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The distribution of square-free numbers

by

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1. Introduction. Let

$$\Delta(x) = \sum_{n \leqslant x} \mu^2(n) - \frac{6}{\pi^2} x.$$

We write

$$\psi(\theta) = \theta - [\theta] - \tfrac{1}{2} \quad (\theta \text{ real}).$$

Let $\varepsilon > 0$. Montgomery and Vaughan [3] showed, on the Riemann hypothesis, that

$$\Delta(x) = -\sum_{n \leqslant M} \mu(n) \psi\left(\frac{x}{n^2}\right) + O(x^{1/2+\varepsilon} M^{-1/2} + M^{1/2+\varepsilon}), \tag{1}$$

for any $M > 0$. They deduced that

$$\Delta(x) = O(x^{9/28+\varepsilon}).$$

Graham [1] improved the exponent 9/28 to 8/25. In the present note we sharpen this, proving

THEOREM. *If the Riemann hypothesis is correct, then*

$$\Delta(x) = O(x^{7/22+\varepsilon}).$$

The new idea is contained in Lemma 3, which is quite similar to work of Heath-Brown ([2], Section 4). We shall make several appeals to the exponential sum estimate

$$\sum_{a < n \leqslant b} e(\lambda n^{-2}) \ll |\lambda|^{1/2} a^{-1} + |\lambda|^{-1/2} a^2 \tag{2}$$

($0 < a < b \leqslant 2a$, λ real non-zero). See [4], Theorem 5.9. Here $e(\theta) = e^{2\pi i\theta}$; we also write $L = \log x$. Constants implied by '$\ll$' and 'O' notations depend at most on ε.

LEMMA 1. *For some N, H with*

$$x^{7/22} < N \leqslant x^{4/11}, \quad 1/2 \leqslant H \leqslant x^{1/22}, \tag{3}$$

we have

$$\Delta(x) \ll L^2 \Big|\sum_{N<n\leqslant 2N} \sum_{H<h\leqslant 2H} h^{-1}\mu(n)e(hx/n^2)\Big| + x^{7/22+\varepsilon}.$$

Proof. Let $J = x^{1/22}$. Just as in [2], Section 2, we have

(4) $$\psi(\theta) = -\sum_{0<|h|\leqslant J} \frac{1}{2\pi i h} e(\theta h) + O\left(\min\left(1, \frac{1}{J\|\theta\|}\right)\right).$$

Moreover,

$$\min\left(1, \frac{1}{J\|\theta\|}\right) = \sum_{h=-\infty}^{\infty} a_h e(\theta h),$$

where

(5) $$a_h \ll \min(LJ^{-1}, Jh^{-2}).$$

We apply (1) with $M = x^{4/11}$. By a simple splitting up argument for the interval $[1, M]$ we have

(6) $$\Delta(x) \ll L\Big|\sum_{N<n\leqslant 2N} \mu(n)\psi(x/n^2)\Big| + x^{7/22+\varepsilon},$$

where $1/2 \leqslant N \leqslant M/2$. We may evidently suppose that

(7) $$N > x^{7/22}.$$

Now (4) gives

(8) $$\sum_{N<n\leqslant 2N} \mu(n)\psi(x/n^2)$$
$$= -\frac{1}{2\pi i}\sum_{0<|h|\leqslant J}\frac{1}{h}\sum_{N<n\leqslant 2N}\mu(n)e\left(\frac{hx}{n^2}\right) + O\left(\sum_{N<n\leqslant 2N}\min\left(1, \frac{1}{J\|x/n^2\|}\right)\right)$$
$$= -\frac{1}{2\pi i}\sum_{0<|h|\leqslant J}\frac{1}{h}\sum_{N<n\leqslant 2N}\mu(n)e\left(\frac{hx}{n^2}\right) + O\left(\sum_{h=-\infty}^{\infty} a_h \sum_{N<n\leqslant 2N} e\left(\frac{hx}{n^2}\right)\right).$$

An application of (5) and (2) yields

(9) $$\sum_{h=-\infty}^{\infty} a_h \sum_{N<n\leqslant 2N} e\left(\frac{hx}{n^2}\right) \ll LNJ^{-1} + \sum_{h=1}^{\infty}\min\left(\frac{L}{J}, \frac{J}{h^2}\right)\left(\frac{(hx)^{1/2}}{N} + \frac{N^2}{(hx)^{1/2}}\right)$$
$$\ll Lx^{7/22} + LJ^{1/2}x^{1/2}N^{-1} + LJ^{-1/2}x^{-1/2}N^2$$
$$\ll Lx^{7/22},$$

in view of (7). The lemma follows on combining (6), (8), (9) and applying a further splitting argument to the interval $[1, J]$.

LEMMA 2. *Let $1 \leqslant U \leqslant N^{1/3}$. For any complex function f on $(N, 2N]$, the sum*

$$\sum_{N<n\leqslant 2N} \mu(n) f(n)$$

may be decomposed into $O((\log N)^2)$ sums of the form

(I) $$\sum_{\substack{X<m\leqslant X_1 \\ N<mn\leqslant 2N}} a_m \sum_{Y<n\leqslant Y_1} f(mn)$$

with $|a_m| \ll N^{\varepsilon/8}$, $X_1 \leqslant 2X$, $Y_1 \leqslant 2Y$ and

(10) $$Y > 2NU^{-1};$$

(II) $$\sum_{\substack{X<m\leqslant X_1 \\ N<mn\leqslant 2N}} b_m \sum_{Y<n\leqslant Y_1} c_n f(mn)$$

with $|b_m|, |c_n| \ll N^{\varepsilon/8}$, $X_1 \leqslant 2X$, $Y_1 \leqslant 2Y$ and

(11) $$U/8 \leqslant Y \ll N^{1/2}.$$

Proof. According to Montgomery and Vaughan [3],

$$\sum_{N<n\leqslant 2N} \mu(n) f(n) = S_1 + S_2,$$

where

$$S_1 = -\sum_{m\leqslant U^2} \sum_{Nm^{-1}<n\leqslant 2Nm^{-1}} a_m f(mn), \quad a_m = \sum_{\substack{de=m \\ d,e\leqslant U}} \mu(d)\mu(e),$$

$$S_2 = -\sum_{\substack{m>U \\ N<mn\leqslant 2N}} \sum_{n>U} \mu(m) c_n f(mn), \quad c_n = \sum_{\substack{e|n \\ e\leqslant U}} \mu(e).$$

By a splitting up argument applied to $1 \leqslant m \leqslant 2N$, $1 \leqslant n \leqslant 2N$ we decompose S_1 and S_2 into $O((\log N)^2)$ nonempty subsums S_{1j} $(j = 1, 2, \ldots)$, S_{2k} $(k = 1, 2, \ldots)$ with domains of summation of the form

$$X < m \leqslant X_1, \quad Y < n \leqslant Y_1, \quad N < mn \leqslant 2N$$

with $X_1 \leqslant 2X$, $Y_1 \leqslant 2Y$. Evidently $\min(X, Y) < (2N)^{1/2}$. Moreover, $X \geqslant U$ and $Y \geqslant U$ in the case of sums S_{2k}. Since the coefficients $\mu(m)$, c_n are clearly $O(N^{\varepsilon/8})$, each sum S_{2k} is of type (II). (We may have to reverse the roles of m and n.)

For a sum S_{1j} it may be the case that $Y > 2NU^{-1}$; in this case S_{1j} is of type (I). Suppose now that $Y \leqslant 2NU^{-1}$, then

$$U \leqslant 2NY^{-1} < 8X \leqslant 8U^2,$$

also

$$Y > NU^{-2}/4 \geqslant U/4.$$

Evidently

$$U/8 \leqslant \min(X,\ Y) \ll N^{1/2},$$

and S_{1j} is seen to be of type (II).

2. Estimation of type (II) sums.

LEMMA 3. *Let*

$$S = \sum_{\substack{X<m\leqslant X_1\\ N<mn\leqslant 2N}} a_m \sum_{Y<n\leqslant Y_1} b_n \sum_{H<h\leqslant 2H} c_h e\left(\frac{hx}{(mn)^2}\right),$$

where all a_m, b_n, c_h have modulus $\leqslant 1$. Suppose that (3) *holds and that*

(12) $$N^2 x^{-7/11} H^{-1} \leqslant Y \ll N^{1/2}.$$

Then

(13) $$S \ll Hx^{7/22+\varepsilon/2}.$$

Proof. Let Q be a positive integer, to be specified below. Let T_q be the set of (n, h), $Y < n \leqslant Y_1$, $H < h \leqslant 2H$ with

$$2HY^{-2}(q-1) \leqslant Qhn^{-2} \leqslant 2HY^{-2}q.$$

Then

$$S = \sum_{X<m\leqslant X_1} a_m \sum_{q=1}^{Q} \left\{ \sum_{\substack{(n,h)\in T_q\\ N<mn\leqslant 2N}} b_n c_h e\left(\frac{hx}{(mn)^2}\right)\right\}.$$

By Cauchy's inequality,

(14) $$|S|^2 \leqslant XQ \sum_{X<m\leqslant X_1} \sum_{q=1}^{Q} \sum_{\substack{(n,h)\in T_q,(r,k)\in T_q\\ N<mn,mr\leqslant 2N}} b_n c_n \bar{b}_n \bar{c}_r e\left(\frac{(hn^{-2}-kr^{-2})x}{m^2}\right)$$
$$\leqslant XQ \sum_{n,r,h,k;(15)} \Big| \sum_{\substack{X<m\leqslant X_1\\ N<mn,mr\leqslant 2N}} e\big(x(hn^{-2}-kr^{-2})m^{-2}\big)\Big|.$$

Here $\sum_{n,r,h,k;(15)}$ indicates a sum over quadruples with

(15) $$Y < n,\ r \leqslant Y_1, \quad H < h,\ k \leqslant 2H,$$
$$|hn^{-2}-kr^{-2}| \leqslant 2HY^{-2}Q^{-1}.$$

The contribution to $\sum_{n,r,h,k;(15)}$ from quadruples with $hr^2 = kn^2$ is

(16) $$\ll X(HY)^{1+\varepsilon/2}$$

by a divisor argument. The remaining quadruples can be split into $O(L)$ sets defined by (15) and

(17) $$\Delta/2 < |hn^{-2}-kr^{-2}| \leqslant \Delta;$$

here

(18) $$Y^{-4} \ll \Delta \leqslant 2HY^{-2}Q^{-1}.$$

Combining (14), (16) we have

$$|S|^2 \ll X^2 Q(HY)^{1+\varepsilon/2} + LXQ \sum_{n,r,h,k;(15),(17)} \Big| \sum_{\substack{X<m\leqslant X_1\\ N<mn,mr\leqslant 2N}} e\big(x(hn^{-2}-kr^{-2})m^{-2}\big)\Big|$$

for one such Δ.

Now the number of quadruples with (15) and (17) is

$$O(L^2 HY + \Delta HY^4)$$

by the argument of Heath-Brown [2] after his equation (16). We also have the bound

$$\min\left(X,\ (x\Delta)^{1/2} X^{-1} + (x\Delta)^{-1/2} X^2\right)$$

for the exponential sum in (14), by (2). Hence

(19) $$|S|^2 \ll X^2 Q(HY)^{1+\varepsilon/2} +$$
$$+ LXQ(L^2 HY + \Delta HY^4)\min\left(X,\ (x\Delta)^{1/2} X^{-1} + (x\Delta)^{-1/2} X^2\right)$$
$$\ll X^2 Q(HY)^{1+\varepsilon/2} + LQHY^4 x^{1/2} \Delta^{3/2} + LQHY^4 x^{-1/2} X^3 \Delta^{1/2}$$
$$\ll X^2 Q(HY)^{1+\varepsilon/2} + LQHY^4 x^{1/2} (HY^{-2}Q^{-1})^{3/2} +$$
$$+ LQHY^4 x^{-1/2} X^3 (HY^{-2}Q^{-1})^{1/2}$$

in view of (18). We now set

$$Q = [HN^{-2} Yx^{7/11}];$$

note that $Q \geqslant 1$ from (12). Since $N \ll XY \ll N$ (for $S \neq 0$) we have

(20) $$X^2 Q(HY)^{1+\varepsilon/2} \ll X^2 Y^2 N^{-2} H^2 x^{7/11+\varepsilon} \ll H^2 x^{7/11+\varepsilon},$$

(21) $$LH^{3/2} X^3 Y^3 x^{-1/2} Q^{1/2} \ll LH^2 N^2 Y^{1/2} x^{-2/11} \ll H^2 x^{7/11+\varepsilon}.$$

(Here we use the upper bound in (12) together with (3).) Similarly,

(22) $$LQ^{-1/2} H^{5/2} Yx^{1/2} \ll LH^2 NY^{1/2} x^{2/11} \ll H^2 x^{7/11+\varepsilon}.$$

Combining (19)–(22), we obtain the bound (13).

Proof of the Theorem. By Lemma 1 it suffices to show that

(23) $$T = \sum_{N<n\leqslant 2N} \mu(n) \sum_{H<h\leqslant 2H} h^{-1} e\left(\frac{hx}{n^2}\right) \ll x^{7/22+3\varepsilon/4}$$

whenever N, H satisfy (3). We apply Lemma 2 with

$$f(n) = \sum_{H<h\leqslant 2H} h^{-1} e\left(\frac{hx}{n^2}\right),$$
$$U = \max(1,\ 8N^2 x^{-7/11} H^{-1}).$$

Note that $N^{5/3} \leqslant x^{20/33}$, and so $U < N^{1/3}$. The sum in (23) can be decomposed into $O(L^2)$ sums T_i $(i = 1, 2, \ldots)$ each of type (I) or type (II) in the sense of Lemma 2. Suppose $|T_1| \geqslant |T_2| \geqslant \ldots$, then

(24) $$|T| \ll L^2 T_1.$$

Suppose for a moment that T_1 is of type (II); then

(25) $$T_1 \ll N^{\varepsilon/4} H^{-1} S,$$

where S is a sum of the form that appears in Lemma 3; the condition (12) is a consequence of (11). (23) follows on combining (24), (25) and (13), and the theorem is proved in this case.

Now suppose that T_1 is of type (I). If $U = 1$, then T_1 is an empty sum, so we may suppose that $U > 1$. Now

$$T_1 \ll \Bigg|\sum_{\substack{X<m\leqslant X_1\\ N<mn\leqslant 2N}} a_m \sum_{Y<n\leqslant Y_1} e\left(\frac{hx}{(mn)^2}\right)\Bigg|$$

$$\ll XN^{\varepsilon/4} \max_{X<m\leqslant X_1} \Bigg|\sum_{\substack{Y<n\leqslant Y_1\\ N<mn\leqslant 2N}} e\left(\frac{hx}{(mn)^2}\right)\Bigg|$$

for some h, $H < h \leqslant 2H$. We apply (2) one last time, obtaining

$$T_1 \ll XN^{\varepsilon/4}\left((hxX^{-2})^{1/2} Y^{-1} + (hxX^{-2})^{-1/2} Y^2\right)$$
$$\ll N^{\varepsilon/4} h^{1/2} x^{1/2} Y^{-1} + N^{\varepsilon/4} (hx)^{-1/2} N^2.$$

Applying the lower bound (10) we obtain

(26) $$T_1 \ll N^{\varepsilon/4-1} h^{1/2} x^{1/2} U + x^{5/22+\varepsilon}$$
$$\ll N^{\varepsilon/4+1} x^{-3/22} + x^{5/22+\varepsilon} \ll x^{5/22+\varepsilon}.$$

(23) follows on combining (24) and (26), and the proof of the theorem is complete.

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