Concerning a conjecture of R. L. Graham

by

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Graham [1] has conjectured that if \( a_1 < \ldots < a_n \) are positive integers then

\[
\max_{1 \leq i,j \leq n} \frac{a_i}{(a_i, a_j)} \geq n.
\]

Many special cases of this result are known (see Wong [3]). Simpson [2] has shown that no counter example can contain a prime in the sequence.

It is the purpose of this note to show

**Theorem.** If \( a_i < \ldots < a_n \) and \( \frac{a_i}{(a_i, a_j)} < n \) for all \( i, j \) then no \( a_i \) can be a power of a prime.

**Proof.** Suppose that g.c.d. \( (a_1, \ldots, a_n) = 1 \), \( a_k = p^m \) for some prime \( p \), \( 1 \leq k \leq n \) and

\[
\frac{a_i}{(a_i, a_j)} < n, \quad 1 \leq i, j \leq n.
\]

Then the numbers \( a_1, \ldots, a_n \) are all of the form

\[
s_{ij} = ip^{m-i+1}, \quad 1 \leq i \leq n-1, \quad 1 \leq j \leq m+1.
\]

Form the \((n-1) \times (m+1)\) matrix \( S = (s_{ij})\).

Our proof will be complete if we can find a matrix \( T \), a permutation of \( S \), so that at most one distinct element from each row of \( T \) can lie in our sequence. Let \( \sigma \) denote the following function defined on \( 1, \ldots, n-1 \).

\[
\sigma(i) = \begin{cases} 
pi & \text{if } 1 \leq i \leq \left\lfloor \frac{n-1}{p} \right\rfloor \\
\frac{i+1}{p} & \text{where } p^i \| i+1, \text{ if } \left\lfloor \frac{n-1}{p} \right\rfloor < i < n-1 \\
\frac{n-1}{p} + 1 & \text{where } p^i \| \left\lfloor \frac{n-1}{p} \right\rfloor + 1.
\end{cases}
\]
It is easily seen that \( \sigma \) is in fact a permutation of \( \{1, \ldots, n-1\} \). Put \( T_{ij} = (\sigma^{j-1}(i))p^{n-1-i} \); then \( T = (T_{ij}) \) is a permutation of \( S \). Fix \( i \); then at most one number of the form \( T_{ij} \) can be in our sequence.

Put

\[
k_i = \sigma^{i-1}(i).
\]

**Lemma.** If \( i > j \) then either

\[
k_i p^{n-1-i} = k_j p^{n-1-j}
\]

or

\[
\frac{k_j p^{n-1-j+1}}{(k_j p^{n-1-j+1}, k_i p^{n-1-i+1})} \geq n.
\]

**Proof.** By the definition of \( \sigma \), we may assume without loss of generality

\[
k_j > \left\lceil \frac{n-1}{p} \right\rceil \quad \text{and} \quad (k_i, p) = 1.
\]

Now

\[
\frac{k_j p^{n-1-j+1}}{(k_j p^{n-1-j+1}, k_i p^{n-1-i+1})} = \frac{k_j p^{n-1-j}}{(k_j p^{n-1-j}, k_i)} = \frac{k_j p^{n-1-j}}{(k_i, k_j)}.\]

We distinguish two cases:

(i) If \( k_r = n-1 \) for some \( j \leq r < i \).

(ii) Not (i).

We estimate \((k_i, k_j)\).

(i) \( k_j = n-r_1 \) where \( 1 \leq r_1 \leq r-j+1 \),

\[
k_i = \left\lfloor \frac{n-1}{p} \right\rfloor + s \quad \text{where} \quad 1 \leq s \leq i-r.
\]

Then

\[
(k_i, k_j) \leq (p^{r+1}, k_i, k_j)
\]

\[
= \left( ps + p \left\lceil \frac{n-1}{p} \right\rceil, n-r_1 \right)
\]

\[
= (n+p^{s}-a, n-r_1) \quad \text{some} \quad 1 \leq a \leq p
\]

\[
\leq ps-a+r_1 \leq p(i-r)+(r-j)
\]

\[
\leq p(i-j).
\]

Therefore

\[
\frac{k_j}{(k_i, k_j)} p^{i-j} > \frac{(n-(i-j)) p^{i-j}}{p(i-j)} \geq n-1
\]

since \((R_i, p) = 1\) for all \( p, i-j \) except \( p = i-j = 2 \), by considering the derivative of \((n-x)p^{n-1}/x\). For the case \( p = i-j = 2 \) it is easily verified that

\[
\frac{n_j p^{i-j}}{(k_i, k_j)} \geq n.
\]

(ii) \( k_i = \frac{k_j + s}{p^s} \) where \( 1 \leq s \leq i-j \).

Then

\[
(k_i, k_j) \leq (p^s, k_i, k_j) \leq i-j.
\]

Hence

\[
\frac{k_j p^{i-j}}{(k_i, k_j)} \geq k_j p > n-1.
\]

This completes the proof.


**References**


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