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Improvement on Davenport's iterative method and new results in additive number theory II
Proof that $G(5) \leqslant 22$

by

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1. Introduction. Combining the proofs in Part I of this series of papers (which will be referred to by [1])(1), together with some additional arguments, it will be shown here that $G(5) \leqslant 22$.(2) The result $H(5) \leqslant 23$ would follow as indicated in § 17 in [1]. Before using the new method in [1] for obtaining better bounds for α_s in $U_s^{(5)}(N) > N^{\alpha_s - \varepsilon}$ with $s \geqslant 7$, we introduce some new arguments for retaining the use of admissible exponents. In doing so, the bounds obtained for α_4 and α_5 will be slightly weaker than those given by Davenport's method.

Most of the definitions and lemmas will be referred directly to [1]. We recall the following definitions. With $P_i = P^{\lambda_i}$,

$$f = f(\alpha) = \sum_{P < x < 2P} e(\alpha x^k), \quad f_i = f_i(\alpha) = \sum_{P_i < x < 2P_i} e(\alpha x^k). \tag{1.1}$$

Write (uniformly)

$$F(\alpha) = \sum_t \sum_x e(\Delta_t(x^k)\alpha), \quad F_r(\alpha) = \sum_t \sum_{t_1} \dots \sum_{t_r} \sum_x e(\Delta_{t,t_1,\dots,t_r}(x^k)\alpha) \tag{1.2}$$

(where $t, t_1, \dots, t_r$ and x are taken in appropriate intervals).

Most of the preliminary results in this paper will be used for $k = 6$ also, and, as much as possible, the arguments are given with general k.

2. Preliminary results. The next lemma will be used repeatedly.

Lemma 2.1. *Let $\mathscr{B}$ be a subset of $\{1, \dots, s\}$, and*

$$M = \int_0^1 |f_1 \dots f_s|^2 \, d\alpha, \tag{2.1}$$

$$M_1 = \int_0^1 \Big|\prod_{\substack{i=1 \\ i \notin \mathscr{B}}}^{s} f_i\Big|^2 d\alpha, \quad M_2 = \int_0^1 |f_1 \dots f_s|^2 \Big|\prod_{i \in \mathscr{B}} f_i\Big|^2 d\alpha. \tag{2.2}$$

(1) Acta Arith. 46 (1985), pp. 1–31.
(2) For the later improvement $G(5) \leqslant 21$, we take $\delta_i = 118$ for $6 \leqslant i \leqslant 8$.

Further let $F_r(\alpha)$ be defined as in (1.2), where $t, t_1, \dots, t_r$ and x are positive integers satisfying (for some $\tau > 0$)

(2.3) $$\mathrm{Card}\,[\{t, t_1, \dots, t_r\}] \ll P^{\tau} \quad \text{and} \quad P < x < 2P.$$

Then (with $1 \leqslant r \leqslant k-3$)

(2.4) $$\int_0^1 F_r(\alpha)|f_1 \dots f_s|^2 d\alpha \ll \{P^{2\tau}(PM_1) + P^{\tau+\varepsilon}(\prod_{\substack{i=1\\ i\notin\mathscr{B}}}^{s} P_i)^2\}^{1/2}(M_2)^{1/2}.$$

Also,

(2.5) $$\int_0^1 F_r(\alpha)|f_1 \dots f_s|^2 d\alpha \ll \{P^{2\tau}(PM) + P^{\tau+\varepsilon}(P_1 \dots P_s)^2\}^{1/2}(M)^{1/2}.$$

Proof. By Schwarz's inequality, the integral in (2.4) is

(2.6) $$\ll \{\int_0^1 |F_r(\alpha)|^2 |\prod_{\substack{i=1\\ i\notin\mathscr{B}}}^{s} f_i|^2 d\alpha\}^{1/2} \{\int_0^1 |f_1 \dots f_s|^2 |\prod_{i\in\mathscr{B}} f_i|^2 d\alpha\}^{1/2};$$

and, using part of the argument in the proof of the Fundamental Lemma (in [1]), (by estimating $|F_r(\alpha)|^2$ using Cauchy's inequality in the standard way) the first factor in (2.6) is (cf. (2.2))

(2.7) $$\ll \{P^{2\tau} PM_1 + P^{\tau} \int_0^1 F_{r+1}(\alpha) |\prod_{\substack{i=1\\ i\notin\mathscr{B}}}^{s} f_i|^2 d\alpha\}^{1/2},$$

where, in the definition of $F_{r+1}(\alpha)$, $t, t_1, \dots, t_r, x$ satisfy (2.3), and $0 < t_{r+1} < P$ (cf. (1.2)).

Now, the number of solutions of the equation

(2.8) $$\Delta_{t,t_1,\dots,t_{r+1}}(x^k) = \sum_{\substack{i=1\\ i\notin\mathscr{B}}}^{s} (x_i^k - y_i^k) \quad \text{with} \quad x_i, y_i \in (P_i, 2P_i)$$

(and $tt_1 \dots t_{r+1} \neq 0$) is

$$\ll (\prod_{\substack{i=1\\ i\notin\mathscr{B}}}^{s} P_i)^2 P^{\varepsilon},$$

since for given x_i's and y_i's, there are $\ll P^{\varepsilon}$ choices for $t, t_1, \dots, t_{r+1}$ (and x is then determined uniquely). Hence,

(2.9) $$\int_0^1 F_{r+1}(\alpha) |\prod_{\substack{i=1\\ i\notin\mathscr{B}}}^{s} f_i|^2 d\alpha \ll (\prod_{\substack{i=1\\ i\notin\mathscr{B}}}^{s} P_i)^2 P^{\varepsilon}.$$

(2.4) now follows from (2.6), (2.7), and (2.9). (2.5) follows the same way on taking $\mathscr{B}$ to be the empty set.

The next lemma is the same as Lemma 4.1 in [1].

LEMMA 2.2. *With $\delta = \delta_s$ (where $0 \leqslant \delta \leqslant 1$), $\lambda_s = \lambda_s^{(s)} = (k-1+\delta)/k$ (and $\lambda_i = \lambda_i^{(s)}$ defined as before for $1 \leqslant i \leqslant s-1$), and f, f_i as in (1.1), we have (for $1 \leqslant l \leqslant k-2$)*

(2.10) $$\int_0^1 |f|^2 |f_1 \dots f_s|^2 d\alpha$$
$$\ll PM + P^{1+\delta+\varepsilon} M \{P^{-1} + P^{-\delta-l-1}(P_1 \dots P_s)^2 M^{-1}\}^{1/2^l},$$

where M is given by (2.1).

The next lemma is a modification of Lemma 2.2 (in the same way as Lemma 8.2 in [1] is a modified form of the Fundamental Lemma).

LEMMA 2.3. *With the same premises as in Lemma 2.2, and M_1, M_2 defined by (2.2), we have (for $1 \leqslant l \leqslant k-2$)*

(2.11) $$\int_0^1 |f|^2 |f_1 \dots f_s|^2 d\alpha$$
$$\ll PM + P^{1+\delta+\varepsilon} M \{P^{-1}(M_1 M_2) M^{-2} + P^{-\delta-l-1}(\prod_{\substack{i=1\\ i\notin\mathscr{B}}}^{s} P_i^2)(M_2 M^{-2})\}^{1/2^l}.$$

Proof. In the proof of Lemma 2.2, we use (at each iterative step) the inequality (2.6) with $\tau = \delta + r$ for $1 \leqslant r \leqslant l$. Here, we follow the same arguments at the iterative steps with $1 \leqslant r \leqslant l-1$; but, at the last step (with $r = l$) we use (2.4) (with $\tau = \delta + r$). Comparing (2.4) and (2.5), it is easily seen that the terms on the right-side of (2.10) can be replaced by those in (2.11). [Note: (2.6) is used with the set $\mathscr{B}$ empty.] The next lemma is essentially Davenport's result (with $0 \leqslant \delta \leqslant 1/2^l$), but is given in that form to retain the use of admissible exponents.

LEMMA 2.4. *Let $\{\lambda_1', \dots, \lambda_s'\}$ $(0 < \lambda_i' \leqslant 1)$ form admissible exponents with*

(2.12) $$(\lambda_1' + \dots + \lambda_s')/k = \alpha \quad (\text{so that, } U_s^{(k)}(N) > N^{\alpha-\varepsilon});$$

and that, for some l with $1 \leqslant l \leqslant k-2$,

(2.13) $$(l+1) - (k-1)\alpha \geqslant (2^l - 1 + \alpha)/2^l.$$

Then, with

(2.14) $$0 \leqslant \delta \leqslant 1/2^l, \quad \lambda_i = (k-1+\delta)\lambda_i'/k \quad (1 \leqslant i \leqslant s),$$

$\{\lambda_1, \dots, \lambda_s, 1\}$ form admissible exponents. Furthermore, $U_{s+1}^{(k)}(N) > N^{\beta-\varepsilon}$, where

(2.15) $$\beta = (\lambda_1 + \dots + \lambda_s + 1)/k.$$

Proof. We use Lemma 2.2, where M satisfies $P_1 \ldots P_s \ll M \ll (P_1 \ldots P_s) P^{\varepsilon}$. Accordingly, the number of solutions S of the equation

$$x^k + \Big(\sum_{i=1}^{s} x_i^k\Big) = y^k + \Big(\sum_{i=1}^{s} y_i^k\Big) \quad \text{with} \quad x, y \in (P, 2P); \quad x_i, y_i \in (P_i, 2P_i)$$

is (cf. (2.10))

$$(2.16) \qquad \ll P(P_1 \ldots P_s) P^{\varepsilon} + P^{1+\delta+\varepsilon-(1/2^l)}(P_1 \ldots P_s) + \\ + P^{1+\delta+\varepsilon-(\delta+l+1)/2^l}(P_1 \ldots P_s)^{1+1/2^l}.$$

From (2.14), the second term in (2.16) is $\ll P^{1+\varepsilon}(P_1 \ldots P_s)$; and, since $P^{(k-1+\delta)\alpha} \gg P_1 \ldots P_s$, the third term in (2.16) is

$$(2.17) \qquad \ll P^{1+\varepsilon}(P_1 \ldots P_s)(P^{((k-1+\delta)\alpha-(\delta+l+1))/2^l+\delta}).$$

The exponent of P in the last factor in (2.17) is equal to

$$\{(k-1)\alpha - (l+1) + (2^l - 1 + \alpha)\delta\}/2^l,$$

which, with (2.13) and (2.14), is easily verified to be $\ll 0$. It now follows that

$$S \ll P^{1+\varepsilon}(P_1 \ldots P_s),$$

proving the result.

The next lemma will be used in estimating α_3 and α_4 (at the same time retaining the use of admissible exponents).

Lemma 2.5. *Let $x_i, y_i, z_i \in (P_i, 2P_i)$, and for given m, n, the integers u_i, u_j, u_r are of the form $x_m^k + x_n^k$; $x, y \in (P, 2P)$; the set of positive integers $t, t_1, \ldots, t_{l-1}$ satisfy (for some $\tau > 0$)*

$$(2.18) \qquad \mathrm{Card}\,[\{t, t_1, \ldots, t_{l-1}\}] \ll P^{\tau}$$

(where, if $l = 1$, we simply consider the set of integers t).

Then, the number of solutions M_1 of the equation

$$(2.19) \qquad \Delta_{t,t_1,\ldots,t_{l-1}}(x^k) + u_i = u_r$$

satisfies

$$(2.20) \qquad M_1 \ll P^{\tau+\varepsilon}(P_m P_n) + (P^{\tau+\varepsilon} P_m P_n M_2)^{1/2},$$

where M_2 is the number of solutions of

$$(2.21) \qquad \Delta_{t,t_1,\ldots,t_l}(x^k) + u_i = u_r$$

with $t, t_1, \ldots, t_{l-1}$ subject to (2.18), and $0 < t_l < P$.

Also,

$$(2.22) \qquad M_1 \ll P^{\tau+\varepsilon}(P_m P_n) + (P^{\tau+\varepsilon})^{1/2}(P_m P_n)^{(3/2)+\varepsilon}.$$

Proof. With $u_r = z_m^k + z_n^k$, let $M_1(t, t_1, \ldots, t_{l-1}, z_1, z_2)$ denote the

number of solutions of (2.19) for given $t, t_1, \ldots, t_{l-1}, z_1$ and z_2. Then, by Cauchy's inequality,

$$(2.23) \qquad M_1 = \sum_t \sum_{t_1} \ldots \sum_{t_{l-1}} \sum_{z_1} \sum_{z_2} M_1(t, t_1, \ldots, t_{l-1}, z_1, z_2) \\ \ll \{\sum_t \sum_{t_1} \ldots \sum_{t_{l-1}} \sum_{z_1} \sum_{z_2} 1\}^{1/2} \times \\ \times \{\sum_t \sum_{t_1} \ldots \sum_{t_{l-1}} \sum_{z_1} \sum_{z_2} M_1^2(t, t_1, \ldots, t_{l-1}, z_1, z_2)\}^{1/2}.$$

By (2.18), the first factor in (2.23) is $\ll \{P^{\tau}(P_m P_n)\}^{1/2}$, and the second factor is $\ll (M_3)^{1/2}$, where M_3 is the number of solutions of

$$(2.24) \qquad \Delta_{t,t_1,\ldots,t_{l-1}}(x^k) + u_i = \Delta_{t,t_1,\ldots,t_{l-1}}(y^k) + u_j = u_r$$

(with $x, y \in (P, 2P)$).

(a) As already defined, M_3 denotes the number of solutions of (2.24). Let

(b) M_4 denote the number of solutions of (2.24) with $x = y$, $u_i = u_j$;

(c) M_5 denote the number of solutions of (2.24) with $x \neq y$.

With $u_j = y_m^k + y_n^k$, for every given u_i, the number of choices for y_m, y_n satisfying

$$(2.25) \qquad u_i = y_m^k + y_n^k$$

is $\ll P^{\varepsilon}$ since the number of representations $r(u_i)$ of u_i in the form (2.25) is uniformly bounded by P^{ε}. Hence, it follows that (cf. (b))

$$(2.26) \qquad M_4 \ll P^{\varepsilon} M_1.$$

Now, let M_6 denote the number of solutions of

$$(2.27) \qquad \Delta_{t,t_1,\ldots,t_{l-1}}(x^k) + u_i = \Delta_{t,t_1,\ldots,t_{l-1}}(y^k) + u_j \quad \text{with} \quad x \neq y.$$

With $u_r = z_m^k + z_n^k$, to every solution of (2.27), there correspond $\ll P^{\varepsilon}$ solutions of (2.24), since $r(u_r) \ll P^{\varepsilon}$ (uniformly). Hence, (cf. (c))

$$(2.28) \qquad M_5 \ll P^{\varepsilon} M_6.$$

Now, putting $y = x + t_l$ $(0 < t_l < P)$, we see that

$$(2.29) \qquad M_6 \ll M_2 \quad \text{(cf. (2.21))};$$

so that, from (2.26) and (2.28) (cf. (a)),

$$(2.30) \qquad M_3 \ll P^{\varepsilon}(M_1 + M_2).$$

Thus, from (2.23),

$$M_1 \ll (P^{\tau} P_m P_n)^{1/2} \{P^{\varepsilon}(M_1 + M_2)\}^{1/2} \\ \ll (P^{\tau+\varepsilon} P_m P_n M_1)^{1/2} + (P^{\tau+\varepsilon} P_m P_n M_2)^{1/2}.$$

(2.20) now follows easily from this. (2.22) is deduced from (2.20) by using the fact that $M_2 \ll (P_m P_n)^2 P^\varepsilon$, since there are $\ll (P_m P_n)^2$ choices for u_i, u_r, and in (2.21), for given u_i, u_r there are $\ll P^\varepsilon$ choices for t, $t_1, \ldots, t_l$ and x.

3. Further auxiliary results. In this, and some of the remaining sections, we prove some inequalities required in the estimates of $U_s^{(5)}(N)$. Similar inequalities will be used for $k = 6$ also (so that, as much as possible, the arguments are given with general k). Some of these inequalities can be improved on slightly (but that will not be required for our purposes). The values of the δ_i's given below will play a role in the proofs. The δ_i's are chosen iteratively to satisfy

(3.1) $$\int_0^1 |f_1 \ldots f_r|^2 \, d\alpha \ll (P_1 \ldots P_r) P^\varepsilon,$$

where, we recall that (while the λ_i's will vary proportionately at different iterative steps)

(3.2) $P_i = P_{i+1}^{\mu_i}$ with $\mu_i = (k-1+\delta_i)/k$ (at each iterative step).

Let

(3.3) $$\delta_1 = 1, \quad \delta_2 = (7/17), \quad \delta_3 = 0.2218, \quad \delta_4 = 0.1629, \quad \delta_5 = (1/8),$$
$$\delta_6 = 0.1018, \quad \delta_7 = 0.1075, \quad \delta_8 = 0.086.$$

We note that (at each iterative step)

(3.4) $$\lambda_1^{(s)} = \lambda_2^{(s)} \quad \text{and} \quad f_1 = f_2.$$

For convenience (in repeated usage), we write

(3.5) $$M_{r,s} = \int_0^1 |f_r f_{r-1} \ldots f_3|^2 |f_2|^s \, d\alpha \quad (\text{for } r \geq 3).$$

The next lemma is implied by Hua's inequality.

LEMMA 3.1. *For* $1 \leq l \leq k$,

(3.6) $$\int_0^1 |f_i|^{2^l} \, d\alpha \ll P_i^{2^l - l + \varepsilon}.$$

LEMMA 3.2.

(3.7) $$\int_0^1 |f_2|^6 \, d\alpha \ll P_2^{(7/2)+\varepsilon},$$

(3.8) $$\int_0^1 |f_2|^{10} \, d\alpha \ll P_2^{(27/4)+\varepsilon},$$

(3.9) $$\int_0^1 |f_2|^{12} \, d\alpha \ll P_2^{(17/2)+\varepsilon},$$

and

(3.10) $$\int_0^1 |f_2|^{14} \, d\alpha \ll P_2^{(41/4)+\varepsilon}.$$

Proof. By Hölder's inequality,

(3.11) $$\int_0^1 |f_2|^6 \, d\alpha \ll \Big\{\int_0^1 |f_2|^4 \, d\alpha\Big\}^{1/2} \Big\{\int_0^1 |f_2|^8 \, d\alpha\Big\}^{1/2},$$

(3.12) $$\int_0^1 |f_2|^{10} \, d\alpha \ll \Big\{\int_0^1 |f_2|^8 \, d\alpha\Big\}^{3/4} \Big\{\int_0^1 |f_2|^{16} \, d\alpha\Big\}^{1/4},$$

(3.13) $$\int_0^1 |f_2|^{12} \, d\alpha \ll \Big\{\int_0^1 |f_2|^8 \, d\alpha\Big\}^{1/2} \Big\{\int_0^1 |f_2|^{16} \, d\alpha\Big\}^{1/2}$$

and

(3.14) $$\int_0^1 |f_2|^{14} \, d\alpha \ll \Big\{\int_0^1 |f_2|^8 \, d\alpha\Big\}^{1/4} \Big\{\int_0^1 |f_2|^{16} \, d\alpha\Big\}^{3/4}.$$

The lemma follows from these and (3.6) (with $l = 2$, 3 and 4).

LEMMA 3.3. *With* $k = 5$, *and* δ_2 *as in* (3.3),

(3.15) $$M_{3,6} = \int_0^1 |f_3|^2 |f_2|^6 \, d\alpha \ll S_{3,6},$$

where

(3.16) $$S_{3,6} = P_3 P_2^{(7/2)+\varepsilon}.$$

Proof. We have

(3.17) $$M_{3,6} \ll P_3 \int_0^1 |f_2|^6 \, d\alpha + \int_0^1 F(\alpha) |f_2|^6 \, d\alpha,$$

where $F(\alpha)$ is defined as in (1.2) with $0 < t < P_3^{\delta_2}$, and $P_3 < x < 2P_3$. Also,

(3.18) $$\int_0^1 F(\alpha) |f_2|^6 \, d\alpha \ll \Big\{\int_0^1 |F(\alpha)|^2 |f_2|^4 \, d\alpha\Big\}^{1/2} \Big\{\int_0^1 |f_2|^8 \, d\alpha\Big\}^{1/2};$$

and (as in the proof of Lemma 2.1), (using $\int_0^1 |f_2|^4 \, d\alpha \ll P_2^{2+\varepsilon}$)

(3.19) $$\int_0^1 |F(\alpha)|^2 |f_2|^4 \, d\alpha \ll (P_3^{2\delta_2} P_3) P_2^{2+\varepsilon} + P_3^{\delta_2} \Big\{\int_0^1 F_1(\alpha) |f_2|^4 \, d\alpha\Big\},$$

where $F_1(\alpha)$ is defined as in (1.2) with $0 < t < P_3^{\delta_2}$, $0 < t_1 < P_3$ and $P_3 < x < 2P_3$.

Furthermore, $\int_0^1 F_1(\alpha) |f_2|^4 \, d\alpha$ is the number of solutions M_1 of $\Delta_{t,t_1}(x^k) + u_i = u_j$ with u_i, u_j of the form $x_2^k + y_2^k$.

To estimate M_1, we can use Lemma 2.5 with $P_m = P_n = P_2$, $P = P_3$, $\tau = 1+\delta_2$ and $l = 2$. Accordingly, using (2.22), we have

$$\int_0^1 F_1(\alpha)|f_2|^4\,d\alpha \ll P_3^{1+\delta_2+\varepsilon} P_2^2 + \{P_3^{1+\delta_2+\varepsilon}\}^{1/2} P_2^{3+\varepsilon}. \tag{3.20}$$

Thus, from (3.6) (with $l = 2, 3$) and (3.7), we have from (3.17), (3.18) and (3.19) that

$$\begin{aligned} M_{3,6} &\ll P_3 P_2^{(7/2)+\varepsilon} + \{P_3^{1+2\delta_2} P_2^{2+\varepsilon} P_3^{\delta_2} (P_3^{1+\delta_2+\varepsilon})^{1/2} P_2^{3+\varepsilon}\}^{1/2} (P_2^{5+\varepsilon})^{1/2} \\ &\ll P_3 P_2^{(7/2)+\varepsilon} + P_3^{(1/2)+\delta_2} P_2^{(7/2)+\varepsilon} + \{P_3^{(1+3\delta_2)/4}\} P_2^{4+\varepsilon}. \end{aligned} \tag{3.21}$$

Now, it can be verified with $k = 5$ (and δ_2 as in (3.3)) that each term in (3.21) is $\ll P_3 P_2^{(7/2)+\varepsilon}$, proving the lemma.

LEMMA 3.4. *With δ_2 as in* (3.3),

$$M_{3,8} = \int_0^1 |f_3|^2 |f_2|^8\,d\alpha \ll S_{3,8}, \tag{3.22}$$

where

$$S_{3,8} = P_3^{(3/4)+\delta_2} P_2^{(81/16)+\varepsilon} \ll P_3 P_2^{(6-\gamma_1)+\varepsilon} \tag{3.23}$$

with

$$\gamma_1 = (181/240). \tag{3.24}$$

Proof. We adapt the proof of Lemma 2.3 with $l = 2$, $\delta = \delta_2$; P_3 replacing P; f_3 replacing f; $|f_2|^8$ replacing $|f_1 \ldots f_s|^2$;

$$\begin{gathered} M = \int_0^1 |f_2|^8\,d\alpha \ll P_2^{5+\varepsilon}; \\ M_1 = \int_0^1 |f_2|^6\,d\alpha \ll P_2^{(7/2)+\varepsilon}; \quad M_2 = \int_0^1 |f_2|^{10}\,d\alpha \ll P_2^{(27/4)+\varepsilon}, \end{gathered} \tag{3.25}$$

and $\big(\prod_{\substack{i=1\\ i\notin\mathscr{B}}}^{s} P_i^2\big) = P_2^6$ (cf. (3.6), (3.7) and (3.8)).

Accordingly, we have (from (2.11) and (3.25))

$$\begin{aligned} M_{3,8} &\ll P_3 M + P_3^{1+\delta_2+\varepsilon} M^{1/2} M_2^{1/4} \{P_3^{-1} M_1 + P_3^{-\delta_2-3} P_2^6\}^{1/4} \\ &\ll P_3 P_2^{5+\varepsilon} + \{P_3^{(3/4)+\delta_2}\} P_2^{(81/16)+\varepsilon} + \{P_3^{(1+3\delta_2)/4}\} P_2^{(91/16)+\varepsilon}. \end{aligned} \tag{3.26}$$

It is now verified (with the value for δ_2 and using $P_2 = P_3^{\mu_2}$) that the second term in this estimate dominates over the other two terms, proving (3.22). (3.23) is verified from this.

LEMMA 3.5. *With δ_2 as in* (3.3),

$$M_{3,10} = \int_0^1 |f_3|^2 |f_2|^{10}\,d\alpha \ll S_{3,10}, \tag{3.27}$$

where

$$S_{3,10} = P_3^{(3/4)+\delta_2} P_2^{(109/16)+\varepsilon} \ll P_3 P_2^{(8-\gamma_2)+\varepsilon} \tag{3.28}$$

with

$$\gamma_2 = (241/240). \tag{3.29}$$

Proof. Here, we adapt the proof of Lemma 2.3 with $l = 2$, $\delta = \delta_2$; $|f_2|^{10}$ replacing $|f_1 \ldots f_s|^2$;

$$\big(\prod_{\substack{i=1\\ i\notin\mathscr{B}}}^{s} P_i^2\big) = P_2^6; \quad M = \int_0^1 |f_2|^{10}\,d\alpha \ll P_2^{(27/4)+\varepsilon};$$

$$M_1 = \int_0^1 |f_2|^6\,d\alpha \ll P_2^{(7/2)+\varepsilon}; \quad M_2 = \int_0^1 |f_2|^{14}\,d\alpha \ll P_2^{(41/4)+\varepsilon}$$

(cf. (3.10)).

Hence, as in (3.26), we have

$$\begin{aligned} M_{3,10} &\ll P_3 P_2^{(27/4)+\varepsilon} + P_3^{1+\delta_2+\varepsilon} P_2^{(27/8)+\varepsilon} P_2^{(41/16)} \times \\ &\qquad \times \{P_3^{-1} P_2^{(7/2)} + P_3^{-\delta_2-3} P_2^6\}^{1/4} \\ &\ll P_3 P_2^{(27/4)+\varepsilon} + \{P_3^{(3/4)+\delta_2+\varepsilon}\} P_2^{(109/16)} + \\ &\qquad + \{P_3^{(1+3\delta_2)/4}\} P_2^{(119/16)+\varepsilon}. \end{aligned} \tag{3.30}$$

Here again, it is verified that the second term in (3.30) dominates over the other two, and that it is bounded by $P_3 P_2^{(8-\gamma_2)}$ (cf. (3.28)).

LEMMA 3.6. *With δ_2 as in* (3.3),

$$M_{3,12} = \int_0^1 |f_3|^2 |f_2|^{12}\,d\alpha \ll S_{3,12}, \tag{3.31}$$

where

$$S_{3,12} = P_3^{(1+3\delta_2)/4} P_2^{(37/4)+\varepsilon} \ll P_3 P_2^{(10-\gamma_3)+\varepsilon} \tag{3.32}$$

with

$$\gamma_3 = (5/4). \tag{3.33}$$

Proof. Here again, we adapt the proof of Lemma 2.3 with $l = 2$, $\delta = \delta_2$; $|f_2|^{12}$ replacing $|f_1 \ldots f_s|^2$;

$$\Big(\prod_{\substack{i=1\\ i\notin\mathscr{B}}}^{s} P_i^2\Big) = P_2^8; \quad M = \int_0^1 |f_2|^{12}\, d\alpha \ll P_2^{(17/2)+\varepsilon};$$

$$M_1 = \int_0^1 |f_2|^8\, d\alpha \ll P_2^{5+\varepsilon}; \quad M_2 = \int_0^1 |f_2|^{16}\, d\alpha \ll P_2^{12+\varepsilon}$$

(cf. (3.6) and (3.9)).

As in (3.26),

$$M_{3,12} \ll P_3 P_2^{(17/2)+\varepsilon} + P_3^{1+\delta_2+\varepsilon} P_2^{(17/4)+\varepsilon} (P_2^{12})^{1/4} \{P_3^{-1} P_2^5 + P_3^{-\delta_2-3} P_2^8\}^{1/4}$$

$$\ll P_3 P_2^{(17/2)+\varepsilon} + \{P_3^{(3/4)+\delta_2+\varepsilon}\} P_2^{(17/2)} + \{P_3^{(1+3\delta_2)/4}\} P_2^{(37/4)+\varepsilon}. \tag{3.34}$$

It is now verified that the third term in (3.34) dominates over the other two, and that it satisfies the inequality (3.32).

4. Estimation of $U_2^{(5)}(N)$ and $U_3^{(5)}(N)$. The next lemma is a standard result.

LEMMA 4.1. *With $\delta_1 = 1$ (and $\lambda_1^{(1)} = 1$), $\{1, 1\}$ form admissible exponents, and $U_2^{(5)}(N) > N^{\alpha_2-\varepsilon}$, where*

$$\alpha_2 = (2/5). \tag{4.1}$$

LEMMA 4.2. *With δ_2 as in (3.3), and*

$$\lambda_1^{(2)} = \lambda_2^{(2)} = (4+\delta_2)/5, \tag{4.2}$$

$\{\lambda_1^{(2)}, \lambda_2^{(2)}, 1\}$ form admissible exponents, and $U_3^{(5)}(N) > N^{\alpha_3-\varepsilon}$, where

$$\alpha_3 = (\lambda_1^{(2)} + \lambda_2^{(2)} + 1)/5 = (47/85). \tag{4.3}$$

Proof. The values of δ_2 and α_3 given above are precisely those given by Davenport's method (cf. Lemmas 4.2 and 4.3 in [1]) with $k = 5$, $\alpha = \alpha_2$, $l = 2$. The fact that the integers $x_1^k + x_2^k$ are not necessarily distinct will not affect the conclusion in this lemma, since, arguing as in Lemma 2.5 (cf. (3.20) and (3.22)), the same estimate as in Davenport's method (where $x_1^k + x_2^k$ are taken to be distinct) is obtained for the number of solutions of the equation

$$x^k + (x_1^k + x_2^k) = y^k + (y_1^k + y_2^k) \quad \text{with} \quad x, y \in (P, 2P); \quad x_i, y_i \in (P_i, 2P_i)$$

(where we use the fact that the number of representations of an integer in the form $x_1^k + x_2^k$ is uniformly bounded by P^ε).

5. Estimation of $U_4^{(5)}(N)$.

LEMMA 5.1. *With δ_i $(1 \leqslant i \leqslant 3)$ as in (3.3), let $\lambda_i = \lambda_i^{(3)}$ $(1 \leqslant i \leqslant 3)$, where*

$$\lambda_3^{(3)} = (4+\delta_3)/5, \quad \lambda_2^{(3)} = \lambda_1^{(3)} = (4+\delta_2)\lambda_3^{(3)}/5. \tag{5.1}$$

Then, $\{\lambda_1^{(3)}, \lambda_2^{(3)}, \lambda_3^{(3)}, 1\}$ form admissible exponents, and $U_4^{(5)}(N) > N^{\alpha_4-\varepsilon}$,

where

$$\alpha_4 = (\lambda_1^{(3)} + \lambda_2^{(3)} + \lambda_3^{(3)} + 1)/5 = (1/5) + \lambda_3^{(3)}\alpha_3, \tag{5.2}$$

satisfying

$$0.666881 < \alpha_4 < 0.666883. \tag{5.3}$$

Proof. For proving the result, it is sufficient to show that (with f, P_i's and f_i's as in (1.1))

$$\int_0^1 |f|^2 |f_3 f_2 f_1|^2\, d\alpha \ll P(P_3 P_2 P_1) P^\varepsilon. \tag{5.4}$$

With S denoting the integral in (5.4), we have

$$S \ll P\int_0^1 |f_3 f_2 f_1|^2\, d\alpha + \int_0^1 F(\alpha) |f_3 f_2 f_1|^2\, d\alpha \tag{5.5}$$

$$\ll P(P_3 P_2 P_1) P^\varepsilon + \int_0^1 F(\alpha) |f_3 f_2 f_1|^2\, d\alpha,$$

(using the fact that $\{\lambda_1, \lambda_2, \lambda_3\}$ form admissible exponents) where $F(\alpha)$ is as in (1.2) with $0 < t < P^{\delta_3}$, $P < x < 2P$; and, by Schwarz's inequality (cf. $f_1 = f_2$),

$$\int_0^1 F(\alpha) |f_3 f_2 f_1|^2\, d\alpha \ll \Big\{\int_0^1 |F(\alpha)|^2 |f_3 f_2|^2\, d\alpha\Big\}^{1/2} \Big\{\int_0^1 |f_3|^2 |f_2|^6\, d\alpha\Big\}^{1/2}. \tag{5.6}$$

Using (3.15) and the fact that

$$\int_0^1 |f_3 f_2|^2\, d\alpha \ll (P_3 P_2) P^\varepsilon,$$

we have (from (5.5) and (5.6))

$$S \ll P(P_3 P_2 P_1) P^\varepsilon + \tag{5.7}$$
$$+ \Big\{P^{2\delta_3+1}(P_3 P_2) P^\varepsilon + P^{\delta_3} \int_0^1 F_1(\alpha) |f_3 f_2|^2\, d\alpha\Big\}^{1/2} \{P_3 P_2^{(7/2)+\varepsilon}\}^{1/2},$$

where $F_1(\alpha)$ is as in (1.2) with $0 < t < P^{\delta_3}$, $0 < t_1 < P$, and $P < x < 2P$.

Now, $\int_0^1 F_1(\alpha) |f_3 f_2|^2\, d\alpha$ is the number of solutions of

$$\Delta_{t,t_1}(x^k) + u_i = u_j \quad \text{with } u_i, u_j \text{ of the form } x_3^k + x_2^k; \tag{5.8}$$

and, this is estimated by using Lemma 2.5 with $l = 2$, $P_m = P_3$, $P_n = P_2$, $\tau = 1 + \delta_3$; so that, from (2.22),

$$\int_0^1 F_1(\alpha) |f_3 f_2|^2\, d\alpha \ll P^{1+\delta_3+\varepsilon}(P_3 P_2) + (P^{1+\delta_3+\varepsilon})^{1/2} (P_3 P_2 P^\varepsilon)^{3/2}. \tag{5.9}$$

Thus, from (5.7), we have

$$(5.10)\qquad S \ll P(P_3 P_2 P_1) P^{\varepsilon} + \\ + \{P^{1+2\delta_3+\varepsilon}(P_3 P_2) + P^{\delta_3}(P^{1+\delta_3+\varepsilon})^{1/2}(P_3 P_2)^{3/2}\}^{1/2}(P_3 P_2^{7/2})^{1/2} \\ \ll P(P_3 P_2 P_1) P^{\varepsilon} + \{P^{(1/2)+\delta_3+\varepsilon}\} P_3 P_2^{9/4} + \{P^{(1+3\delta_3)/4}\} P_3^{5/4} P_2^{(5/2)+\varepsilon}.$$

Now, it is verified (with $k = 5$, and the values for δ_2, δ_3) that each of the last two terms in (5.10) is bounded by $PP_3 P_2^{2+\varepsilon}$, proving the result (cf. $P_1 = P_2$).

The remaining lemmas in this section will be used for estimating α_i with $i \geq 5$.

LEMMA 5.2. *With* δ_2, δ_3 *as in* (3.3),

$$(5.11)\qquad M_{4,6} = \int_0^1 |f_4 f_3|^2 |f_2|^6 \, d\alpha \ll S_{4,6},$$

where

$$(5.12)\qquad S_{4,6} = \{P_4^{(1+3\delta_3)/4}\} P_3^{(19/16)+(1/4)\delta_2} P_2^{(257/64)+\varepsilon} \ll (P_4 P_3) P_2^{(4-\gamma_4)}$$

with

$$(5.13)\qquad \gamma_4 = 0.43855.$$

Proof. Here, we use Lemma 2.3 with $l = 2$, $\delta = \delta_3$; P_4 replacing P; $|f_3|^2 |f_2|^6$ replacing $|f_1 \ldots f_s|^2$; f_4 replacing f; $(\prod_{i=1, i \notin \mathscr{B}}^{s} P_i^2) = P_3^2 P_2^4$;

$$(5.14)\qquad M = M_{3,6} = \int_0^1 |f_3|^2 |f_2|^6 \, d\alpha; \\ M_1 = \int_0^1 |f_3|^2 |f_2|^4 \, d\alpha; \quad M_2 = M_{3,8} = \int_0^1 |f_3|^2 |f_2|^8 \, d\alpha.$$

Since $\{\lambda_2, \lambda_2, \lambda_3\}$ form admissible exponents, $M_1 \ll P_3 P^{2+\varepsilon}$ (cf. Lemma 4.2). Hence, from (5.14), as in (3.26) (cf. (3.15) and (3.22)),

$$(5.15)\qquad M_{4,6} \ll P_4(S_{3,6}) + P_4^{1+\delta_3+\varepsilon}(S_{3,6})^{1/2}(S_{3,8})^{1/4} \times \\ \times \{P_4^{-1}(P_3 P_2^2) + P_4^{-\delta_3-3}(P_3^2 P_2^4)\}^{1/4}.$$

It is easily verified (with the values for δ_2, δ_3) that the last term on the right-side of (5.15) dominates over the other two; so that,

$$(5.16)\qquad M_{4,6} \ll P_4^{(1+3\delta_3)/4+\varepsilon}(P_3^{1/2} P_2)(S_{3,6})^{1/2}(S_{3,8})^{1/4}.$$

(5.12) is now verified from (5.16), (5.14), (3.15) and (3.22).

[The following combinations of terms may be used in verifying (5.12):

(Such combinations will be especially useful in dealing with estimates that involve large number of factors, and will be indicated as they occur).

$$(S_{3,6})^{1/2}(S_{3,8})^{1/4}(P_3 P_2^2)^{1/4} = P_3 P_2^{(7/4)+(6-\gamma_1)/4+(1/2)+\varepsilon}$$

(from (3.16) and (3.23)), and

$$(S_{3,6})^{1/2}(S_{3,8})^{1/4}(P_3^2 P_2^4)^{1/4} = P_3 (P_3 P_2^2)^{1/4} P_2^{(7/4)+(6-\gamma_1)/4+(1/2)+\varepsilon};$$

also, $(P_3 P_2^2)^{1/4} = P_4^{5\lambda_3\alpha_3/4}$, which can be estimated by using (4.3). Now, the terms in P_4 can be estimated to be $\ll P_4 P_2^{\gamma}$ (for suitable γ) by using $P_2 = P_4^{\lambda_2}$].

LEMMA 5.3. *With* δ_2, δ_3 *as in* (3.3),

$$(5.17)\qquad M_{4,8} = \int_0^1 |f_4 f_3|^2 |f_2|^8 \, d\alpha \ll S_{4,8},$$

where

$$(5.18)\qquad S_{4,8} = \{P_4^{(1+3\delta_3)/4}\} P_3^{(17/16)+(3/4)\delta_2} P_2^{(363/64)+\varepsilon} \ll (P_4 P_3) P_2^{(6-\gamma_5)}$$

with

$$(5.19)\qquad \gamma_5 = 0.69065.$$

Proof. Here, we use the argument in Lemma 2.1 twice as follows:

$$(5.20)\qquad M_{4,8} \ll P_4(M_{3,8}) + \int_0^1 F(\alpha) |f_3|^2 |f_2|^8 \, d\alpha,$$

where $F(\alpha)$ is as in (1.2) with $0 < t < P_4^{\delta_3}$, and $P_4 < x < 2P_4$.

Also (by Schwarz's inequality)

$$(5.21)\qquad \int_0^1 F(\alpha)|f_3|^2|f_2|^8 \, d\alpha \ll \Big\{\int_0^1 |F(\alpha)|^2 |f_3|^2 |f_2|^6 \, d\alpha\Big\}^{1/2} \Big\{\int_0^1 |f_3|^2 |f_2|^{10} \, d\alpha\Big\}^{1/2};$$

and, (with the standard way of estimating $|F(\alpha)|^2$)

$$(5.22)\qquad \int_0^1 |F(\alpha)|^2 |f_3|^2 |f_2|^6 \, d\alpha \ll P_4^{2\delta_3+1}(M_{3,6}) + P_4^{\delta_3} \int_0^1 F_1(\alpha) |f_3|^2 |f_2|^6 \, d\alpha,$$

where $F_1(\alpha)$ is as in (1.2) (with $0 < t < P_4^{\delta_3}$, $0 < t_1 < P_4$, $P_4 < x < 2P_4$).

Repeating the above argument,

$$(5.23)\qquad \int_0^1 F_1(\alpha)|f_3|^2|f_2|^6 \, d\alpha \ll \Big\{\int_0^1 |F_1(\alpha)|^2 |f_3|^2 |f_2|^4 \, d\alpha\Big\}^{1/2} \Big\{\int_0^1 |f_3|^2 |f_2|^8 \, d\alpha\Big\}^{1/2},$$

and,

$$(5.24)\qquad \int_0^1 |F_1(\alpha)|^2 |f_3|^2 |f_2|^4 \, d\alpha \ll P_4^{3+2\delta_3}(P_3 P_2^{2+\varepsilon}) + P_4^{1+\delta_3} \Big\{\int_0^1 F_2(\alpha) |f_3|^2 |f_2|^4 \, d\alpha\Big\}$$

(using the fact that $\int_0^1 |f_3|^2 |f_2|^4\, d\alpha \ll P_3 P_2^{2+\varepsilon}$, and estimating $|F_1(\alpha)|^2$ in the standard way), where $F_2(\alpha)$ is as in (1.2) with $0 < t < P_4^{\delta_3}$, $0 < t_1 < P_4$, $0 < t_2 < P_4$, and $P_4 < x < 2P_4$.

Also, $\int_0^1 F_2(\alpha) |f_3|^2 |f_2|^4\, d\alpha$ is equal to the number of solutions of

(5.25) $$\Delta_{t,t_1,t_2}(x^k) + u_i = u_j,$$

where u_i, u_j are of the form $x_3^k + x_2^k + y_2^k$ with $x_i, y_i \in (P_i, 2P_i)$.

Now, the number of solutions of (5.25) is $\ll P_3^2 P_2^{4+\varepsilon}$ since, there are $\ll P_3^2 P_2^4$ choices for u_i, u_j (and then, $\ll P_2^{\varepsilon}$ choices for t, t_1, t_2 and x in (5.25)); so that,

(5.26) $$\int_0^1 F_2(\alpha) |f_3|^2 |f_2|^4\, d\alpha \ll P_3^2 P_2^{4+\varepsilon}.$$

From (5.26), (5.24) and (5.23), it follows that

(5.27) $$\int_0^1 F_1(\alpha) |f_3|^2 |f_2|^6\, d\alpha \ll \{P_4^{1+\delta_3} (P_3^2 P_2^{4+\varepsilon})\}^{1/2} (M_{3,8})^{1/2}$$

(the other term occurring in the estimate being bounded by the estimate in (5.27) since, $P_4^{2+\delta_3} \ll P_3 P_2^2$ with the values for δ_2 and δ_3).

From (5.22) and (5.27), it is verified that (cf. (3.15) and (3.22))

(5.28) $$\int_0^1 |F(\alpha)|^2 |f_3|^2 |f_2|^6\, d\alpha \ll P_4^{\delta_3} \{P_4^{1+\delta_3} (P_3^2 P_2^{4+\varepsilon})\}^{1/2} (S_{3,8})^{1/2}$$

(as the right-side term in (5.28) dominates over $P_4^{1+2\delta_3}(S_{3,6})$).

Accordingly, from (5.21),

(5.29) $$\int_0^1 F(\alpha) |f_3|^2 |f_2|^8\, d\alpha \ll P_4^{(1/2)\delta_3} \{P_4^{1+\delta_3} (P_3^2 P_2^{4+\varepsilon})\}^{1/4} (S_{3,8})^{1/4} (S_{3,10})^{1/2}.$$

The term on the right-side of (5.29) is verified to satisfy (5.18) (cf. (3.23) and (3.28)) and that, it dominates over $P_4(S_{3,8})$. Hence, result follows from (5.20).

Lemma 5.4. *With δ_2, δ_3 as in* (3.3),

(5.30) $$M_{4,12} = \int_0^1 |f_4 f_3|^2 |f_2|^{12}\, d\alpha \ll S_{4,12},$$

where

(5.31) $$S_{4,12} = (P_4 P_3) P_2^{(10-\gamma_6)}$$

with

(5.32) $$\gamma_6 = 1.1125.$$

Proof. Here, we use Lemma 2.2 with $l = 3$, $\delta = \delta_3$, $P = P_4$, $f = f_4$;

$$(P_1 \ldots P_s)^2 = P_3^2 P_2^{12}, \qquad M = M_{3,12} = \int_0^1 |f_3|^2 |f_2|^{12}\, d\alpha.$$

Accordingly, we have (cf. (2.10) and (3.31))

(5.33) $$M_{4,12} \ll P_4(S_{3,12}) + P_4^{(7/8)+\delta_3+\varepsilon}(S_{3,12}) + P_4^{(1/2)+(7/8)\delta_3+\varepsilon} (P_3^2 P_2^{12})^{1/8} (S_{3,12})^{7/8}.$$

The third term on the right-side of (5.33) dominates over the other two, and (5.31) is verified from this.

[Note: From (3.32), $(P_3^2 P_2^{12})^{1/8} (S_{3,12})^{7/8} = P_3 (P_3 P_2^2)^{1/8} P_2^{7(10-\gamma_3)/8+(5/4)}$, and $(P_3 P_2^2)^{1/8} = P_4^{5\lambda_3\alpha_3/8}$].

6. Estimation of $U_5^{(5)}(N)$.

Lemma 6.1. *With δ_i $(1 \leqslant i \leqslant 4)$ as in* (3.3), *let $\lambda_i = \lambda_i^{(4)}$ $(1 \leqslant i \leqslant 4)$, where*

(6.1) $$\lambda_4^{(4)} = (4+\delta_4)/5, \qquad \lambda_i^{(4)} = (4+\delta_i)\lambda_{i+1}^{(4)}/5 \qquad (1 \leqslant i \leqslant 3).$$

Then, $\{\lambda_1^{(4)}, \ldots, \lambda_4^{(4)}, 1\}$ form admissible exponents, and $U_5^{(5)}(N) > N^{\alpha_5 - \varepsilon}$, where

(6.2) $$\alpha_5 = (\lambda_1^{(4)} + \ldots + \lambda_4^{(4)} + 1)/5 = (1/5) + \lambda_4^{(4)} \alpha_4,$$

satisfying

(6.3) $$0.755231 < \alpha_5 < 0.755235.$$

Proof. It suffices to show that

(6.4) $$\int_0^1 |f|^2 |f_4 f_3 f_2 f_1|^2\, d\alpha \ll P (P_4 \ldots P_1) P^{\varepsilon},$$

and, for this, we use Lemma 2.3 with $l = 2$, $\delta = \delta_4$, $s = 4$;

(6.5) $$\Big(\prod_{\substack{i=1\\ i\neq \text{[illegible]}}}^{s} P_i^2\Big) = (P_4 P_3 P_2)^2; \qquad M = \int_0^1 |f_4 f_3 f_2 f_1|^2\, d\alpha;$$
$$M_1 = \int_0^1 |f_4 f_3 f_2|^2\, d\alpha \quad \text{and} \quad M_2 = M_{4,6} = \int_0^1 |f_4 f_3|^2 |f_2|^6\, d\alpha.$$

Accordingly, with S denoting the integral in (6.4), we have (cf. (2.11))

(6.6) $$S \ll PM + P^{1+\delta_4+\varepsilon} M^{1/2} (M_{4,6})^{1/4} \{P^{-1} M_1 + P^{-\delta_4-3} (P_4 P_3 P_2)^2\}^{1/4}.$$

Now, $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ form admissible exponents (cf. Lemma 5.1), and $\{\lambda_2, \lambda_3, \lambda_4\}$ being a subset of this, also form admissible exponents (cf. Lemma 7.1 in [1]). Hence, from (6.5),

(6.7) $$M \ll (P_4 \ldots P_1) P^{\varepsilon} \quad \text{and} \quad M_1 \ll (P_4 P_3 P_2) P^{\varepsilon};$$

so that, from (6.6) (cf. (5.11))

$$S \ll P(P_4 \dots P_1)P^{\varepsilon} + (P_4 \dots P_1)^{1/2}(S_{4,6})^{1/4}(P_4 P_3 P_2)^{1/4} \times \times \{P^{(3/4)+\delta_4+\varepsilon} + P^{(1+3\delta_4)/4+\varepsilon}(P_4 P_3 P_2)^{1/4}\}. \tag{6.8}$$

Now (with $P_4 = P^{\lambda_4}$, and the values of the δ's), it is verified that the second term on the right-side of (6.8) dominates over the third term (since $P^{2+\delta_4} \gg P_4 P_3 P_2$), and that, it is bounded by the first term, proving the lemma. [Here, from (5.12) (with $P_1 = P_2$), the second term in the estimate is

$$\ll (P_4 \dots P_1) P_2^{(4-\gamma_4)/4-(3/4)} P^{(3/4)+\delta_4} P^{\varepsilon};$$

so that, it suffices to show that

$$P_2^{(1-\gamma_4)/4} P^{(3/4)+\delta_4} \ll P^{1-\varepsilon}$$

(using $P_2 = P^{\lambda_2}$)].

The next three lemmas will be used in the remaining sections.

LEMMA 6.2. *Subject to* (3.3),

$$M_{5,6} = \int_0^1 |f_5 f_4 f_3|^2 |f_2|^6 \, d\alpha \ll S_{5,6}, \tag{6.9}$$

where

$$S_{5,6} = (P_5 P_4 P_3) P_2^{(4-\gamma_7)} \tag{6.10}$$

with

$$\gamma_7 = 0.3774. \tag{6.11}$$

Proof. We use Lemma 2.2 with $l = 3$, $\delta = \delta_4$, $P = P_5$, $f = f_5$;

$$(P_1 \dots P_s)^2 = (P_4 P_3)^2 P_2^6 \quad \text{and} \quad M = M_{4,6} = \int_0^1 |f_4 f_3|^2 |f_2|^6 \, d\alpha.$$

Hence (cf. (2.10) and (5.11)),

$$M_{5,6} \ll P_5(S_{4,6}) + P_5^{(7/8)+\delta_4+\varepsilon}(S_{4,6}) + + P_5^{(1/2)+(7/8)\delta_4+\varepsilon}(P_4^2 P_3^2 P_2^6)^{1/8}(S_{4,6})^{7/8}. \tag{6.12}$$

(6.10) is now verified from (6.12) and (5.12).

[Note:

$$(P_4^2 P_3^2 P_2^6)^{1/8}(S_{4,6})^{7/8} \ll (P_4 P_3)(P_4 P_3 P_2^2)^{1/8} P_2^{7(4-\gamma_4)/8+(1/2)};$$

$$(P_4 P_3 P_2^2)^{1/8} = P_5^{5\lambda_4\alpha_4/8},$$

so that, we can use (5.3) (and $P_2 = P_5^{\lambda_2}$)].

LEMMA 6.3. *Subject to* (3.3),

$$M_{5,8} = \int_0^1 |f_5 f_4 f_3|^2 |f_2|^8 \, d\alpha \ll S_{5,8}, \tag{6.13}$$

where

$$S_{5,8} = (P_5 P_4 P_3) P_2^{(6-\gamma_8)} \tag{6.14}$$

with

$$\gamma_8 = 0.6211. \tag{6.15}$$

Proof. We use Lemma 2.2 with $l = 3$, $\delta = \delta_4$, $P = P_5$, $f = f_5$;

$$(P_1 \dots P_s)^2 = (P_4 P_3)^2 P_2^8 \quad \text{and} \quad M = M_{4,8} = \int_0^1 |f_4 f_3|^2 |f_2|^8 \, d\alpha;$$

so that, from (2.10) and (5.17),

$$M_{5,8} \ll P_5(S_{4,8}) + P_5^{(7/8)+\delta_4+\varepsilon}(S_{4,8}) + + P_5^{(1/2)+(7/8)\delta_4+\varepsilon}(P_4^2 P_3^2 P_2^8)^{1/8}(S_{4,8})^{7/8}. \tag{6.16}$$

(6.14) is now verified from (6.16) and (5.18).

[Note:

$$(P_4^2 P_3^2 P_2^8)^{1/8}(S_{4,8})^{7/8} = (P_4 P_3)(P_4 P_3 P_2^2)^{1/8} P_2^{7(6-\gamma_5)/8+(3/4)},$$

and

$$(P_4 P_3 P_2^2)^{1/8} = P_5^{5\lambda_4\alpha_4/8}].$$

LEMMA 6.4. *Subject to* (3.3),

$$M_{5,12} = \int_0^1 |f_5 f_4 f_3|^2 |f_2|^{12} \, d\alpha \ll S_{5,12}, \tag{6.17}$$

where

$$S_{5,12} = (P_5 P_4 P_3) P_2^{(10-\gamma_9)} \tag{6.18}$$

with

$$\gamma_9 = 0.9901. \tag{6.19}$$

Proof. We apply Lemma 2.2 as in the preceding lemma with the difference that

$$(P_1 \dots P_s)^2 = (P_4 P_3)^2 P_2^{12} \quad \text{and} \quad M = M_{4,12} = \int_0^1 |f_4 f_3|^2 |f_2|^{12} \, d\alpha.$$

Accordingly (cf. (5.30)),

$$M_{5,12} \ll P_5(S_{4,12}) + P_5^{(7/8)+\delta_4+\varepsilon}(S_{4,12}) + \\ + P_5^{(1/2)+(7/8)\delta_4+\varepsilon}(P_4^2 P_3^2 P_2^{12})^{1/8}(S_{4,12})^{7/8}. \tag{6.20}$$

(6.18) is now verified from (6.20) and (5.31).

[Note: $(P_4^2 P_3^2 P_2^{12})^{1/8}(S_{4,12})^{7/8} = (P_4 P_3)(P_4 P_3 P_2^2)^{1/8} P_2^{7(10-\gamma_6)/8+(5/4)}$].

7. Estimation of $U_6^{(5)}(N)$.

LEMMA 7.1. *With δ_i $(1 \leqslant i \leqslant 5)$ as in (3.3), let $\lambda_i = \lambda_i^{(5)}$ $(1 \leqslant i \leqslant 5)$, where*

$$\lambda_5^{(5)} = (4+\delta_5)/5, \quad \lambda_i^{(5)} = (4+\delta_i)\lambda_{i+1}^{(5)}/5 \quad (1 \leqslant i \leqslant 4). \tag{7.1}$$

Then, $\{\lambda_1^{(5)}, \ldots, \lambda_5^{(5)}, 1\}$ form admissible exponents, and $U_6^{(5)}(N) > N^{\alpha_6-\varepsilon}$, where

$$\alpha_6 = (\lambda_1^{(5)} + \ldots + \lambda_5^{(5)} + 1)/5 = (1/5) + \lambda_5^{(5)}\alpha_5, \tag{7.2}$$

satisfying

$$0.823065 < \alpha_6 < 0.823069. \tag{7.3}$$

Proof. In Lemma 2.4, we take $\alpha = \alpha_5$, $l = 3$. It is verified from (6.3) that (2.13) holds; so that, we can take $\delta = \delta_5 = (1/2^3)$, and the result follows.

The next two lemmas will be used in the remaining sections.

LEMMA 7.2. *Subject to* (3.3),

$$M_{6,8} = \int_0^1 |f_6 f_5 f_4 f_3|^2 |f_2|^8 \, d\alpha \ll S_{6,8}, \tag{7.4}$$

where

$$S_{6,8} = (P_6 P_5 P_4 P_3) P_2^{(6-\gamma_{10})} \tag{7.5}$$

with

$$\gamma_{10} = 0.588. \tag{7.6}$$

Proof. We use Lemma 2.3 with $l = 3$, $\delta = \delta_5$, $P = P_6$;

$$\Big(\prod_{\substack{i=1 \\ i \neq 8}}^{s} P_i^2\Big) = (P_5 P_4 P_3)^2 P_2^4; \quad (f_1 \ldots f_s)^2 = (f_5 f_4 f_3)^2 f_2^8;$$

$$M = M_{5,8}; \quad M_1 = \int_0^1 |f_5 f_4 f_3|^2 |f_2|^4 \, d\alpha \quad \text{and} \quad M_2 = M_{5,12} \tag{7.7}$$

(cf. (6.13) and (6.17)). Accordingly, from (2.11),

$$M_{6,8} \ll P_6(M_{5,8}) + \\ + P_6^{1+\delta_5+\varepsilon}(M_{5,8})^{3/4}(M_{5,12})^{1/8}\{P_6^{-1} M_1 + P_6^{-\delta_5-4}(P_5 P_4 P_3)^2 P_2^4\}^{1/8}. \tag{7.8}$$

Now, $\{\lambda_2, \lambda_2, \lambda_3, \lambda_4, \lambda_5\}$ form admissible exponents by Lemma 6.1 (cf. $\lambda_1 = \lambda_2$); so that, $M_1 \ll (P_5 P_4 P_3) P_2^{2+\varepsilon}$. Hence, from (7.8) (cf. (6.13) and (6.17)), we have

$$M_{6,8} \ll P_6(S_{5,8}) + (S_{5,8})^{3/4}(S_{5,12})^{1/8}(P_5 P_4 P_3 P_2^2)^{1/8} \times \\ \times \{P_6^{(7/8)+\delta_5+\varepsilon} + P_6^{(1/2)+(7/8)\delta_5+\varepsilon}(P_5 P_4 P_3 P_2^2)^{1/8}\}. \tag{7.9}$$

The result is now verified from (7.9), (6.14) and (6.18).

[Note:

$$(P_5 P_4 P_3 P_2^2)^{1/8}(S_{5,8})^{3/4}(S_{5,12})^{1/8} = (P_5 P_4 P_3) P_2^{3(6-\gamma_8)/4+(10-\gamma_9)/8+(1/4)};$$

also,

$$(P_5 P_4 P_3 P_2^2)^{1/8} = P_6^{5\lambda_5\alpha_5/8},$$

so that, we can use (6.3) (and $P_2 = P_6^{\lambda_2}$)].

The next lemma is essentially Lemma 9.3 in [1] (with the I_j's as in the Fundamental Lemma).

LEMMA 7.3. *Let $j \geqslant k-1$, and*

$$\alpha_j < 1 - 1/k(2^{k-2}). \tag{7.10}$$

Then,

$$I_j = \int_0^1 |f_j|^2 |f_1 \ldots f_j|^2 \, d\alpha \ll P_j^{2-2\sigma+\delta_0}(P_1 \ldots P_j), \tag{7.11}$$

where

$$\sigma = 1/2^{k-1}. \tag{7.12}$$

[In the next three sections, (6.2) in [1] may be used to verify $S_0 \ll S_1$].

8. Estimation of $U_7^{(5)}(N)$.

LEMMA 8.1. *With δ_i $(1 \leqslant i \leqslant 6)$ as in (3.3), let $\lambda_i = \lambda_i^{(6)}$ $(1 \leqslant i \leqslant 6)$, where*

$$\lambda_6^{(6)} = (4+\delta_6)/5, \quad \lambda_i^{(6)} = (4+\delta_i)\lambda_{i+1}^{(6)}/5 \quad (1 \leqslant i \leqslant 5). \tag{8.1}$$

Then, $\{\lambda_1^{(6)}, \ldots, \lambda_6^{(6)}, 1\}$ form admissible exponents, and $U_7^{(5)}(N) > N^{\alpha_7-\varepsilon}$, where

$$\alpha_7 = (\lambda_1^{(6)} + \ldots + \lambda_6^{(6)} + 1)/5 = (1/5) + \lambda_6^{(6)}\alpha_6 > 0.875207. \tag{8.2}$$

Proof. We apply Lemma 8.1 in [1] with $l = 3$, $s = 6$; so that, from (2.7), (2.8), (2.9), (2.10), (8.3) and (8.4) in [1],

$$S \ll S_0 + S_1 + S_2'' + T'', \tag{8.3}$$

where

$$S_0 = (P^{(1/2)+\delta_6+\varepsilon} U) P_6^{1/2} P^{-\sigma_6+\delta_0}, \tag{8.4}$$

(8.5) $$S_1 = (P^{\delta_6/2+\varepsilon} U) P^{(1/4+\tau_1/2)} (P_6^{1/2} P_5^{1/4}) P^{-(\sigma_6+\sigma_5/2)+\delta_0},$$

(8.6) $$S_2'' = (P^{\delta_6/2+\varepsilon} U) P^{(1/8+\tau_1/4+\tau_2/4)} (P_6^{1/2} P_5^{1/4}) P^{-(\sigma_6+\sigma_5/2)+\delta_0},$$

(8.7) $$T'' = (P^{\delta_6/2+\varepsilon} U) P^{(\tau_1/4+\tau_2/8)} \times (P_6^{1/2} P_5^{1/4} P_4^{1/8})(P_1 P_2 P_3)^{1/8} P^{-(\sigma_6+\sigma_5/2)+\delta_0},$$

with

(8.8) $$\sigma_6 = \lambda_6/16, \quad \sigma_5 = \lambda_5/16, \quad \tau_1 = 5\lambda_5-3, \quad \tau_2 = 5\lambda_4-2$$

(cf. (2.4) in [1]).

(In choosing σ_5 and σ_6, we use the fact that α_5 and α_6 satisfy (7.10).) From these, it is verified that $S \ll P^{1+\varepsilon} U$ (with $U = P_1 \ldots P_6$), proving the result. (Note that $(P_1 \ldots P_4)^{1/8} = P^{5\lambda_4\alpha_4/8}$, and this can be estimated by using (5.3).)

9. Estimation of $U_8^{(5)}(N)$.

Lemma 9.1. *With δ_i $(1 \leqslant i \leqslant 7)$ as in (3.3), let $\lambda_i = \lambda_i^{(7)}$ $(1 \leqslant i \leqslant 7)$, where*

(9.1) $$\lambda_7^{(7)} = (4+\delta_7)/5, \quad \lambda_i^{(7)} = (4+\delta_i)\lambda_{i+1}^{(7)}/5 \quad (1 \leqslant i \leqslant 6).$$

Then, $\{\lambda_1^{(7)}, \ldots, \lambda_7^{(7)}, 1\}$ form admissible exponents, and $U_8^{(5)}(N) > N^{\alpha_8-\varepsilon}$, where

(9.2) $$\alpha_8 = (\lambda_1^{(7)} + \ldots + \lambda_7^{(7)} + 1)/5 = (1/5) + \lambda_7^{(7)}\alpha_7 > 0.918981.$$

Proof. Here, we apply Lemma 8.2 in [1] with $l = 3$, $s = 7$. In the notation of that lemma, $\mathscr{B}$ will consist of the single integer 2, and (with $f_1 = f_2$) we take (cf. (8.11) in [1])

(9.3) $$I' = M_{5,6} = \int_0^1 |f_5 f_4 f_3|^2 |f_2|^6 \, d\alpha \ll S_{5,6} \quad \text{(cf. (6.9))}.$$

The number ξ in (8.11) (in [1]) can be taken to be $\lambda_2\gamma_7$, where γ_7 is given by (6.11) (since $P_2^{\gamma_7} = P^{\lambda_2\gamma_7}$). Accordingly, from (8.12) and (8.13) (in [1]), we get the estimate

(9.4) $$S \ll S_0 + S_1 + S_2''' + T''',$$

where

(9.5) $$S_0 = (P^{(1/2)+\delta_7+\varepsilon} U) P_7^{1/2} P^{-\sigma_7+\delta_0},$$

(9.6) $$S_1 = (P^{\delta_7/2+\varepsilon} U) P^{(1/4+\tau_1/2)} (P_7^{1/2} P_6^{1/4}) P^{-\theta},$$

(9.7) $$S_2''' = (P^{\delta_7/2+\varepsilon} U)(P^{(1/8+\tau_1/4+\tau_2/4)})(P_7^{1/2} P_6^{1/4} P_2^{1/8}) P^{-\theta-(\lambda_2\gamma_7/8)},$$

(9.8) $$T''' = (P^{\delta_7/2+\varepsilon} U) P^{(\tau_1/4+\tau_2/8)} (P_7^{1/2} P_6^{1/4} P_5^{1/8})(P_1 \ldots P_4)^{1/8} P^{-\theta-(\lambda_2\gamma_7/8)},$$

with

(9.9) $$\sigma_7 = \lambda_7/16, \quad \sigma_6 = \lambda_6/16, \quad \tau_1 = 5\lambda_6-3, \quad \tau_2 = 5\lambda_5-2$$

(using the fact that α_6, α_7 satisfy (7.10)), and

(9.10) $$\theta = \sigma_7 + \sigma_6/2 - \delta_0.$$

The lemma now follows on verifying $S \ll P^{1+\varepsilon} U$ with $U = P_1 \ldots P_7$. (Here, $(P_1 \ldots P_5)^{1/8} = P^{5\lambda_5\alpha_5/8}$, which is estimated by using (6.3).)

10. Estimation of $U_9^{(5)}(N)$.

Lemma 10.1. *With $\delta_i (1 \leqslant i \leqslant 8)$ as in (3.3), let $\lambda_i = \lambda_i^{(8)}$ $(1 \leqslant i \leqslant 8)$, where*

(10.1) $$\lambda_8^{(8)} = (4+\delta_8)/5, \quad \lambda_i^{(8)} = (4+\delta_i)\lambda_{i+1}^{(8)}/5 \quad (1 \leqslant i \leqslant 7).$$

Then, $\{\lambda_1^{(8)}, \ldots, \lambda_8^{(8)}, 1\}$ form admissible exponents, and $U_9^{(5)}(N) > N^{\alpha_9-\varepsilon}$, where

(10.2) $$\alpha_9 = (\lambda_1^{(8)} + \ldots + \lambda_8^{(8)} + 1)/5 = (1/5) + \lambda_8^{(8)}\alpha_8 > 0.95098.$$

Proof. We use Lemma 8.2 in [1] with $l = 3$, $s = 8$, $\mathscr{B} = \{1, 2\}$; so that, (with $f_1 = f_2$) we take

(10.3) $$I' = M_{6,8} = \int_0^1 |f_6 f_5 f_4 f_3|^2 |f_2|^8 \, d\alpha \ll S_{6,8} \quad \text{(cf. (7.4))}.$$

The number ξ in (8.11) (in [1]) is now taken to be $\lambda_2\gamma_{10}$, where γ_{10} is given by (7.6); so that, from (8.12) and (8.13) (in [1]), we have the estimate

(10.4) $$S \ll S_0 + S_1 + S_2''' + T''',$$

where

(10.5) $$S_0 = (P^{(1/2)+\delta_8+\varepsilon} U) P_8^{1/2} P^{-\sigma_8+\delta_0},$$

(10.6) $$S_1 = (P^{\delta_8/2+\varepsilon} U) P^{(1/4+\tau_1/2)} (P_8^{1/2} P_7^{1/4}) P^{-\theta},$$

(10.7) $$S_2''' = (P^{\delta_8/2+\varepsilon} U) P^{(1/8+\tau_1/4+\tau_2/4)} (P_8^{1/2} P_7^{1/4} P_2^{1/4}) P^{-\theta-(\lambda_2\gamma_{10}/8)}$$

(using $(P_1 P_2)^{1/8} = P_2^{1/4}$),

(10.8) $$T''' = (P^{\delta_8/2+\varepsilon} U) P^{(\tau_1/4+\tau_2/8)} (P_8^{1/2} P_7^{1/4})(P_1 \ldots P_6)^{1/8} P^{-\theta-(\lambda_2\gamma_{10}/8)},$$

with

(10.9) $$\sigma_8 = \lambda_8/16, \quad \sigma_7 = \lambda_7/16, \quad \tau_1 = 5\lambda_7-3, \quad \tau_2 = 5\lambda_6-2$$

(using the fact that α_7, α_8 satisfy (7.10)), and

(10.10) $$\theta = \sigma_8 + \sigma_7/2 - \delta_0.$$

Noting that $(P_1 \ldots P_6)^{1/8} = P^{5\lambda_6\alpha_6/8}$, and using (7.3), it is now verified that $S \ll P^{1+\varepsilon} U$ with $U = P_1 \ldots P_8$, and the result follows.

11. Proof that $G(5) \leqslant 22$. With $P = N^{1/5}$ and $Q = P^{4+\delta_0}$, divide the unit interval $Q^{-1} < \alpha < 1+Q^{-1}$ into intervals $\mathfrak{m}$ and supplementary intervals m

as in the proof of Lemma 4.8 in [1] (so that, the basic intervals $\mathfrak{m}_{a,q}$'s are defined with $1 \leqslant q \leqslant P^{(5/16)}$, $|\alpha - a/q| \leqslant (qQ)^{-1}$). Let the λ_i's be as in Lemma 10.1 (and the f_i's be defined correspondingly). As already established in [1] (by using Weyl's inequality),

$$f(\alpha) \ll P^{1-(1/16)+\delta_0} \quad \text{if} \quad \alpha \in m. \tag{11.1}$$

Write

$$r_5(N) = \int_{Q^{-1}}^{1+Q^{-1}} f^4(\alpha)\{f(\alpha)f_1(\alpha)\ldots f_8(\alpha)\}^2 e(-N\alpha)\,d\alpha. \tag{11.2}$$

With α_9 defined by (10.2), we see that (from Lemma 10.1) the contribution to $r_5(N)$ from m is

$$\ll P^{4(1-1/16+\delta_0)}(PP_1\ldots P_8)^2 P^{-5\alpha_9+\delta_0} \ll P^{-5-\delta_0}(P^4)(PP_1\ldots P_8)^2 \tag{11.3}$$

since $5\alpha_9 + (4/16) > 5$.

For the treatment of $\mathfrak{m}$, and the transition to the singular series, we make the obvious modifications in [2]. Davenport uses 7 fifth powers in the singular series. (This can however, be simplified by using a larger number of fifth powers.) In place of estimating $(f^7 - g^7)$, and its integral over $\mathfrak{m}$, we estimate $(f^6 f_8 - g^6 g_8)$ and its integral by starting with

$$f^6 f_8 - g^6 g_8 = (f^6 - g^6) f_8 + g^6 (f_8 - g_8),$$

and using (4.30), (4.31), (4.32) in [1]. (Note that $\mathfrak{m}_{a,q}$'s are defined with $q \leqslant P^{1/2}$ in [2], and, this hardly makes any difference.) It would then follow (as in [2]) from (11.3), that $r_5(N) \gg P(P_1 \ldots P_8)^2$, proving that $G(5) \leqslant 22$.

Reference

[2] H. Davenport, *On Waring's problem for fifth and sixth powers*, Amer. J. Math. 64 (1942), pp. 199–207.

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Some problems involving powers of integers

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1. Introduction. There are a number of famous problems which appear to be questions about the distribution of powers of integers. For example, Catalan's conjecture that 8 and 9 are the only powers which differ by 1, and even Fermat's last theorem, have implications of this sort. Many such questions, including Catalan's conjecture, can now be resolved in principle by invoking lower bounds for linear forms in the logarithms of algebraic numbers. (This technique and its applications, which include many powerful results on Diophantine equations and inequalities, are surveyed in [5].) The main point of this paper is a lower bound for simultaneous linear forms in logarithms with some interesting applications to powers and integers which are almost powers.

Consider first the problem of estimating the number of perfect powers in a short interval. J. Turk [7] has shown that the interval $[N, N+N^{1/2}]$ can contain at most $c(\log N)^{1/2}$ powers, for some positive constant c. This is the appropriate interval to examine because any longer interval $[N, N+N^{(1/2)+\varepsilon}]$, with $\varepsilon > 0$, trivially contains $\frac{1}{2}N^{\varepsilon}(1+o(1))$ squares. This question is also discussed in [6], where it arises in the context of exponential Diophantine equations. An appendix to [6] explains how to use linear forms in logarithms, continued fractions and some brute force computation to find all 21 solutions of the inequality $|p^a - q^b| < p^{a/2}$ in positive integers a and b and primes p and q with $p < q < 20$. Probably, the number of prime powers in any interval $[N, N+N^{1/2}]$ is bounded. The computations in [6] give just one example, namely 11^2, 5^3 and 2^7, of three prime powers in such an interval. In this direction, we shall prove:

THEOREM 1. *The interval* $[N, N+N^{1/2}]$ *with* $N \geqslant 16$, *say, contains at most*

$$\exp\left(40(\log\log N \log\log\log N)^{1/2}\right)$$

perfect powers.

Similar questions can be posed about numbers which are almost powers, or have various other assigned multiplicative structures. Such numbers still tend to be sparse. We give the following definition, from a number of