q) = 1$, and all non-negative integers $m_0, \ldots, m_s$ with $m_0+ \ldots +m_s \leqslant q^{-J}L$, where

$$g^{(J)}(z) = \sum_{(\lambda;J)} p^{(J)}(\lambda)\psi^{(J)}(\lambda; z)\alpha_1^{\lambda_1 z}\ldots\alpha_n^{\lambda_n z}\gamma_1^{m_1}\ldots\gamma_s^{m_s},$$

the sum is taken over all integral vectors $\lambda = (\lambda_{-1}, \ldots, \lambda_n)$ with $0 \leqslant \lambda_j \leqslant L_j^{(J)}$ $(-1 \leqslant j \leqslant n)$,

$$L_{-1}^{(J)} = L_{-1}, \quad L_0^{(J)} = L_0, \quad L_j^{(J)} \leqslant q^{-J}L_j \quad (1 \leqslant j \leqslant n),$$

and

$$\psi^{(J)}(\lambda; z) = \psi(\lambda_{-1}, \lambda_0, \lambda_1 q^J, \ldots, \lambda_n q^J; z/q^J).$$

We can suppose that we have established this proposition for $J = N$ satisfying $L_{s+i} < q^N$ $(1 \leqslant i \leqslant t)$. Otherwise,

$$\Omega'\log\Omega' \leqslant (\Omega\log\Omega')^{1/t}/\log A_{s+i} \quad \text{for some } i$$

and

$$|\Lambda_i| < \exp\big(-C\Omega'\log A_{s+i}\log\Omega'\log(B\Omega)\big),$$

and already $\alpha_1^{1/q}, \ldots, \alpha_s^{1/q}, \alpha_{s+i}^{1/q}$ do not generate an extension of K of degree q^{s+1} by the basic conclusion of [1] Section 7, in contradiction to our assumptions. The equations $g^{(N)}(l) = 0$ now take the form

$$\sum_{\lambda_s=0}^{L_s^{(N)}}\Big(\sum_{\lambda_{-1}=0}^{L_{-1}^{(N)}}\ldots\sum_{\lambda_{s-1}=0}^{L_{s-1}^{(N)}} p^{(N)}(\lambda)\psi^{(N)}(\lambda; l)\alpha_1^{\lambda_1 l}\ldots\alpha_s^{\lambda_s l}\gamma_1^{m_1}\ldots\gamma_{s-1}^{m_{s-1}}\Big)\gamma_s^{m_s} = 0,$$

valid, in particular, for all integers l with $1 \leqslant l \leqslant hq^N$, $(l, q) = 1$, and for all non-negative integers $m_0, \ldots, m_s$ with $m_0 \leqslant \frac{1}{2}q^{-N}L$ and $m_j \leqslant L_j^{(N)}$ $(1 \leqslant j \leqslant s)$. Since the Vandermonde determinant of order $L_s^{(N)}+1$ with $\gamma_s^{m_s}$ in the (λ_s+1)-th row and (m_s+1)-th column is non-zero, the sum in parentheses above is equal to zero. Repeated application of this argument gives the equations

$$\sum_{\lambda_{-1}=0}^{L_{-1}} \sum_{\lambda_0=0}^{L_0} p^{(N)}(\lambda)\psi^{(N)}(\lambda; l) = 0 \quad (0 \leqslant \lambda_1 \leqslant L_1^{(N)}, \ldots, 0 \leqslant \lambda_n \leqslant L_n^{(N)})$$

for the same values of l and m_0. Thus the function

$$\exp\{(\lambda_{s+1}\beta_{10} + \ldots + \lambda_n \beta_{t0})z\} \sum_{\lambda_{-1}=0}^{L_{-1}} \sum_{\lambda_0=0}^{L_0} p^{(N)}(\lambda)\big(\Delta(\lambda_{-1} + z/q^N; h)\big)^{\lambda_0+1}$$

has at least $(hq^N - hq^{N-1})(\frac{1}{2}q^{-N}L) > h(L_0+1)$ zeros counted with multiplicities. On the other hand, the polynomial given by the double sum above has degree at most $h(L_0+1)$ and it follows, as in [1], Section 7, that the $p^{(N)}(\lambda)$ are all 0, contrary to construction. This shows that $\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}$ cannot generate an extension of K of degree q^n and completes the first part of the proof.

Next, we assert that if $d \geqslant 8$, say, and we replace multiplicative independence of $\alpha_1, \ldots, \alpha_n$ by the assumption that $\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}$ generate an extension of K of degree q^n, then the theorem holds with $C = (nd)^{84n}$. To achieve a set of linear forms of the special shape considered earlier in this section, we pick a non-singular $t \times t$ submatrix of the given matrix of coefficients (β_{ij}) and multiply the vector of linear forms by the inverse of this matrix. The determinant of this matrix has absolute value at least $(ndB)^{-2n^2d}$, since it is an algebraic number in K whose conjugates have absolute values at most $(ndB)^n$ and its denominator is at most B^{n^2}. Thus we obtain a set of linear forms of the required special shape in which the coefficients have heights at most $(ndB)^{8n^2d^2}$; for each conjugate has absolute value at most $(ndB)^{4n^2d}$ and the denominator is at most B^{2n^2}. If every choice for the $t \times t$ non-singular submatrix used above necessarily contains the column (β_{i0}), we can assume that one of the new linear forms is $\Lambda_t(\text{say}) = \beta_{t0}$ and then $|\Lambda_t| > (ndB)^{-8n^2d^2}$. Otherwise, the proposition proved in the first part of this section gives the lower bound

$$\max_{1 \leqslant i \leqslant t} |\Lambda_i| > \exp\big(-(nd)^{80n}(\Omega \log \Omega')^{1/t} \log\big((ndB)^{8n^2d^2}\Omega\big)\big)$$

$$> \exp\big(-(nd)^{82n}(\Omega \log \Omega')^{1/t} \log(B\Omega)\big)$$

for the new linear forms. Since this procedure could have increased the maximal absolute value of the original linear forms by a factor of at most $(ndB)^{4n^2d}$, we have verified the theorem, under the present assumptions, with $C = (nd)^{84n}$.

We now use the preliminary proposition just proved to establish the theorem. To fix the notation, we arrange the α's so that $A_1 \leqslant A_2 \leqslant \ldots \leqslant A_n$. We can assume $d \geqslant 8$ and that K contains $e^{i\pi/q}$ with $q = 7$, provided that we prove the theorem with $C = (2nd)^{200n}$. We can further assume that $B \geqslant \Omega$ and $\Omega \geqslant nd$, provided that we obtain a reduced value for C, say $C = (2nd)^{198n}$.

It is convenient to allow a slightly more general situation in which α_1 is a root of unity, $\alpha_2, \ldots, \alpha_n$ are multiplicatively independent and the matrix formed by the coefficients β_{ij} with the column (β_{i1}) omitted has rank t. We recover the theorem by taking the β_{i1} to be zero, provided that we obtain a reduced value of C, say $C = (nd)^{100n}$. In fact, we shall take $\alpha_1 = e^{i\pi/q^k}$, where k is the largest integer for which α_1 is in K; then $\alpha_1^{1/q}$ generates an extension of K of degree q. Let m be an integer with $1 \leqslant m < n$ such that $\alpha_{m+1}^{1/q}$ does not generate an extension of $K(\alpha_1^{1/q}, \ldots, \alpha_m^{1/q})$ of degree q. By Kummer theory, as in Lemma 4 of [1],

$$\alpha_{m+1} = \alpha_1^{r_1} \ldots \alpha_m^{r_m}\gamma^q$$

for some γ in K and some integers $r_1, \ldots, r_m$ with $0 \leqslant r_j < q$. We construct, as far as possible, a sequence $\gamma_1 = \gamma, \gamma_2, \gamma_3, \ldots$ of elements of K such that

$$\gamma_l = \alpha_1^{r_{l1}} \ldots \alpha_m^{r_{lm}}\gamma_{l+1}^q \quad (l = 1, 2, \ldots),$$

where the r_{lj} are integers with $0 \leqslant r_{lj} < q$. From these equations

$$\alpha_{m+1} = \alpha_1^{s_{l1}} \ldots \alpha_m^{s_{lm}}\gamma_l^{q^l},$$

where the s_{lj} are integers with $0 \leqslant s_{lj} < q^l$, and hence

$$\log\alpha_{m+1} = s_1 \log\alpha_1 + s_{l2}\log\alpha_2 + \ldots + s_{lm}\log\alpha_m + q^l \log\gamma_l,$$

where the logarithms have their principal values and s_1 is an integer. As in [1], the height of γ_l is at most $(2dA)^{2nd}$, where $A = \max A_j$, and s_1 has absolute value at most $10nd^3q^l\log A$.

Let $H = \big((nd)^9 \log A\big)^n$. We distinguish two cases according as the above construction terminates for some l with $q^l \leqslant H$, or does not. In the latter case, let l be the least integer with $q^l > H$. From the construction, $\alpha_2, \ldots, \alpha_{m+1}$ and γ_l are multiplicatively dependent. The results of [3] enable us to find a second relation of multiplicative dependence, say

$$\alpha_2^{b_2'} \ldots \alpha_{m+1}^{b_{m+1}'}\gamma_l^{b'} = 1,$$

in which $b_2', \ldots, b_{m+1}'$ and b' are integers, not all zero, with absolute values at most H. In fact, $b' \neq 0$ because we have assumed $\alpha_2, \ldots, \alpha_{m+1}$ multiplicatively independent. On eliminating γ_l between these two relations, we obtain a further relation

$$\alpha_1^{b_1''} \ldots \alpha_{m+1}^{b_{m+1}''} = 1,$$

in which the b_j'' are integers and $b_{m+1}'' = q^l b_{m+1}' - b'$ is non-zero. However, this is impossible because $\alpha_2, \ldots, \alpha_{m+1}$ are multiplicatively independent.

Consequently, the construction must terminate for some l with $q^l \leqslant H$. We now replace $\log \alpha_{m+1}$ in the linear forms by $\log \gamma_l$ to get a set of linear forms whose coefficients have heights at most B^{12nd}. In addition, γ_l has height at most $(2dA_{m+1})^{2nd} \leqslant A_{m+1}^{n^2d^2}$ and $\gamma_l^{1/q}$ generates an extension of $K(\alpha_1^{1/q}, \ldots, \alpha_m^{1/q})$ of degree q. After at most n such substitutions, we can ensure we have a set of α's such that $\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}$ generate an extension of K of degree q^n, and the new β's will have heights at most $B^{(12nd)^n}$. We can apply the lower bound for $\max|\Lambda_i|$ proved earlier to this new situation and, on writing the estimate in terms of the original $A_1, \ldots, A_n$ and B, we see that the value of C, namely $(nd)^{84n}$, is increased by a factor of at most $(2nd)^{6n}$. This completes the proof.

4. Applications

Proof of Theorem 1. We suppose $N \geqslant 20$ and set

$$n = \left[\frac{1}{10}\left(\frac{\log\log N}{\log\log\log N}\right)^{1/2}\right].$$

Suppose we have n powers $x_1^{a_1}, \ldots, x_n^{a_n}$ in the interval $[N, N+N^{1/2}]$ and that each of the a's exceeds

$$M = \exp(20(\log\log N \log\log\log N)^{1/2})\log\log N.$$

We choose the notation so that $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$. We now consider two cases according as $x_1, \ldots, x_n$ are multiplicatively independent, or not.

If $x_1, \ldots, x_n$ are multiplicatively independent, we can apply Theorem 4 to the $n-1$ linear forms

$$\Lambda_i = a_i \log x_i - a_n \log x_n \quad (1 \leqslant i \leqslant n-1).$$

Since

$$|\Lambda_i| = |\log(x_i^{a_i}/x_n^{a_n})| \leqslant \log((N+N^{1/2})/N) \leqslant N^{-1/2},$$

the lower bound in the theorem gives the inequality

$$\tfrac{1}{2}\log N < (16n)^{200n}(\Omega\log\Omega)^{1/(n-1)}\log(a_n\Omega),$$

where $\Omega = \prod_{1\leqslant j\leqslant n} \log x_j$. Here $\log x_j < 2\log N/a_j$ and $a_n < 2\log N$, so we obtain

$$M^n < \prod_{j=1}^{n} a_j < (16n)^{200n^2}\log N(\log\log N)^n < (\log N)^2(\log\log N)^n,$$

contrary to the definition of M.

Suppose, on the other hand, that $x_1, \ldots, x_n$ are multiplicatively dependent. To fix the notation, suppose $x_1, \ldots, x_m$ form a maximal multiplicatively independent set. Then $x_1, \ldots, x_{m+1}$ are multiplicatively dependent and, as in [3], there must be a relation

$$b_1\log x_1 + \ldots + b_{m+1}\log x_{m+1} = 0,$$

where $b_1, \ldots, b_{m+1}$ are integers not all zero and satisfy

$$|b_i| < (2m)^m \prod_{j=1}^{m+1} \log x_j.$$

In fact, b_{m+1} is necessarily non-zero and we can apply Theorem 4 to the m linear forms

$$\Lambda_i = \sum_{j=1}^{m} (a_j\delta_{ij} - a_{m+1}b_j/b_{m+1})\log x_j \quad (1 \leqslant i \leqslant m),$$

where δ_{ij} is 1 if $i=j$ and 0 otherwise. Again,

$$|\Lambda_i| = |a_i\log x_i - a_{m+1}\log x_{m+1}| \leqslant N^{-1/2},$$

so we obtain

$$\tfrac{1}{2}\log N < (16m)^{200m}(\Omega\log\Omega)^{1/m}\log(2a_{m+1}\log x_{m+1}(2m)^m\Omega^2)$$

with $\Omega = \prod_{1\leqslant j\leqslant m} \log x_j$, and in the same way as before, this leads to

$$M^m < \prod_{j=1}^{m} a_j < (16m)^{200mn}(\log\log N)^n,$$

again contrary to the choice of M.

Now let $x_1^{a_1}, x_2^{a_2}, \ldots$ be all the powers in the interval $[N, N+N^{1/2}]$. Clearly, there is at most one ath power in this interval for each a, so we can suppose the a's are distinct primes. From the previous work, there are at most n powers with exponents exceeding M. So the total number of powers in the interval is at most $n+\pi(M)$, and this establishes Theorem 1.

Proof of Theorem 2. Suppose $N \geqslant 20$ and set

$$n = \left[\frac{1}{20}\left(\frac{\log\log N}{\log\log\log N}\right)^{1/2}\right].$$

Suppose we have n elements $z_1, \ldots, z_n$ of P_δ in the interval $[N, N+N^{1/2}]$ and that, when each z_j is written as $z_j = m_j x_j$ as in the definition of P_δ, each a_j exceeds

$$M = \exp(40\delta^{-1}(\log\log N \log\log\log N)^{1/2})\log\log N.$$

We choose the notation so that $a_1 \leqslant a_2 \leqslant \ldots \leqslant a_n$.

Suppose first that $x_1, \ldots, x_n$ are multiplicatively independent and consider the $n-1$ linear forms

$$\Lambda_i = \log(m_i/m_1) + a_i\log x_i - a_1\log x_1 \quad (2 \leqslant i \leqslant n).$$

If the $2n-1$ numbers m_i/m_1 and x_i are multiplicatively dependent, then there is a linear relation between their logarithms having integer coefficients with absolute values at most $(4n\log N)^{2n}$, say. We use these relations to eliminate as many of the $\log(m_i/m_1)$ as possible from the linear forms Λ_i. After at most n steps, the Λ_i become linear forms in the remaining $\log(m_i/m_1)$ and the $\log x_i$ with rational coefficients whose heights are at most $(8n\log N)^{2n^2}$. The linear forms still have rank $n-1$, and as before, the lower bound in Theorem 4 gives the inequality

$$\tfrac{1}{2}\log N < \big(16(2n-1)\big)^{200(2n-1)}(\Omega\log\Omega)^{1/(n-1)}\log\big((8n\log N)^{2n^2}\Omega\big),$$

where

$$\Omega = \prod_{1\leqslant j\leqslant n}\log m_j\log x_j.$$

Here $\log x_j < 2\log N/a_j$ and $\log m_j < a_j^{1-\delta}$, so

$$M^{\delta n} < \prod_{j=1}^{n} a_j^{\delta} < (32n)^{400n^2}\log N(\log\log N)^n < (\log N)^2(\log\log N)^n,$$

contrary to the definition of M.

If $x_1, \ldots, x_n$ are multiplicatively dependent, we can modify the argument as in the proof of Theorem 1 to obtain the same conclusion. Consequently, with at most n exceptions, the elements $z = mx^a$ of P_δ in the interval $[N, N+N^{1/2}]$ have $a < M$ and, consequently, $m < \exp(M^{1-\delta})$. Since there can be at most one such $z = mx^a$ with a given a and m in the interval, we obtain the estimate asserted in the theorem.

Proof of Theorem 3. Suppose $N \geqslant 20$ and set

$$n = \left[\frac{1}{20}\left(\frac{\log\log N}{\log\log\log N}\right)^{1/2}\right].$$

We suppose, slightly more generally than required for the theorem, that we have n solutions of the inequalities

$$N \leqslant (x_i^{a_i}-1)/(x_i-1) \leqslant N+N^{1/2} \qquad (1\leqslant i\leqslant n),$$

where the x's are integers greater than 1 and the a's are integers greater than

$$M = (\log N)^{(1/2)+\varepsilon}.$$

Consider the linear forms

$$\Lambda_i = \log\frac{x_i-1}{x_1-1} - a_i\log x_i + a_1\log x_1 \qquad (2\leqslant i\leqslant n).$$

Then $|\Lambda_i| < 2N^{-1/2}$ and we can apply Theorem 4 in the same way as before. For example, if the x's are multiplicatively independent, we obtain the inequality

$$\tfrac{1}{2}\log N < \big(16(2n-1)\big)^{200(2n-1)}(\Omega^2\log\Omega^2)^{1/(n-1)}\log\big((8n\log N)^{2n^2}\Omega^2\big),$$

with $\Omega = \prod_{1\leqslant j\leqslant n}\log x_j$, as in the proof of Theorem 2, and this leads to

$$M^{2n} < \prod_{j=1}^{n} a_j^2 < (32n)^{400n^2}(\log N)^{n+1}(\log\log N)^n,$$

giving a contradiction whenever N is sufficiently large.

Now, once a is fixed, there is at most one choice for the integer x for which

$$N \leqslant (x^a-1)/(x-1) \leqslant N+N^{1/2}.$$

Consequently, the above inequalities have at most $(\log N)^{(1/2)+\varepsilon}$ solutions with $x>1$ and $a>2$, whenever N is sufficiently large.

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