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The Hausdorff dimension of systems of linear forms

by

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Let $\sum_{i=1}^{m} x_i a_{ij}$, $1 \leqslant j \leqslant n$, be a system of n real linear forms in m variables. Let $f(x)$ be a positive function of a positive real variable such that the integral

$$\int_1^\infty f(x)^n x^{m-1}\,dx$$

converges. Groshev has proved that the system of inequalities

$$\Big\|\sum_{i=1}^{m} q_i a_{ij}\Big\| < f(q), \quad 1 \leqslant j \leqslant n,$$

where $q = \max\{|q_1|, \ldots, |q_m|\}$, has infinitely many solutions $(q_1, \ldots, q_m)$ in $\boldsymbol{Z}^m$ for "almost no" matrices (a_{ij}) (see [12], p. 33, Theorem 12). As usual, $\|x\| = \inf\{|x-k|: k \in \boldsymbol{Z}\}$, the least distance of the real number x from the integer nearest to x, while 'almost no' matrices (a_{ij}) means that the set of matrices (a_{ij}) (identified with the points $(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{mn})$ in $\boldsymbol{R}^{mn}$) has Lebesgue measure 0. Groshev also proved the complementary result that when the integral $\int_1^\infty f(x)^n x^{m-1}\,dx$ diverges and $f(x)$ satisfies certain convergence and monotonicity conditions, 'almost all' systems of linear forms satisfy the above inequalities for infinitely many integral vectors.

Let $\alpha > m/n$ and for each vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ in $\boldsymbol{R}^n$ write

$$|\boldsymbol{x}| = \max\{|x_1|, \ldots, |x_n|\}.$$

Then the integral $\int_1^\infty f(x)^n x^{m-1}\,dx$ converges for $f(x) = x^{-\alpha}$ $(x > 0)$ and the set of matrices (a_{ij}) satisfying the inequalities

$$\Big\|\sum_{i=1}^{m} q_i a_{ij}\Big\| < |\boldsymbol{q}|^{-\alpha}, \quad 1 \leqslant j \leqslant n, \tag{1}$$

for infinitely many vectors $q = (q_1, \ldots, q_m)$ in Z^m has Lebesgue measure 0. In this paper it is shown that the Hausdorff or fractional dimension of this set of matrices or systems of linear forms is

$$(m-1)\,n+\frac{m+n}{\alpha+1}. \tag{2}$$

The system of linear forms $\sum_{i=1}^{m} x_i a_{ij}$, $1 \leqslant j \leqslant n$, can be written more concisely as the n-dimensional vector xA, where $x = (x_1, \ldots, x_m)$ and $A = (a_{ij})$, and the object of this paper can be regarded as the determination of the Hausdorff dimension of the set of matrices A which send infinitely many lattice points q in Z^m to points qA in R^n which are a distance of at most $|q|^{-\alpha}$ from Z^n. Alternatively the paper can be regarded as studying linear maps from Z^m to the torus R^n/Z^n which send infinitely many points q to within $|q|^{-\alpha}$ of the origin of the torus. Also by doubling the size of the torus, the associated problem, where qA is reflected from the sides of the torus, can be shown to be equivalent to the present one where qA is "translated" to the opposite side.

To simplify the notation further, write $I = (-\frac{1}{2}, \frac{1}{2}]$ and for each vector x in R^n, define $\langle x\rangle$ to be the unique vector $x-r$ in I^n, where $r \in Z^n$. Then the system of inequalities (1) can be written more concisely as

$$|\langle qA\rangle| < |q|^{-\alpha}$$

and we show that the Hausdorff dimension of the set

$$W(m, n) = \{A \in R^{mn}\colon |\langle qA\rangle| < |q|^{-\alpha} \text{ for infinitely many } q \text{ in } Z^m\},$$

where $\alpha > m/n$, is given by (2). Note that for any real number x, $\|x\| = |\langle x\rangle|$. The set

$$W(1, 1) = \{x \in R\colon \|qx\| < |q|^{-\alpha} \text{ for infinitely many } q \text{ in } Z\},$$

where $\alpha > 1$, is the set of well-approximable numbers and was shown by Jarník [9] and Besicovitch [2] to have Hausdorff dimension $2/(\alpha+1)$. Jarník [10] (p. 508, Satz 1) and later Eggleston [8] (p. 60, Theorem 7) extended this result to the set

$$W(1, n) = \{x \in R^n\colon |\langle qx\rangle| < |q|^{-\alpha} \text{ for infinitely many } q \text{ in } Z\},$$

where $\alpha > 1/n$, of simultaneously well-approximable numbers and showed that its Hausdorff dimension is $(n+1)/(\alpha+1)$. The methods used in the present paper for $W(m, n)$ have some features in common with those of Jarník and Besicovitch but also include a 'variance' or 'second moment' argument which has some similarities with an argument due to Littlewood and used by Cassels in his discussion of metrical Diophantine approximation

in Chapter VII of [4]; see also Chapter I, § 7 of [12]. It is not hard to prove that (2) is an upper bound for the Hausdorff dimension of $W(m, n)$ and the difficulty lies in showing that (2) is also a lower bound. The arguments needed are fairly elaborate and rely upon the variance being small. This suggests a parallel with the notion of independence in probability. Indeed the metrical theories of Diophantine approximation just cited and the present paper bear an interesting resemblance to the Borel–Cantelli Lemmas (see [11], pp. 337–340).

The Hausdorff dimension (also called Hausdorff–Besicovitch or fractional dimension) of a set X in R^k will be denoted by $\dim X$ and can be defined as follows. Let $\mathscr{C}$ by any finite or countable cover of X by k-dimensional hypercubes C. For each real number s define the s-volume of a cover $\mathscr{C}$ to be

$$L^s(\mathscr{C}) = \sum_{C\in\mathscr{C}} L(C)^s$$

where $L(C)$ is the length of a side of the hypercube C. For each positive ϱ and real s write

$$\Lambda_\varrho^s(X) = \inf L^s(\mathscr{C})$$

where the infimum is taken over covers $\mathscr{C}$ of X with $L(C) \leqslant \varrho$ for all C in $\mathscr{C}$. Clearly $\Lambda_\varrho^s(X)$ cannot increase as ϱ decreases, and if $s' \geqslant s$,

$$\Lambda_\varrho^{s'}(X) \leqslant \varrho^{s'-s}\Lambda_\varrho^s(X).$$

Thus if $\Lambda^s(X) = \sup_{\varrho>0} \Lambda_\varrho^s(X)$ is finite and if $s' > s$, then $\Lambda^{s'}(X)$ vanishes. The Hausdorff dimension $\dim X$ of X is the supremum of all real s for which $\Lambda^s(X)$ is positive, i.e.

$$\dim X = \sup\{s\colon \Lambda^s(X) > 0\} = \sup\{s\colon \sup_{\varrho>0}\inf_{\mathscr{C}} L^s(\mathscr{C}) > 0\}.$$

It follows that if X can be covered by a collection $\mathscr{C}$ with arbitrarily small s-volume $L^s(\mathscr{C})$, then $\dim X \leqslant s$. On the other hand if for each positive ε, there exists a positive number $\varrho = \varrho(\varepsilon)$ such that every cover $\mathscr{C}$ of X with $L(C) \leqslant \varrho$ satisfies $L^s(\mathscr{C}) > \varepsilon$, then $\dim X \geqslant s$; roughly speaking if the s-volume of covers of small hypercubes of X is large, then $\dim X \geqslant s$. An equivalent condition is that if there exists a positive ε such that for any positive ϱ, collections $\mathscr{C}$ with $L(C) \leqslant \varrho$ and satisfying $L^s(\mathscr{C}) < \varepsilon$ cannot cover X, then $\dim X \geqslant s$. In other words, if collections of small hypercubes and of small s-volume cannot cover X, then $\dim X \geqslant s$. This form will be used to establish that $\dim W(m, n) \geqslant (m-1)\,n+(m+n)/(\alpha+1)$.

Clearly a cover $\mathscr{C}$ of X will be a cover for any subset X' of X and it follows from the definition that if $X' \subseteq X \subseteq R^k$, then

$$\dim X' \leqslant \dim X \leqslant k. \tag{3}$$

When $q \in Z^m$, $\langle qA \rangle$ depends only on the coefficients a_{ij} modulo 1 and so it suffices to consider a_{ij} in the interval $I = (-\frac{1}{2}, \frac{1}{2}]$, i.e. to consider $A \in I^{mn}$. Indeed, since R^{mn} is a countable union of translates of I^{mn},

$$\dim W(m, n) = \dim W,$$

where

$$W = \{A \in I^{mn}\colon |\langle qA \rangle| < |q|^{-\alpha} \text{ infinitely often}\} = W(m, n) \cap I^{mn}.$$

Note that when there is no risk of confusion the Hausdorff dimension will be referred to simply as the dimension.

LEMMA 1. *Let $\alpha > m/n$. Then*

$$\dim W \leqslant (m-1)n + \frac{m+n}{\alpha+1} < mn.$$

Proof. Let $\varepsilon > 0$ be given and let $t > (m-1)n + (m+n)/(\alpha+1)$. For each q in Z^m, r in Z^n, the number of mn-dimensional hypercubes C of width $4|q|^{-(\alpha+1)}$ with centres on the $(m-1)n$-dimensional hyperplane $\{R \in (2I)^{mn}\colon qR = r\}$ at integral multiples of $|q|^{-(\alpha+1)}$ apart is

$$\leqslant |q|^{(\alpha+1)(m-1)n}$$

($(2I)^{mn}$ $\{R = (r_{ij})\colon -1 < r_{ij} \leqslant 1,\ 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$). The collection $\mathscr{C}(q, r)$ of such hypercubes C covers

$$B(q, r) = \{A \in I^{mn}\colon |qA - r| < |q|^{-\alpha}\}$$

and for each $N = 1, 2, \ldots$, the collection

$$\mathscr{C}_N = \{\mathscr{C}(q, r)\colon |r| < \tfrac{1}{2}|q|,\ |q| > N\}$$

covers

$$W = \{A \in I^{mn}\colon |\langle qA \rangle| < |q|^{-\alpha} \text{ infinitely often}\}.$$

The t-volume of $\mathscr{C}_N$ is given by

$$L^t(\mathscr{C}_N) = \sum_q \sum_r \sum_C 4^t |q|^{-(\alpha+1)t},$$

where the sums are over q in Z^m with $|q| > N$, r in Z^n with $|r| < \frac{1}{2}|q|$ and C in $\mathscr{C}(q, r)$. Thus

$$L^t(\mathscr{C}_N) \ll \sum_{q > N} \sum_{|q| = q} \sum_r \sum_C q^{-(\alpha+1)t}$$

$$\ll \sum_{q > N} q^{-(\alpha+1)t + (m-1) + n + (\alpha+1)(m-1)n} < \varepsilon$$

for N sufficiently large since $t > (m-1)n + (m+n)/(\alpha+1)$. Hence $\inf_N L^t(\mathscr{C}_N) = 0$ and the lemma follows by the definition of Hausdorff dimension.

The complementary inequality is much harder to establish and some additional definitions and notation are needed. Let $\alpha > m/n$, let δ be any positive number $< 1/n$ and write

$$t = (m-1)n + \frac{m+n}{\alpha+1} - \delta.$$

Suppose that for some positive ε and any positive ϱ, the countable or finite collection $\mathscr{C}$ of mn-dimensional open hypercubes C with $L(C) \leqslant \varrho$ satisfies

$$L^t(\mathscr{C}) = \sum_{C \in \mathscr{C}} L(C)^t < \varepsilon. \tag{4}$$

It will be shown that no such $\mathscr{C}$ can cover W, so that from the definition of Hausdorff dimension

$$\dim W \geqslant (m-1)n + \frac{m+n}{\alpha+1}.$$

For each q in Z^m and r in Z^n, let $H(q, r)$ be the $(m-1)n$-dimensional hyperplane in I^{mn} given by

$$H(q, r) = \{R \in I^{mn}\colon qR = r\}.$$

($H(q, r)$ is a more general form of the resonant hyperplanes considered in [1]).

Let N be a sufficiently large positive integer, let η satisfy

$$0 < \eta < \min\left\{\frac{m}{n}, \alpha - \frac{m}{n}, \frac{(1+\alpha)\delta}{n}\right\}$$

and let τ in $(0, 1)$ satisfy

$$(m-1)\tau < n\eta \quad (< m).$$

Denote the set of primes by P and the set of vectors p in Z^m which satisfy

$$|p| = p_1 \in P, \quad N < p_1 < 2N, \quad |p_i| < N^{1-\tau}, \quad 2 \leqslant i \leqslant m,$$

by P_N (any vector in P_N will be written p). Then

$$\sum_p 1 = \frac{2^{m-1} N^{m-\tau(m-1)}}{\log N}\left(1 + O\left(\frac{1}{\log N}\right)\right). \tag{5}$$

Let S_N be the collection of hyperplanes given by

$$S_N = \{H(p, r)\colon p \in P_N, |r| < \tfrac{1}{2}|p| = \tfrac{1}{2}p_1\};$$

S_N will be called the *N-skeleton of W*. By definition, each R in $H(p, r)$ satisfies $pR = r$ and so for each k in Z, $(kp)R = (kr)$, whence $R \in W$. Thus $S_N \subseteq W$ (strictly speaking $\{R \in I^{mn}\colon R \in H(p, r) \in S_N\} \subseteq W$). The number of hyperplanes $H(p, r)$ in S_N or the cardinality $|S_N|$ of S_N is given by

$$|S_N| = \sum_q \sum_r 1 = \frac{2^{m-1}(2^{n+1}-1)}{n+1} \cdot \frac{N^{m+n-\tau(m-1)}}{\log N}\left(1+O\left(\frac{1}{\log N}\right)\right)$$

where as always the second sum is over those r in Z^n with $|r| < \frac{1}{2}p_1$. Note that the unit normal to each $H(p, r)$ in S_N is

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ . & . & . & . \\ 1 & 0 & \dots & 0 \end{bmatrix} + O(N^{-\tau}),$$

i.e. the hyperplanes $H(p, r)$ are all almost parallel and almost orthogonal to each x_{1j}-axis, $1 \leqslant j \leqslant n$.

Let

$$D(v, \varrho) = \{x \in R^n: |x-v| < \varrho\}$$

be the n-dimensional open cube of side $2\varrho > 0$, centred at v in R^n and with volume

$$\operatorname{vol} D(v, \varrho) = 2^n \varrho^n.$$

For each q in Z^m the function Φ_q: $I^{mn} \to I^n$ given by

$$\Phi_q(A) = \langle qA \rangle$$

satisfies

$$\Phi_p^{-1}(D(0, \varrho/p_1)) = B_\varrho(p, 0) \cap I^{mn},$$

where $0 < \varrho < \frac{1}{2}$ and $B_\varrho(p, r) = \{A \in R^{mn}: |pA-r| < \varrho\}$. Let $\varrho = \varrho(N)$ be given by

(6) $$\varrho = \varrho(N) = N^{-m/n+\eta}$$

since $n\eta < m$, $\varrho \to 0$ as $N \to \infty$ and $\varrho^{-n} \ll N^m/\log N$. For each x in R^n let

$$\theta(x) = \begin{cases} 1 & \text{if } |x| < \varrho, \\ 0 & \text{otherwise,} \end{cases}$$

i.e. let θ be the characteristic function of $D(0, \varrho)$. Then volume considerations give

(7) $$\int_{R^n} \theta(x)\,dx = 2^n \varrho^n = 2^n N^{-m+n\eta},$$

where $dx = dx_1 \dots dx_n$, and

(8) $$\operatorname{vol}(B_\varrho(p, 0) \cap I^{mn}) = \int_{I^{mn}} \theta(pA)\,dA = 2^n \varrho^n p_1^{-n},$$

where $dA = da_{11} \dots da_{1n} \dots d_{m1} \dots d_{mn}$, whence

$$\operatorname{vol} \Phi_p^{-1}(D(0, \varrho/p_1)) = \operatorname{vol} D(0, \varrho/p_1)$$

(more general results are given in [12], pp. 34–35).

Suppose $\theta(pA-r) = 1$, i.e. suppose that $|pA-r| < \varrho$. Then since the distance function $|x| = \max\{|x_1|, \dots, |x_n|\}$ satisfies the triangle inequality, it follows that any r' in Z^n distinct from r satisfies

$$|pA-r'| \geqslant |r-r'| - |pA-r| \geqslant 1-\varrho.$$

Hence, because N is sufficiently large, $\theta(pA-r') = 0$ and so

(9) $$\sum_r \theta(pA-r) = \begin{cases} 1 & \text{if } |pA-r| < \varrho \text{ for some } r,\ |r| < \frac{1}{2}p_1, \\ 0 & \text{otherwise.} \end{cases}$$

Define ν_N: $I^{mn} \to \{0, 1, 2, \dots\}$ by

$$\nu_N(A) = \sum_p \sum_r \theta(pA-r),$$

so that $\nu_N(A)$ is the number of hyperplanes $H(p, r)$ in S_N such that $|pA-r| < \varrho$ or which are close to A (the Euclidean distance of A from $H(p, r)$ is at most $\varrho/(p_1^2+\dots+p_m^2)^{1/2}$).

Let

$$\mu_N = \int_{I^{mn}} \nu_N(A)\,dA$$

and

$$\sigma_N^2 = \int_{I^{mn}} (\nu_N(A)-\mu_N)^2\,dA = \int_{I^{mn}} \nu_N(A)^2\,dA - \mu_N^2.$$

We now sketch how the lower bound for dim W is obtained. First it is shown that the hyperplanes in the N-skeleton S_N are asymptotically regularly distributed or independent in the sense that the variance σ_N^2 of ν_N is small, satisfying $\sigma_N^2 = o(\mu_N^2)$. This permits the construction of a subset $T(N)$ of I^{mn} which is well enough distributed to ensure that the volume of the intersection of $T(N)$ with hypercubes C from $\mathscr{C}$ can be estimated quite effectively. Moreover $T(N)$ in closely related to W: the intersection $\bigcap_{j=1}^{\infty} T(N_j)$, where $\{N_j: j = 1, 2, \dots\}$ is an increasing sequence of integers, is a subset of W. For sufficiently rapidly increasing sequences, the intersection is not empty and a volume calculation shows that any collection of hypercubes satisfying (4) cannot cover the intersection and hence W. It follows that (2) is a lower bound for dim W.

LEMMA 2.

$$\mu_N = 2^n \varrho^n \sum_{p} \left(1+O\left(\frac{1}{\log N}\right)\right)$$
$$= 2^{m+n-1} N^{n\eta-(m-1)\tau} (\log N)^{-1} \left(1+O\left(\frac{1}{\log N}\right)\right).$$

Proof. By definition

$$\mu_N = \int_{I^{mn}} \nu_N(A)\, dA = \sum_{p}\sum_{r} \int_{I^{mn}} \theta(pA-r)\, dA.$$

Since $N < p_1 < 2N$ and $|p_i| < N^{1-\tau}$, $2 \leqslant i \leqslant m$, and $|r| < \frac{1}{2}p_1$,

$$\mu_N = \sum_{p}\sum_{r} 2^n \varrho^n p_1^{-n}(1+O(N^{-\tau}))$$
$$= 2^n \varrho^n \sum_{p} \left(1-\frac{1}{p_1}\right)^n (1+O(N^{-\tau})) = 2^n \varrho^n \sum_{p} \left(1+O\left(\frac{1}{\log N}\right)\right).$$

The rest of the lemma follows from (5) and (6).

Note that since $n\eta > (m-1)\tau$, $\mu_N \to \infty$ as $N \to \infty$.

The estimate $\sigma_N^2 = o(\mu_N^2)$ is an immediate consequence of the following second moment estimate.

LEMMA 3.

$$\int_{I^{mn}} \nu_N(A)^2\, dA \leqslant \mu_N + \mu_N^2 \left(1+O\left(\frac{1}{\log N}\right)\right).$$

Proof. First, by (9)

$$\left(\sum_{r} \theta(pA-r)\right)\left(\sum_{r'} \theta(pA-r')\right) = \sum_{r} \theta(pA-r),$$

whence

$$\int_{I^{mn}} \nu_N(A)^2\, dA = \int_{I^{mn}} \sum_{p}\sum_{p'}\sum_{r}\sum_{r'} \theta(pA-r)\theta(p'A-r')\, dA$$
$$= \int_{I^{mn}} \sum_{p}\sum_{r} \theta(pA-r)\, dA +$$
$$+ \int_{I^{mn}} \sum_{p} \sum_{r\neq r'} \theta(pA-r)\theta(pA-r')\, dA +$$
$$+ \int_{I^{mn}} \sum_{p\neq p'} \sum_{r}\sum_{r'} \theta(pA-r)\theta(p'A-r')\, dA$$
$$= \mu_N + \sum_{p} \sum_{r\neq r'} \int_{I^{mn}} \theta(pA-r)\theta(pA-r')\, dA +$$
$$+ \sum_{p\neq p'} \sum_{r}\sum_{r'} \int_{I^{mn}} \theta(pA-r)\theta(p'A-r')\, dA.$$

The integral $\int_{I^{mn}} \theta(pA-r)\theta(p'A-r')\, dA$ is the mn-dimensional volume of the set U of points A in I^{mn} which satisfy $|pA-r| < \varrho$ and $|p'A-r'| < \varrho$, or in other words the integral gives the volume of points in I^{mn} close to both the hyperplanes $H(p, r)$ and $H(p', r')$. When $p = p'$ and $r \neq r'$, these hyperplanes are parallel and at least $1/2N$ apart and hence $> \varrho/N$ apart, so that when $r \neq r'$,

$$\int_{I^{mn}} \theta(pA-r)\theta(pA-r')\, dA = 0. \tag{10}$$

Translating the hyperplane $H(p, r)$ by the mn-vector $(-1/p_1)(r_1, \ldots, r_n, 0, \ldots, 0)$ gives $H(p, 0)$, the hyperplane parallel to $H(p, r)$ and passing through the origin. Under the same translation the hyperplane $H(p', r')$ becomes $H(p', r'-p_1' r/p_1)$. Since the choice of p and p' ensures that the hyperplanes $H(p, r)$ and $H(p', r')$ are almost parallel, the translation cannot decrease the volume of U. Hence

$$\int_{I^{mn}} \theta(pA-r)\theta(p'A-r')\, dA \leqslant \int_{I^{mn}} \theta(pA)\theta\left(p'A-\frac{p_1 r'-p_1' r}{p_1}\right) dA.$$

When the primes p_1, p_1' are distinct, the system of Diophantine equations

$$p_1 r' - p_1' r = c,$$

where $c \in Z^n$, has at most one solution. Hence

$$\sum_{p_1\neq p_1'} \sum_{r}\sum_{r'} \int_{I^{mn}} \theta(pA)\theta\left(p'A-\frac{p_1 r'-p_1' r}{p_1}\right) dA$$
$$\leqslant \sum_{p_1\neq p_1'} \int_{I^{mn}} \theta(pA)\left(\sum_{c\in R^n} \theta(p'A-c/p_1)\right) dA$$
$$\leqslant \sum_{p_1\neq p_1'} \int_{I^{mn}} \theta(pA)\left(\int_{R^n} \theta(p'A-v/p_1)\, dv\right) dA$$

since $\int_{I^{mn}} \theta(pA)\theta(p'A-w)\, dA$ decreases as each $w_j \geqslant 0$ increases and decreases as each $w_j < 0$ decreases, $1 \leqslant j \leqslant n$. Thus by (7) and (8),

$$\sum_{p_1\neq p_1'} \sum_{r}\sum_{r'} \int_{I^{mn}} \theta(pA)\theta\left(p'A-\frac{p_1 r'-p_1' r}{p_1}\right) dA \tag{11}$$
$$\leqslant \sum_{p_1\neq p_1'} \left(\frac{2\varrho}{p_1}\right)^n p_1^n (2\varrho)^n \leqslant (2\varrho)^{2n} \sum_{p_1\neq p_1'} 1.$$

When $p \neq p'$ but $p_1 = p'_1$, then

$$a = \min\{|p_i - p'_i|:\ p_i \neq p'_i,\ 2 \leqslant i \leqslant m\} \geqslant 1.$$

Volume considerations give

$$\int_{I^{mn}} \theta(\boldsymbol{p}A-\boldsymbol{r})\,\theta(\boldsymbol{p}'A-\boldsymbol{r}')\,dA \begin{cases} \leqslant \left(\frac{2\varrho}{p_1}\right)^n\left(\frac{2\varrho}{a}\right)^n & \text{when } |\boldsymbol{r}-\boldsymbol{r}'| \leqslant a(\frac{1}{2}+2\varrho), \\ = 0 & \text{otherwise.}\end{cases}$$

Hence

$$\sum_{p_1=p'_1}\sum_{\boldsymbol{r}}\sum_{\boldsymbol{r}'}\int_{I^{mn}} \theta(\boldsymbol{p}A-\boldsymbol{r})\,\theta(\boldsymbol{p}'A-\boldsymbol{r}')\,dA \leqslant (2\varrho)^{2n}\sum_{p_1=p'_1} p_1^{-n}a^n\sum_{\boldsymbol{r}}\sum_{\boldsymbol{r}'} 1$$

where the third sum is over all $\boldsymbol{r}' \in \boldsymbol{Z}^n$ satisfying $|\boldsymbol{r}-\boldsymbol{r}'| \leqslant a(\frac{1}{2}+2\varrho)$. Thus

$$(12)\qquad \sum_{p_1=p'_1}\sum_{\boldsymbol{r}}\sum_{\boldsymbol{r}'}\int_{I^{mn}} \theta(\boldsymbol{p}A-\boldsymbol{r})\,\theta(\boldsymbol{p}'A-\boldsymbol{r}')\,dA$$
$$\leqslant (2\varrho)^{2n}\sum_{p_1=p'_1} p_1^{-n}a^{-n}\sum_{\boldsymbol{r}} a^n\big(1+O(\varrho)\big) \leqslant (2\varrho)^{2n}\sum_{p_1=p'_1}\big(1+O(\varrho)\big).$$

By (10), (11) and (12),

$$\int_{I^{mn}} \nu_N^2(A)\,dA \leqslant \mu_N + 0 + (2\varrho)^{2n}\sum_{\boldsymbol{p}}\sum_{\boldsymbol{p}'}\big(1+O(\varrho)\big)$$
$$\leqslant \mu_N + \mu_N^2\big(1+O(1/\log N)\big)$$

by Lemma 2.

The above argument can be simplified when $m \geqslant 2$ since when $m \geqslant 2$ it follows from Lemma 9 of Chapter I in [12] that

$$(13)\qquad \sum_{\boldsymbol{r}}\sum_{\boldsymbol{r}'}\int_{I^{mn}} \theta(\boldsymbol{q}A-\boldsymbol{r})\,\theta(\boldsymbol{q}'A-\boldsymbol{r}')\,dA = 2^{2n}\varrho^{2n}$$

for any distinct $\boldsymbol{q}$, $\boldsymbol{q}'$ in $\boldsymbol{Z}^m$. It can then be verified that

$$\mu_N = 2^n\varrho^n\sum_{\boldsymbol{p}\in S_N} 1$$

and that the second moment of ν_N satisfies

$$\int_{I^{mn}} \nu_N^2(A)\,dA \leqslant \mu_N + \mu_N^2.$$

COROLLARY.

$$\sigma_N^2 \leqslant \mu_N + O(\mu_N^2/\log N) = o(\mu_N^2).$$

Proof. The variance σ_N^2 satisfies

$$\sigma_N^2 = \int_{I^{mn}} \nu_N(A)^2\,dA - \mu_N^2,$$

and μ_N tends to infinity with N.

Let

$$Z_N = \{A \in I^{mn}:\ \nu_N(A) = 0\} = \nu_N^{-1}(0),$$

so that Z_N is the set of points in I^{mn} which are not close to any hyperplane $H(\boldsymbol{p}, \boldsymbol{r})$ in S_N. The relatively small upper bound just obtained for the variance of ν_N implies that the volume of Z_N is small.

LEMMA 4.

$$\operatorname{vol} Z_N \leqslant \mu_N^{-1} + O(1/\log N) = o(1).$$

Proof. Since $\nu_N(A) = 0$ for each A in Z_N,

$$\int_{Z_N} \big(\nu_N(A)-\mu_N\big)^2\,dA = \int_{Z_N} \mu_N^2\,dA = \mu_N^2 \operatorname{vol} Z_N.$$

But

$$\int_{Z_N} \big(\nu_N(A)-\mu_N\big)^2\,dA \leqslant \int_{I^{mn}} \big(\nu_N(A)-\mu_N\big)^2\,dA = \sigma_N^2,$$

whence

$$\operatorname{vol} Z_N \leqslant \sigma_N^2/\mu_N^2$$

and the lemma follows.

This estimate for the volume of Z_N is used to construct the regularly distributed subset $T(N)$ from the N-skeleton S_N. First another regularly subset $F(N)$ is constructed. Divide I^{mn} into $[N/16\varrho]^{mn}$ congruent hypercubes H with $L(H) = [N/16\varrho]^{-1}$ ($[x]$ is the integer part of the real number x). Shrink each hypercube H about its centre by $\frac{1}{2}$ to get a new hypercube H' with the same centre and with $L(H') = \frac{1}{2}L(H)$. Let there be M such hypercubes H' whose intersection with each hyperplane $H(\boldsymbol{p}, \boldsymbol{r})$ from S_N has an $(m-1)n$-dimensional volume strictly less than $(\frac{1}{2}L(H))^{(m-1)n}$ when $m \geqslant 2$ and which are disjoint from S_N when $m = 1$. Let H'' be an mn-dimensional hypercube with the same centre as H' (and H) and with $L(H'') = \frac{1}{2}L(H') = \frac{1}{4}L(H)$. Then since the hyperplanes in the N-skeleton are almost orthogonal to each x_{1j}-axis, $1 \leqslant j \leqslant n$, the distance of any point A in H'' from $H(\boldsymbol{p}, \boldsymbol{r})$ is at least $\frac{1}{8}L(H)\big(1+O(N^{-\tau})\big) > \varrho/N$, whence $\nu_N(A) = 0$ for all A in H''. Thus

$$M\cdot\big(\tfrac{1}{4}L(H)\big)^{(m-1)n} \leqslant \operatorname{vol} Z_N = o(1)$$

or

$$M = o\big(L(H)^{-mn}\big) = o(\varrho/N)^{mn}.$$

It follows that there are $(N/16\varrho)^{mn}\big(1+o(1)\big)$ hypercubes H' whose intersection with a hyperplane $H(\boldsymbol{p}, \boldsymbol{r})$ from S_N has $(m-1)n$-dimensional volume at least $L(H')^{(m-1)n}$. Pick one such intersection (or 'slice' of H') J say from each of the $(N/16\varrho)^{mn}\big(1+o(1)\big)$ hypercubes H' and let $F(N)$ be the collection of such J, so that the cardinality $|F(N)|$ of $F(N)$ satisfies

$$|F(N)| = (N/16\varrho)^{mn}(1+o(1)) = L(H)^{-mn}(1+o(1)).$$

By construction the distance (in the Euclidean and the supremum metric) from the centre of the hypercube H (or H') to J is at most $\frac{1}{2}L(H) = \frac{1}{2}[N/16\varrho)]^{-1}$. Let cl H' be the closure of H' and for each J in $F(N)$ let V be the closed set given by

$$(14)\qquad V = \{A \in \mathrm{cl}\, H'\colon |A-R| \leqslant \frac{1}{m}(2N)^{-\alpha-1} \text{ for some } R \text{ in } J\}.$$

Thus V is J "thickened" slightly. Note that if $J \subset \mathrm{cl}\, H' \subset H$, then $V \subset H$ and that since $J \in H(\boldsymbol{p}, \boldsymbol{r})$ for some $\boldsymbol{p}$, $\boldsymbol{r}$, there exists a real matrix R in J with $\boldsymbol{p}R = \boldsymbol{r}$. Because $\alpha > m/n$ and N is sufficiently large, the mn-dimensional volume of V is given by

$$\mathrm{vol}\, V = \left(\frac{2}{m}(2N)^{-\alpha-1}\right)^n \left(\tfrac{1}{2}L(H)\right)^{(m-1)n}\left(1+O(N^{-\tau})\right)$$
$$= g(N)\left(1+O(N^{-\tau})\right) \text{ say.}$$

Let $T(N)$ be the collection of the sets V, so that $|T(N)| = |F(N)|$ and

$$\mathrm{vol}\, T(N) = \sum_{V \in T(N)} \mathrm{vol}\, V = |F(N)|\, g(N)\left(1+o(1)\right)$$
$$(15)\qquad = L(H)^{-mn} g(N)\left(1+o(1)\right)$$
$$(16)\qquad = (m2^{m+\alpha+3})^{-n} N^{m+n-(\alpha+1)n-mn}\left(1+o(1)\right).$$

The set $T(N)$ is sufficiently regular and numerous to "measure" the volume of a set. More precisely if the boundary of a set X has measure 0, vol X is given asymptotically by

$$\frac{\mathrm{vol}\left(X \cap T(N)\right)}{\mathrm{vol}\, T(N)},$$

as we now show.

Lemma 5. *Let the boundary of a given set $X \subseteq I^{mn}$ be of measure* 0. *Then*

$$\mathrm{vol}\left(X \cap T(N)\right) = \mathrm{vol}\, X \cdot \mathrm{vol}\, T(N) \cdot \left(1+o(1)\right).$$

Proof. Dissect I^{mn} into $[N/16\varrho]^{mn}$ hypercubes H with $L(H) = [N/16\varrho]^{-1}$. Suppose Q hypercubes lie wholly within X and Q' meet X and its complement $I^{mn}\backslash X$. Then

$$Q \cdot L(H)^{mn} \leqslant \mathrm{vol}\, X \leqslant (Q+Q')\, L(H)^{mn}$$

and

$$Q' \cdot L(H)^{mn} = o(1),$$

whence

$$\mathrm{vol}\, X = Q \cdot L(H)^{mn} + o(1).$$

The number of hypercubes which lie in X and contain a set V from $T(N)$ is $Q + o\left(L(H)^{-mn}\right)$ and it follows that

$$\mathrm{vol}\left(X \cap T(N)\right) = \left(Q + o\left(L(H)^{-mn}\right)\right) g(N)\left(1+o(1)\right)$$
$$= \mathrm{vol}\, X \cdot L(H)^{-mn} g(N)\left(1+o(1)\right)$$
$$= \mathrm{vol}\, X \cdot \mathrm{vol}\, T(N) \cdot \left(1+o(1)\right)$$

by (15).

When the set X depends on N the above argument breaks down but the volume can be estimated in the following special case.

Lemma 6. *Let C be an mn-dimensional hypercube with $L(C) \geqslant N^{-(\alpha+1)}$. Then*

$$\mathrm{vol}\left(C \cap T(N)\right) \ll \mathrm{vol}\, C \cdot \mathrm{vol}\, T(N) + L(C)^{(m-1)n} N^{-(\alpha+1)n},$$

where the implied constant depends on ε, m and n but not on N.

Proof. Since $\eta < \alpha - m/n$ and N is sufficiently large,

$$N^{-(\alpha+1)} < [N/16\varrho]^{-1}.$$

By dissecting I^{mn} into $[N/16\varrho]^{mn}$ mn-dimensional congruent hypercubes H with $L(H) = [N/16\varrho]^{-1}$ and using arguments similar to those in the preceding lemma, it is straightforward to verify that the number of sets V which meet C is

$$\ll \left(L(C)/L(H)\right)^{mn}$$

when $L(C) > L(H)$ and

$$\ll 1$$

otherwise. Thus since $L(C) \geqslant N^{-(\alpha+1)} > \dfrac{2}{m}(2N)^{-(\alpha+1)}$,

$$\mathrm{vol}\left(C \cap T(N)\right) \ll L(C)^{mn} L(H)^{-mn} \mathrm{vol}\, V + L(C)^{(m-1)n} N^{-(\alpha+1)n}$$
$$\ll \mathrm{vol}\, C \cdot \mathrm{vol}\, T(N) + L(C)^{(m-1)n} N^{-(\alpha+1)n}$$

by (15).

This result is now applied to hypercubes C from $\mathscr{C}$ for which $L(C)$ lies in a certain range (recall that $\mathscr{C}$ satisfies (4), where $t = (m-1)n + (m+n)/(\alpha+1) - \delta$). Let N_r and N_s be sufficiently large positive integers with $N_r < N_s$. Write

$$E(s, r) = \{C \in \mathscr{C}\colon N_s^{-\alpha-1} < L(C) \leqslant N_r^{-\alpha-1}\}.$$

LEMMA 7. *For each $\mathscr{C}$ satisfying* (4),

$$\operatorname{vol}(E(s,r)\cap T(N_s)) \ll \operatorname{vol} T(N_s)\cdot(\operatorname{vol} T(N_r)\cdot N_r^{n\eta-(\alpha+1)\delta}+N_s^{n\eta-(\alpha+1)\delta})$$

where the implied constant depends on ε, m and n but not on N_r or N_s.

Proof. By Lemma 6,

$$\operatorname{vol}(E(s,r)\cap T(N_s)) \ll \sum_C L(C)^{mn}\cdot \operatorname{vol} T(N_s)+\sum_C L(C)^{(m-1)n}\cdot N_s^{-(\alpha+1)n},$$

where the sums are over those C in $\mathscr{C}$ with $N_s^{-\alpha-1} < L(C) \leqslant N_r^{-\alpha-1}$. Now

$$\sum_C L(C)^{mn} = \sum_C L(C)^{t+n-(m+n)/(\alpha+1)+\delta} < \varepsilon\cdot N_r^{-n(\alpha+1)+m+n-\delta(\alpha+1)}$$

and

$$\sum_C L(C)^{(m-1)n} = \sum_C L(C)^{t-(m+n)/(\alpha+1)+\delta} < \varepsilon\cdot N_s^{m+n-\delta(\alpha+1)}$$

since $\delta < 1/n$.

Hence

$$\begin{aligned}\operatorname{vol}(E(s,r)\cap T(N_s)) &\ll N_r^{m+n-(\alpha+1)n-(\alpha+1)\delta}\operatorname{vol} T(N_s)+N_s^{m+n-(\alpha+1)n-(\alpha+1)\delta}\\ &\ll \operatorname{vol} T(N_s)\cdot(\operatorname{vol} T(N_r)\cdot N_r^{n\eta-(\alpha+1)\delta}+N_s^{n\eta-(\alpha+1)\delta})\end{aligned}$$

by (16).

Note that the partial mn-volume $\sum L(C)^{mn}$, where $L(C)\leqslant N^{-\alpha-1}$, is comparable with the volume of $T(N)$.

Let

$$G_0 = \{A\in \boldsymbol{R}^{mn}\colon |A|\leqslant \tfrac{1}{2}\} = [-\tfrac{1}{2},\tfrac{1}{2}]^{mn}$$

and

$$\beta = (\alpha+1)\delta - n\eta;$$

by the choice of η, β is positive. For each $s = 1, 2, \ldots$, let $N_s > N_{s-1}$ and define G_s inductively by

$$G_s = (G_{s-1}\cap T(N_s))\setminus E(s, s-1),$$

so that each G_s is closed and $G_s \subseteq G_{s-1}$.

LEMMA 8. *Let $\mathscr{C}$ be a collection of hypercubes C satisfying* (4). *For each $s = 1, 2, \ldots$, let $N_0, N_1, \ldots, N_s$ be a sufficiently rapidly increasing sequence of positive integers, with N_0 sufficiently large. Then for each $s = 1, 2, \ldots$*

$$\operatorname{vol} G_s > 2^{-2s}\prod_{j=1}^{s}\operatorname{vol} T(N_j) > 0.$$

Proof. The result is true for $s = 1$ since

$$\begin{aligned}\operatorname{vol} G_1 &= \operatorname{vol} T(N_1)-\operatorname{vol}(E(1,0)\cap T(N_1))\\ &\geqslant \operatorname{vol} T(N_1)(1-K(\operatorname{vol} T(N_0)\cdot N_0^{-\beta}+N_1^{-\beta})),\end{aligned}$$

where K is the implied constant in Lemma 6. As $\beta > 0$ and N_0 and N_1 are sufficiently large,

$$\operatorname{vol} G_1 \geqslant \tfrac{1}{4}\operatorname{vol} T(N_1).$$

Assume inductively that

$$\operatorname{vol} G_s > 2^{-2s}\prod_{j=1}^{s}\operatorname{vol} T(N_j).$$

By definition

$$G_{s+1} = (G_s\cap T(N_{s+1}))\setminus E(s+1, s),$$

whence

$$\operatorname{vol} G_{s+1} = \operatorname{vol}(G_s\cap T(N_{s+1}))-\operatorname{vol}(E(s+1,s)\cap G_s\cap T(N_{s+1})).$$

Choose N_s sufficiently large so that

$$KN_s^{-\beta} < 2^{-2s-3}\prod_{j=1}^{s-1}\operatorname{vol} T(N_j),$$

where K is the implied constant in Lemma 6. Choose N_{s+1} sufficiently large so that

$$KN_{s+1}^{-\beta} < 2^{-2s-3}\prod_{j=1}^{s}\operatorname{vol} T(N_j)$$

and so that Lemma 5 implies

$$\operatorname{vol}(G_s\cap T(N_{s+1})) \geqslant \tfrac{1}{2}\operatorname{vol} G_s\cdot\operatorname{vol} T(N_{s+1}).$$

Then by Lemma 6

$$\begin{aligned}\operatorname{vol} G_{s+1} &> \tfrac{1}{2}\operatorname{vol} G_s\cdot\operatorname{vol} T(N_{s+1})-K\operatorname{vol} T(N_{s+1})(\operatorname{vol} T(N_s)N_s^{-\beta}+N_{s+1}^{-\beta})\\ &> \tfrac{1}{2}\operatorname{vol} G_s\cdot\operatorname{vol} T(N_{s+1})-2^{2s-2}\prod_{j=1}^{s+1}\operatorname{vol} T(N_j)\\ &> 2^{-2(s+1)}\prod_{j=1}^{s+1}\operatorname{vol} T(N_j)\end{aligned}$$

by the inductive hypothesis and the lemma follows.

Note that the factor of $\tfrac{1}{4}$ introduced at each stage could be increased to any number $\lambda < 1$.

Plainly Lemma 8 implies that no G_s is empty for $s = 1, 2, \ldots$ But since G_0 is a closed and bounded subset of $\boldsymbol{R}^{mn}$, G_0 is compact and so satisfies the finite intersection property. By construction

$$G_0 \supseteq G_1 \supseteq \ldots \supseteq G_s \supseteq G_{s+1} \supseteq \ldots$$

so that $\{G_s\colon s = 1, 2, \ldots\}$ is a collection of closed sets in G_0 which satisfy for each $s = 1, 2, \ldots$

$$G_s = \bigcap_{j=1}^{s} G_j \neq \emptyset,$$

i.e. which satisfy the finite intersection property. Hence

$$G_\infty = \bigcap_{j=1}^{\infty} G_j \neq \emptyset.$$

Now G_∞ does not meet any hypercube C in the collection $\mathscr{C}$, since if $A \in C \in \mathscr{C}$, then there exist positive integers N_{s-1}, N_s such that

$$N_s^{-\alpha-1} < L(C) \leqslant N_{s-1}^{-\alpha-1},$$

i.e. $C \in E(s, s-1)$, whence $A \notin G_s \supseteq G_\infty$.

Next suppose $A \in G_\infty$, so that $A \in G_s$, for each $s = 1, 2, \ldots$ Then by definition, $A \in T(N_s)$ for each $s = 1, 2, \ldots$ Hence by (14) there exist a matrix R and vectors $\boldsymbol{p}$ in $\boldsymbol{Z}^m$, $\boldsymbol{r}$ in $\boldsymbol{Z}^n$ with

$$|\boldsymbol{p}| = p_1 \in \boldsymbol{P}, \quad N_s < p_1 < 2N_s, \quad |\boldsymbol{r}| < \tfrac{1}{2} p_1,$$

satisfying

$$\boldsymbol{p}R = \boldsymbol{r}$$

and such that

$$|A - R| \leqslant \frac{1}{m}(2N_s)^{-\alpha-1}.$$

(Note that R is in the N_s-skeleton S_{N_s} or more accurately in a hyperplane $H(\boldsymbol{p}, \boldsymbol{r})$ in the N_s-skeleton). Thus for each $s = 1, 2, \ldots$, there exist vectors $\boldsymbol{p}$ in $\boldsymbol{Z}^m$ and $\boldsymbol{r}$ in $\boldsymbol{Z}^n$ such that

$$\begin{aligned} |\boldsymbol{p}A - \boldsymbol{r}| &= |\boldsymbol{p}A - \boldsymbol{p}R + \boldsymbol{p}R - \boldsymbol{r}| = |\boldsymbol{p}(A-R)| \\ &\leqslant m|\boldsymbol{p}|\,|A-R| \leqslant (2N_s)^{-\alpha} < p_1^{-\alpha}. \end{aligned}$$

Therefore if $A \in G_\infty$, there are infinitely many $\boldsymbol{q}$ in $\boldsymbol{Z}^m$ such that

$$|\langle \boldsymbol{q}A \rangle| < |\boldsymbol{q}|^{-\alpha}, \quad \text{i.e.} \quad G_\infty \subseteq W.$$

Since no hypercube C in $\mathscr{C}$ meets G_∞, it follows that $\mathscr{C}$ cannot cover W, whence from the definition,

$$\dim W \geqslant (m-1)n + (m+n)/(\alpha+1).$$

Since $\dim W(m, n) = \dim W$, this gives the

THEOREM. *Let*

$$W(m, n) = \{A \in \boldsymbol{R}^{mn}\colon |\langle \boldsymbol{q}A \rangle| < |\boldsymbol{q}|^{-\alpha} \text{ infinitely often}\}.$$

When $\alpha > m/n$,

$$\dim W(m, n) = (m-1)n + (m+n)/(\alpha+1)$$

and when $\alpha \leqslant m/n$, $W(m, n)$ *has Lebesgue measure* 1, *so that* $\dim W(m, n) = mn$.

Proof. Only the statement that $W(m, n)$ has measure 1 when $\alpha \leqslant m/n$ needs to be proved. If $\alpha \leqslant m/n$, then

$$\sum_{\boldsymbol{q}} |\boldsymbol{q}|^{-n\alpha} = \infty,$$

where the summation is over non-zero $\boldsymbol{q}$ in $\boldsymbol{Z}^m$, and the result follows from a general Khintchine type theorem due to Groshev and discussed by Sprindžuk [12].

The theorem still holds when the integral vectors $\boldsymbol{q}$ are restricted to certain subsets Q of $\boldsymbol{Z}^m$ which are not too sparse or irregular. Given any subset Q of $\boldsymbol{Z}^m$, write

$$W_Q(m, n) = \{A \in \boldsymbol{R}^{mn}\colon |\langle \boldsymbol{q}A \rangle| < |\boldsymbol{q}|^{-\alpha} \text{ for infinitely many } \boldsymbol{q} \text{ in } Q\}.$$

Then evidently $W_Q(m, n) \subseteq W(m, n)$ and so when $\alpha > m/n$,

$$\dim W_Q(m, n) \leqslant \dim W(m, n) = (m-1)n + (m+n)/(\alpha+1). \tag{17}$$

If the points in Q are asymptotically reasonably numerous and well distributed, it can be shown by modifying the above method that when $\alpha > m/n$, the complementary inequality

$$\dim W_Q(m, n) \geqslant (m-1)n + (m+n)/(\alpha+1)$$

holds. If the argument is to work for $m \geqslant 1$, a 'coprime' condition of some kind is needed for the second moment estimate (Lemma 3). As an example, the set Q can be taken to consist of vectors $\boldsymbol{q}$ whose coordinates are primes in arithmetic progressions, i.e.

$$Q = \{(q_1, \ldots, q_m) \in \boldsymbol{Z}^m\colon |q_i| \in \boldsymbol{P},\ q_i \equiv a_i \pmod{k_i},\ 1 \leqslant i \leqslant m\}, \tag{18}$$

where $(a_i, k_i) = 1$, $1 \leqslant i \leqslant m$.

To prove the theorem in this case the integral vector $\boldsymbol{p}$, introduced above and satisfying (5), must be restricted further. Let $\boldsymbol{p} = (p_1, \ldots, p_m)$ now denote any vector in $Q \cap P_N$.

Recall that $\boldsymbol{r}$ is any vector in $\boldsymbol{Z}^n$ with $|\boldsymbol{r}| < \frac{1}{2}p_1$. Then

$$\sum_{\boldsymbol{p}} 1 = 2^{m-1} N^{m-(m-1)\varepsilon} \Big(\prod_{i=1}^{m} \varphi(k_i) \cdot (\log N)^m\Big)^{-1} + O\left(\frac{N^{m-(m-1)\varepsilon}}{(\log N)^{m+1}}\right),$$

where φ is Euler's function, and the mean of

$$\nu_N(A) = \sum_{\boldsymbol{p}} \sum_{\boldsymbol{r}} \theta(\boldsymbol{p}A - \boldsymbol{r})$$

is given by

$$\mu_N = \int_{I^{mn}} \nu_N(A)\,dA = \sum_{\boldsymbol{p}} \sum_{\boldsymbol{r}} 2^n \varrho^n p_1^{-1}\big(1+O(N^{-\tau})\big)$$
$$= 2^n \varrho^n \sum_{\boldsymbol{p}} \big(1+o(1)\big).$$

Since $n\eta > (m-1)\tau$, $\mu_N \to \infty$ as $N \to \infty$.

The method used in the second moment estimate (Lemma 3) is unaffected by the additional restrictions on the vectors $\boldsymbol{p}$ and it can be verified that

$$\int_{I^{mn}} \nu_N(A)^2\,dA \leqslant \mu_N + (2\varrho)^{2n}\Big(\sum_{\boldsymbol{p}} \big(1+O(1/\log N)\Big)^2 \leqslant \mu_N + \mu_N^2\big(1+o(1)\big).$$

Thus the variance σ_N^2 satisfies

$$\sigma_N^2 = o(\mu_N^2)$$

and the volume of the set $Z_N = \{A \in I^{mn}\colon\ \nu_N(A) = 0\}$ satisfies

$$\operatorname{vol} Z_N = o(1).$$

The construction of a regularly distributed subset $T(N)$ proceeds as before, with vol $T(N)$ satisfying (15) and (16), and with Lemmas 5, 6 and 7 holding for $T(N)$. Let $G_0 = [-\frac{1}{2}, \frac{1}{2}]^{mn}$ and for each $s = 1, 2, \ldots$, let

$$G_s = \big(G_{s-1} \cap T(N_s)\big) \setminus E(s, s-1).$$

Then it follows as in Lemma 8 that

$$\operatorname{vol} G_s > 2^{-2s} \prod_{j=1}^{s} \operatorname{vol} T(N_j) > 0,$$

so that $G_\infty = \bigcap_{s=1}^{\infty} G_s \neq \emptyset$. Again by construction G_∞ does not meet any C in $\mathscr{C}$. But $G_\infty \subseteq W_{\boldsymbol{Q}}(m, n) \cap I^{mn}$ since if $A \in G_\infty$, then $A \in T(N_s)$ for each $s = 1, 2, \ldots$ Thus for each $s = 1, 2, \ldots$, there exist a matrix R and vectors $\boldsymbol{p}$ in $\boldsymbol{Q}$ and $\boldsymbol{r}$ in $\boldsymbol{Z}^n$ such that

$$\boldsymbol{p}R = \boldsymbol{r}$$

and

$$|A - R| < \frac{1}{m}(2N_s)^{-\alpha-1},$$

whence

$$|\boldsymbol{p}A - \boldsymbol{r}| < |\boldsymbol{p}|^{-\alpha},$$

and $A \in W_{\boldsymbol{Q}}(m, n) \cap I^{mn}$. Hence $\mathscr{C}$ cannot cover $W_{\boldsymbol{Q}}(m, n) \cap I^{mn}$ and so

$$\dim W_{\boldsymbol{Q}}(m, n) = \dim W_{\boldsymbol{Q}}(m, n) \cap I^{mn} \geqslant (m-1)\,n + (m+n)/(\alpha+1).$$

Thus by (17), the theorem also holds for $W_{\boldsymbol{Q}}(m, n)$ when $\boldsymbol{Q}$ is given by (18). In particular the theorem holds when the coordinates of the vectors in $\boldsymbol{Q}$ have prime modulus (see [8], p. 69 for the case $m = 1$); this allows the simpler proof based on (13) to be used for $m \geqslant 2$.

When $\boldsymbol{Q} = \{(q_1, \ldots, q_m) \in \boldsymbol{Z}^m\colon\ q_i \equiv a_i \pmod{k_i},\ 1 \leqslant i \leqslant m\}$, the cases $m = 1$ and $m \geqslant 2$ have to be considered separately. The first case has been dealt with by Eggleston [8] (Theorem 7) or it can be proved by modifying and specialising the above arguments. We have to show that the dimension of the set

$$W_Q(1, n) = \{\boldsymbol{x} \in \boldsymbol{R}^n\colon\ |\langle q\boldsymbol{x}\rangle| < q^{-\alpha} \text{ for infinitely many } q \equiv a \pmod{k}\},$$

where $Q = \{q \in \boldsymbol{Z}\colon\ q \equiv a \pmod{k}\}$ and $\alpha > 1/n$ is $(n+1)/(\alpha+1)$. When $m = 1$ the hyperplanes $H(\boldsymbol{q}, \boldsymbol{r})$ reduce to rational points of the form $(r_1/q, \ldots, r_n/q)$ but the N-skeleton S_N must be modified slightly, as follows. Let $b = (a, k)$, the highest common factor of a and k, and write $a' = a/b$, $k' = k/b$. Let the integer p now be a prime satisfying

$$N < p < 2N \quad \text{and} \quad p \equiv a' \pmod{k'},$$

so that

$$\sum_p 1 = \frac{N}{\varphi(k')\log N}\left(1 + O\left(\frac{1}{\log N}\right)\right).$$

Next let

$$S_N = \{\boldsymbol{r}/p\colon\ p,\ |\boldsymbol{r}| < \tfrac{1}{2}p\},$$

so that

$$|S_N| = \sum_p \sum_{\boldsymbol{r}} 1 = \frac{(2^{n+1}-1)\,N^{n+1}}{(n+1)\,\varphi(k')\log N}\left(1 + O\left(\frac{1}{\log N}\right)\right).$$

For each N, the mean μ_N of $\nu_N(\boldsymbol{x}) = \sum_p \sum_{\boldsymbol{r}} \theta(p\boldsymbol{x} - \boldsymbol{r})$ is given by

$$\mu_N = 2^n N^{n\eta}\big(\varphi(k')(\log N)\big)^{-1}\big(1 + O(1/\log N)\big)$$

and the variance $\sigma_N^2 = o(\mu_N^2)$.

The construction of $T(N)$ proceeds as before except that the 'slices' V reduce to hypercubes $D\big(\boldsymbol{r}/p, (2bN)^{-\alpha-1}\big)$, where $\boldsymbol{r}/p \in J \subseteq S_N$, shrunk by an additional factor $b^{-\alpha-1}$. Thus instead of being given by (15) and (16),

$$\operatorname{vol} T(N) = (2bN)^{-(\alpha+1)n} L(H)^{-n}\big(1 + o(1)\big)$$
$$= (2^{\alpha+3} b^{1+\alpha})^{-n} N^{n+1-(\alpha+1)n-n\eta}\big(1 + o(1)\big)$$

and so is still comparable with the partial volume $\sum_C L(C)^n$ where the sum is over those C in $\mathscr{C}$ satisfying $L(C) \leqslant N^{-\alpha-1}$. The argument is not affected by each hypercube in $T(N)$ being shrunk by $b^{-\alpha-1}$ nor by the resulting smaller constant factor in the volume of $T(N)$. As before, it follows that the set G_∞, which meets no C in $\mathscr{C}$, is not empty. Also $G_\infty \subseteq W_Q(1, n) \cap I^n$ since as before, given x in G, there exists an infinite sequence of vectors r/p such that

$$|x - r/p| \leqslant (2bN_s)^{-\alpha-1} < (bp)^{-\alpha-1}.$$

But $p \equiv a'(\bmod k')$, whence there are infinitely many integers $q = bp$ satisfying $q \equiv a(\bmod k)$ and integer vectors $u = br$ such that

$$|qx - u| < q^{-\alpha},$$

i.e. $x \in W_Q(1, n) \cap I^n$. It follows that $\dim W_Q(1, n) = (n+1)/(\alpha+1)$.

When $m \geqslant 2$, (13) can be used and p will now be a vector in Z^m satisfying $p_i \equiv a_i(\bmod k_i)$, $1 \leqslant i \leqslant m$, and satisfying

$$N < p_1 < 2N, \quad |p_i| < N^{1-\tau}, \quad 2 \leqslant i \leqslant m.$$

Then the mean μ_N of $\nu_N(A)$, the number of hyperplanes $H(p, r)$, where $|r| < \frac{1}{2}p_1$, close to A is given by

$$\mu_N = 2^n \varrho^n \sum_p (1+o(1)) = 2^{m+n-1} N^{-(m-1)\tau+m} \Big(\prod_{i=1}^{m} k_i\Big)^{-1} (1+o(1)).$$

The second moment

$$\int_{I^{mn}} \nu_N(A)^2 \, dA = \mu_N + \sum_{p \neq p'} \sum_r \sum_{r'} \int_{I^{mn}} \theta(pA - r)\, \theta(p'A - r') \, dA$$
$$= \mu_N + \sum_{p \neq p'} (2\varrho)^{2n}$$

by (13), whence

$$\int_{I^{mn}} \nu_N(A)^2 \, dA = \mu_N + (2\varrho)^{2n} \sum_{p \neq p'} 1 = \mu_N + \mu_N^2 (1+o(1))$$

and so

$$\sigma_N^2 = o(\mu_N^2).$$

The argument now follows the same lines as before.

When the error term $|q|^{-\alpha}$ is replaced by a function ψ: $Z^m \to R$ which satisfies $\psi(q, 0, \ldots, 0) = |q|^{-\alpha}$ and which is 'small' on average over the lattice points $\{q \in Z^m\colon |q| = q\}$ on a large sphere, the Hausdorff dimension of the set corresponding to W is smaller and much easier to determine since it coincides with a simple, general lower bound. In fact the dimension is that of the case when $m = 1$, augmented by $(m-1)n$, the number of degrees of

freedom. For example the dimension of the set

$$\{A \in R^{mn}\colon |\langle qA \rangle| < \prod_{i=1}^{m} (\bar{q}_i)^{-\alpha} \text{ for infinitely many } q \text{ in } Z^m\},$$

where $\bar{x} = \max\{|x|, 1\}$ and $\alpha > 1/n$, is $(m-1)n + (n+1)/(\alpha+1)$ (the dimension when $m = 1$ is $(n+1)/(\alpha+1)$ by [8] or [10]). The same kind of result still holds when instead of $|\langle x \rangle|$, a fairly general distance function $F(x)$ (see § V.10.2 in [5]) is considered [6] or when $F(x) = \big(\prod_{j=1}^{n} |x_j|\big)^{1/n}$ is considered [13] (the dimension when $m = 1$ is obtained in [3]). The reason for this might be connected with a transference principle (see p. 69 in [12]).

We conclude with an application, suggested by Dr James Vickers, to the periodic Kolmogorov–Arnol'd–Moser theorem on invariant tori in perturbed systems (further details and references are given in Appendix 8 of [1] and in [7]). This theorem holds for sets of non-resonant frequencies $(\omega_1, \ldots, \omega_m) = \omega$ for which there exists a positive constant $c = c(\omega)$ and a number $\alpha > m$ such that

$$|k_0 + k_1 \omega_1 + \ldots + k_m \omega_m| > c\, |k|_1^{-\alpha}$$

for all non-zero $k = (k_1, \ldots, k_m)$ in Z^m, k_0 in Z, i.e. such that

$$|\langle k \cdot \omega \rangle| = \|k \cdot \omega\| > c\, |k|_1^{-\alpha}$$

for all non-zero k in Z^m, where $|k|_1 = |k_1| + \ldots + |k_m|$. Since $(1/m)|k|_1 \leqslant |k| \leqslant |k|_1$, the periodic Kolmogorov–Arnol'd–Moser theorem holds when there exists a positive constant C such that

$$|\langle k \cdot \omega \rangle| > C\, |k|^{-\alpha}$$

for all non-zero k in Z^m. When $\alpha > m$, the set Ω of such ω has Lebesgue measure 1 and the complement $E(\alpha) = R^m \setminus \Omega$ has measure 0, so that the theorem holds for almost all ω in R^m. Now it can be verified that for any positive ε,

$$W(m, \alpha+\varepsilon) \subseteq E(\alpha) \subseteq W(m, \alpha),$$

whence

$$\dim E(\alpha) = \dim W(m, \alpha) = m - 1 + (m+1)/(\alpha+1)$$

for $\alpha > m$. In particular when $\alpha = m+1$,

$$\dim E(\alpha) = m - 1/(m+2).$$

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Локальная предельная теорема для мультипликативных арифметических функций

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Одной из основных задач вероятностной теории чисел является изучение локальных предельных законов распределения значений арифметических функций. Локальным предельным теоремам для аддитивных арифметических функций посвящена достаточно обширная литература. В случае мультипликативных функций этот вопрос до 1970 года оставался не изученным, а в последние годы интенсивно исследуется многими авторами (см., например, [4]–[9]).

В 1973 г. Б. В. Левин и А. А. Юдин [3] показали, что для аддитивных функций локальный закон распределения на простых числах индуцирует локальный закон распределения на натуральном ряде. Центральным результатом работы [3], позволяющим точно судить о поведении главного члена, является следующая

Теорема. *Пусть $\psi(n)$ — вещественная аддитивная функция удовлетворяющая условиям:*

1) $\frac{1}{\pi(N)}\sum_{p\leqslant N,\psi(p)=\omega} 1 = \tau_\omega + (\varepsilon_N(\omega)/(\ln N)^{1+\varepsilon})$, *где* $\varepsilon>0$, $\sum_\omega |\varepsilon_N(\omega)| < +\infty$ *равномерно по* N;

2) *не существует* $h\geqslant 2$, $h\in N$ *такого, что*

$$\sum_{k\equiv 0 (\mathrm{mod}\, h)} \tau_k = 1 \quad u \quad \sum_{p,\psi(p)\notin Z} 1/p < +\infty;$$

3) $\sigma^2 = \sum_k \tau_k\cdot k^2 < +\infty$.

Тогда

$$\frac{1}{N}\sum_{n\leqslant N,\psi(n)=\theta} 1 = \frac{A(\theta)}{\sigma\sqrt{2\pi \ln_2 N}}\exp\left\{-\frac{(E\ln_2 N-\theta)^2}{2\sigma^2\ln_2 N}\right\} + o\left(\frac{1}{\ln_2 N}\right),$$

где $E=\sum_k \tau_k\cdot k$, $A(\theta)$ — *константа, зависящая от* ψ *и* θ.