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Progress towards a conjecture on the mean value of Titchmarsh series, III

by

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1. Introduction. In the paper I of this series the second author defined Titchmarsh series and formulated a conjecture. We recall these first and state what he proved in that paper.

Titchmarsh series. Let $A \geqslant 1$ be a constant. Let $1 = \lambda_1 < \lambda_2 < \lambda_3 < \ldots$ where $1/A \leqslant \lambda_{n+1} - \lambda_n \leqslant A$. Let $a_1, a_2, a_3, \ldots$ be a sequence of complex numbers, possibly depending on a parameter H ($\geqslant 10$) such that $a_1 = 1$ and $|a_n| \leqslant (\lambda_n H)^A$. Put $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ where $s = \sigma + it$. Then $F(s)$ is analytic in $\sigma \geqslant A+2$; $F(s)$ is called a *Titchmarsh series* if there exists a constant $A \geqslant 1$ with the above properties and further a system of inifinite rectangles $R(T, T+H)$ defined by $\{\sigma \geqslant 0;\ T \leqslant t \leqslant T+H\}$ where $10 \leqslant H \leqslant T$ and T (which may be related to H) tends to infinity and $F(s)$ admits an analytic continuation into these rectangles and maximum of $|F(s)|$ over $R(T, T+H)$ does not exceed $\exp(H^A)$.

Conjecture. For a Titchmarsh series $F(s)$, we have

$$\frac{1}{H} \int_L |F(s)|^2\, dt > C_A \sum_{\lambda_n \leqslant X} |a_n|^2$$

where $X = 2 + D_A H$, L denotes the side ($T \leqslant t \leqslant T+H$, $\sigma = 0$) of $R(T, T+H)$ and C_A and D_A are positive constants depending only on A.

He proved the following theorems:

Theorem A. *We have*

$$\frac{1}{H} \int_L |F(it)|^2\, dt > C_A$$

where C_A is an effectively computable positive constant depending only on A.

Theorem B. *We have*

$$\frac{1}{H} \int_L |F(it)|^2\, dt > C_A \sum_{\lambda_n \leqslant X} |a_n|^2 \left(1 - \frac{\log \lambda_n}{\log H} + \frac{1}{\log\log H}\right)$$

where $X = 2+D_A H$ and C_A and D_A are positive constants (depending only on A), which are effective.

In the present paper, we replace the quantity $\exp(H^A)$ in the definition of Titchmarsh series by $\exp\exp(H/80A)$ and still prove Theorem B (with refinements). Secondly, we replace the tapering factor $1-\frac{\log\lambda_n}{\log H}+\frac{1}{\log\log H}$ in Theorem B with more refined quantities (see Theorems 1, 2 and 3 below). From now on we replace λ_n by n, for simplifying the notation. All our results are true for the Titchmarsh series $\sum a_n/\lambda_n^s$ also.

In paper I of this series there were some typographical errors and we correct them in the present paper.

2. Let a_n be a sequence of complex numbers, possibly depending upon a parameter H, with $a_1 = 1$ and $|a_n| \leqslant (nH)^A$; we assume that the series $F(s) = \sum a_n/n^s$ has an analytic continuation in $R(T, T+H)$ as explained in Section 1 and that $|F(s)| \leqslant \exp\exp(H/80A)$ there. The constants A_1, A_2 and A_3 appearing below are effectively computable positive constants, depending only on A. We also assume that H is sufficiently large.

THEOREM 1. *There exists A_1 such that*

$$\int_0^\infty d\sigma \int_{T+H/10}^{T+9H/10} |F(s)-1|\,dt \leqslant A_1\Big(H^{A+1}+\int_T^{T+H}|F(it)|\,dt\Big).$$

THEOREM 2. *There exists A_2 such that for any real β $(-1/2 \leqslant \beta \leqslant 1/2)$*

$$\frac{1}{H}\int_T^{T+H}|F(it)|^2\,dt \geqslant A_2 \sum_{n\leqslant H/200}|a_n|^2\left(\frac{|\beta|(\log(H/n))^\beta}{|(\log H)^\beta-1|}\right).$$

THEOREM 3. *There exists A_3 with*

$$\frac{1}{H}\int_T^{T+H}|F(it)|^2\,dt \geqslant A_3 \sum_{n\leqslant H/200}|a_n|^2 f(n)$$

where one can take $f(n)$ to be

(i) $\frac{1}{\log\log H}$,

(ii) $\frac{1}{M}$,

(iii) $\left(\frac{1}{\log M}+\frac{1}{\log N}\right)\frac{1}{\log\log\log H}$,

(iv) $\left(\frac{1}{\log M(\log\log M)^2}+\frac{1}{\log N(\log\log N)^2}\right)$

where $M = \frac{\log H}{\log(H/n)}+10^6$ *and* $N = \log(H/n)+10^6$.

Remarks. (1) From Theorem 3, one can get that for a suitable A_4,

$$\frac{1}{H}\int_T^{T+H}|F(it)|^2\,dt \geqslant A_4 \sum_{n\leqslant H/200}|a_n|^2\Big(\frac{1}{\log\log H}+\frac{1}{M}+ +\frac{1}{\log M\log\log\log H}+\frac{1}{\log N\log\log\log H}+ +\frac{1}{(\log M)(\log\log M)^2}+\frac{1}{\log N(\log\log N)^2}\Big).$$

(2) The numerical constants (e.g. 200 in $\sum_{n\leqslant H/200}$ and 80 in $F(s) \leqslant \exp\exp(H/80A)$ clearly could be improved.

(3) For slightly different set of results on the lower bounds of the mean value of the Titchmarsh series, we refer the reader to Ramachandra [3]. The results contained in [3] also can be slightly improved using the method of this paper.

3. Proof of Theorem 1. Let $G(z) = F(z)-1$. Then for $T+H/10 \leqslant t \leqslant T+9H/10$, $0 < \sigma \leqslant A+2$, we have

$$G(\sigma+it) = \frac{1}{2\pi i}\int_R G(z)\left(\exp\left(\sin\left(\frac{z-\sigma-it}{4A+8}\right)\right)^2\right)\frac{dz}{z-\sigma-it}$$

where the integration is over the boundary R (taken anticlockwise) of the rectangle $0 \leqslant \operatorname{Re} z \leqslant A+4$, $T \leqslant \operatorname{Im} z \leqslant T+H$.

Since $|G(z)| \leqslant \exp\exp(H/80A)$ by assumption and

$$|\exp((\sin z)^2)| \ll \exp(-\exp|\operatorname{Im} z|) \quad \text{if} \quad |\operatorname{Re} z| \leqslant 1/4$$

it follows that the integral over the horizontal sides are $\ll 1$. Since $|a_n| \leqslant (nH)^A$, we have $G(z) \ll H^A$ on $\operatorname{Re} z = A+4$ and hence the integral on the vertical side $\operatorname{Re} z = A+4$ is $O(H^A)$. The integral on the vertical side $\operatorname{Re} z = 0$ is at most

$$\int_T^{T+H}\frac{|G(ix)|}{|ix-\sigma-it|}\,dx.$$

Thus we have, for $T+H/10 \leqslant t \leqslant T+9H/10$,

$$|G(\sigma+it)| \ll \int_T^{T+H}\frac{|G(ix)|\left(\exp\left(\sin\left(\frac{ix-\sigma-it}{4A+8}\right)\right)^2\right)}{|ix-\sigma-it|}\,dx+H^A.$$

Consequently,

$$\int_{T+H/10}^{T+9H/10} |G(\sigma+it)|\,dt$$

$$\ll \int_{T}^{T+H} |G(ix)|\,dx \int_{T+H/10}^{T+9H/10} \frac{\exp\left(\sin\left(\frac{ix-\sigma-it}{4A+8}\right)\right)^2}{|ix-\sigma-it|}\,dt + H^{A+1}$$

$$\ll \log\left(\frac{1}{\sigma}+1\right) \int_{T}^{T+H} |G(ix)|\,dx + H^{A+1}.$$

This yields

LEMMA 1. *We have*

$$\int_{0}^{A+2} d\sigma \int_{T+H/10}^{T+9H/10} |G(\sigma+it)|\,dt \ll \int_{T}^{T+H} |G(ix)|\,dx + H^{A+1}.$$

Now we need

LEMMA 2. *There holds*

$$\int_{A+2}^{\infty} d\sigma \int_{T+H/10}^{T+9H/10} |G(\sigma+it)|\,dt \ll H^{A+1}.$$

Proof. Since $|G(\sigma+it)| \leqslant \sum_{n} \frac{|a_n|}{n^{\sigma}} \leqslant \sum_{n} \frac{(nH)^A}{n^{\sigma}}$, the result follows.

The proof of Theorem 1 follows from Lemmas 1 and 2.

4. Proof of Theorem 2. We introduce the following notation.

Let as usual, $s = \sigma + it$.

Let k be an integer $= [3A]+11$.

$$b_n = \begin{cases} a_n\left(1-\frac{n}{3x}\right)^k - a_n\left(1-\frac{n}{x}\right)^k & \text{if} \quad n \leqslant x, \\ a_n\left(1-\frac{n}{3x}\right)^k & \text{if} \quad x \leqslant n \leqslant 3x, \\ 0 & \text{if} \quad n > 3x. \end{cases}$$

$w = u+iv$, $w_1 = u_1+iv_1$, $w_2 = u_2+iv_2$.

$$g(w) = \frac{3^w-1}{w(w+1)(w+2)\dots(w+k)}.$$

$y = H/100$.

$\delta = 1/\log H$.

LEMMA 3. *For any complex number α, and real $\beta > -1$*

$$\int_{0}^{y} x^{\alpha}(\log(y/x))^{\beta}\,dx = y^{\alpha+1}\frac{1}{(\alpha+1)^{\beta+1}}\Gamma(1+\beta),$$

provided $\mathrm{Re}(1+\alpha) > 0$.

Proof. Since both sides of the equality are analytic functions of α, it suffices to prove the result then α is real. In this case, the substitution $\log(y/x) = t$ yields the result.

LEMMA 4. *Let* $\mathrm{Re}\, w_1 = \mathrm{Re}\, w_2 = -\mathrm{Re}\, s = -\delta$. *Let* $0 \leqslant H_1 - \mathrm{Im}\, w_1 \leqslant H_2 - \mathrm{Im}\, w_1 \leqslant H$, $0 \leqslant H_1 - \mathrm{Im}\, w_2 \leqslant H_2 - \mathrm{Im}\, w_2 \leqslant H$.

Then

$$\left|\int_{T+H_1}^{T+H_2} F(s+w_1)\overline{F(s+w_2)}\,dt\right| \ll \int_{T}^{T+H} |F(it)|^2\,dt.$$

Proof.

$$\left|\int_{T+H_1}^{T+H_2} F(s+w_1)\overline{F(s+w_2)}\,dt\right| < \int_{T+H_1}^{T+H_2} |F(s+w_1)|^2\,dt + \int_{T+H_1}^{T+H_2} |F(s+w_2)|^2\,dt$$

$$< \int_{T+H_1-v_1}^{T+H_2-v_1} |F(it)|^2\,dt + \int_{T+H_1-v_2}^{T+H_2-v_2} |F(it)|^2\,dt$$

$$\ll \int_{T}^{T+H} |F(it)|^2\,dt.$$

LEMMA 5. *We have, uniformly for β in $|\beta| \leqslant 1$,*

$$\int_{\delta-i\infty}^{\delta+i\infty} |dw_1| \left|\int_{\delta-i\infty}^{\delta+i\infty} g(w_1)\overline{g(w_2)}\frac{1}{(w_1+\overline{w_2}+4\delta)^{\beta+1}}\,\overline{dw_2}\right| \ll \left|\frac{(\log H)^{\beta}-1}{\beta}\right|.$$

Proof. Since

$$\left|\frac{1}{(w_1+\overline{w_2}+4\delta)^{\beta+1}}\right| \ll \begin{cases} (\log H)^{\beta+1} & \text{if} \quad |v_1-v_2| \leqslant 1/\log H, \\ |v_1-v_2|^{-(\beta+1)} & \text{if} \quad 1/\log H < |v_1-v_2| < 1, \\ 1 & \text{if} \quad |v_1-v_2| \geqslant 1, \end{cases}$$

it is easy to see that the double integral

$$\ll (\log H)^{\beta} + \left|\frac{(\log H)^{\beta}-1}{\beta}\right| + 1;$$

we note that $|(\log H)^{\beta}| \ll |((\log H)^{\beta}-1)/\beta|$ by discussing the cases

$|\beta| \leqslant 1/\log\log H$, $|\beta| \geqslant 1/\log\log H$ separately. Clearly $1 \ll |((\log H)^{\beta}-1)/\beta|$ and this completes the proof.

LEMMA 6. *We have*

$$\int_{T+3H/10}^{T+7H/10} \Big|\sum_n b_n/n^s\Big|^2 dt \gg H\sum_n |b_n|^2/n^{2\sigma}$$

provided $x \leqslant H/100$.

This is an immediate consequence of the theorem of Montgomery and Vaughan [1]. For a simpler proof see [4].

LEMMA 7. *Let* $T+3H/10 \leqslant t \leqslant T+7H/10$ *and* $X \leqslant H$. *Then there exist* H_3 *and* H_4 *with* $H/10 \leqslant H_3,\ H_4 \leqslant 2H/10$ *such that*

$$\int_{-1/\log H}^{A+2} F\left(\frac{1}{\log H}+it+w\right)X^w g(w)\,du = O(H^{-2})+O\left(H^{-2}\int_T^{T+H} |F(it)|\,dt\right)$$

on the line $\operatorname{Im} w = H_3$ *as well as on the line* $\operatorname{Im} w = -H_4$.

Proof. In order to prove the existence of H_3, it suffices to prove that

$$\int_{H/10}^{2H/10} dv\left|\int_{-1/\log H}^{A+2} F\left(\frac{1}{\log H}+it+w\right)X^w g(w)\,du\right| \ll H^{-2}+H^{-2}\int_T^{T+H} |F(it)|\,dt.$$

Hence it suffices to prove that

$$\int_{H/10}^{2H/10} dv \int_{-1/\log H}^{A+2} \left|F\left(\frac{1}{\log H}+it+w\right)-1\right| |X^w g(w)|\,du \ll H^{-2}+H^{-2}\int_T^{T+H} |F(it)|\,dt.$$

We note that, by a change of variable, it suffices to prove that

$$H^{A+2-k}\int_0^{\infty} d\sigma \int_{T+H/10}^{T+9H/10} |F(s)-1|\,dt \ll H^{-2}+H^{-2}\int_T^{T+H} |F(it)|\,dt$$

and this is a consequence of Theorem 1.

The proof of existence of H_4 is also similar.

LEMMA 8. *If* $\operatorname{Re} s = \delta$, *then*

$$P(x) = \sum_{n=1}^{\infty} \frac{b_n}{n^s} = \frac{k!}{2\pi i}\int_{-\delta-iH_4}^{-\delta+iH_3} F(s+w)g(w)x^w\,dw+E$$

where

$$E = E(x) = O\left(H^{-2}+H^{-2}\int_T^{T+H} |F(it)|\,dt\right).$$

Proof. Since

$$\frac{k!}{2\pi i}\int \frac{y^w}{w(w+1)\dots(w+k)}\,dw = \begin{cases}0 & \text{if } 0<y<1,\\ (1-1/y)^k & \text{if } y \geqslant 1\end{cases}$$

it follows that

$$P(x) = \frac{k!}{2\pi i}\int_{A+3-i\alpha}^{A+3+i\alpha} F(s+w)g(w)x^w\,dw.$$

We break the integral in $-H_4 \leqslant \operatorname{Im} w < H_3$ with an error $O(H^{-2})$. (Here H_3 and H_4 are as defined in Lemma 7.) Now we move the line of integration to $\operatorname{Re} w = -\delta$ and the error is, by Lemma 7,

$$Q\left(H^{-2}+H^{-2}\int_T^{T+H} |F(it)|\,dt\right).$$

This proves the result.

From Lemma 8, there follows

LEMMA 9. *We have*

$$|(P(x)-E(x))|^2$$

$$= \left|\left(\frac{k!}{2\pi}\right)\right|^2 \int_{\delta-iH_4}^{\delta+iH_3} dw_1 \int_{\delta-iH_4}^{\delta+iH_3} F(s+w_1)\overline{F(s+w_2)}\,g(w_1)\overline{g(w_2)}x^{w_1+\overline{w_2}}\,\overline{dw_2}.$$

LEMMA 10. *We have*

$$\int_{T+3H/10}^{T+7H/10} dt\int_0^y x^{4\delta-1}|(P(x)-E(x))|^2(\log(y/x))^{\beta}\,dx$$

$$\ll \left|\frac{(\log H)^{\beta}-1}{\beta}\right| |\Gamma(1+\beta)| \int_T^{T+H} |F(it)|^2\,dt.$$

Proof. From Lemma 9, we have

$$\int_{T+3H/10}^{T+7H/10} dt\int_0^y x^{4\delta-1}|(P(x)-E(x))|^2(\log(y/x))^{\beta}\,dx$$

$$= \int_{\delta-iH_4}^{\delta+iH_3} dw_1 \int_{\delta-iH_4}^{\delta+iH_3} \overline{dw_2}\int_{T+3H/10}^{T+7H/10} F(s+w_1)\overline{F(s+w_2)}\,g(w_1)\overline{g(w_2)}\,dt \times$$

$$\times \int_0^y x^{w_1+\overline{w_2}+4\delta-1}(\log(y/x))^{\beta}\,dx.$$

We estimate the x-integral by Lemma 3, the t-integral by Lemma 4 and the w_1, w_2 integral by Lemma 5. This completes the proof of the lemma.

LEMMA 11. *We have*

$$\int_{T+3H/10}^{T+7H/10} dt \int_{0}^{y} x^{4\delta-1} |(P(x))|^2 (\log(y/x))^\beta \, dx$$

$$\ll \left|\frac{(\log H)^\beta - 1}{\beta}\right| |\Gamma(1+\beta)| \int_{T}^{T+H} |F(it)|^2 \, dt + H^{-1}.$$

Proof. Since

$$|(E(x))|^2 \ll H^{-4}\Big(\int_{T}^{T+H} |F(it)| \, dt\Big)^2 + H^{-4} \ll H^{-3} \int_{T}^{T+H} |F(it)|^2 \, dt + H^{-4},$$

we have, by Lemma 3,

$$\int_{T+3H/10}^{T+7H/10} dt \int_{0}^{y} x^{4\delta-1} |(E(x))|^2 (\log(y/x))^\beta \, dx$$

$$\ll \Big(H^{-2} \int_{T}^{T+H} |F(it)|^2 \, dt + H^{-3}\Big)(\log H)^{\beta+1} |\Gamma(1+\beta)|.$$

Now,

$$\int_{T+3H/10}^{T+7H/10} dt \int_{0}^{y} x^{4\delta-1} |(P(x))|^2 (\log(y/x))^\beta \, dx$$

$$\ll \int_{T+3H/10}^{T+7H/10} dt \int_{0}^{y} x^{4\delta-1} \big(|(P(x)-E(x))|^2 + |(E(x))|^2\big)(\log(y/x))^\beta \, dx$$

and hence the result, using Lemma 10.

LEMMA 12. *We have*

$$\int_{T+3H/10}^{T+7H/10} dt \int_{0}^{y} x^{4\delta-1} |(P(x))|^2 (\log(y/x))^\beta \, dx \gg \sum_{n \leqslant H/200} |a_n|^2 (\log(y/n))^\beta.$$

Proof. Using Lemma 6, the left side of Lemma 12 is

$$\int_{0}^{y} x^{4\delta-1} (\log(y/x))^\beta \, dx \int_{T+3H/10}^{T+7H/10} |P(x)|^2 \, dt \gg H \int_{0}^{y} x^{4\delta-1} (\log(y/x))^\beta \sum_{n} \frac{|b_n|^2}{n^{2\sigma}} \, dx$$

$$\gg H \int_{0}^{y} (\log(y/x))^\beta x^{4\delta-1} \sum_{x \leqslant n \leqslant 2x} |a_n|^2 \, dx$$

$$\gg H \sum_{n \leqslant H/200} |a_n|^2 \int_{n/2}^{n} (\log(y/x))^\beta x^{4\delta-1} \, dx$$

$$\gg H \sum_{n \leqslant H/200} |a_n|^2 (\log(y/n))^\beta.$$

Now Theorem 2 follows from Lemma 11, Lemma 12 and the fact that

$$H^{-1} = o\Big(H \sum_{n \leqslant H/200} |a_n|^2 (\log(y/n))^\beta\Big)$$

(since $a_1 = 1$ and H is sufficiently large).

5. Proof of Theorem 3. We deduce Theorem 3 from Theorem 2. By putting $\beta = 1/\log\log H$ we get (i). By putting $\beta = 1/2$ we get (ii). In order to deduce (iii), let

$$f(\beta) = \begin{cases} 1/|\beta| & \text{if} \quad |\beta| > 1/\log\log H, \\ 0 & \text{otherwise}. \end{cases}$$

We multiply both sides of the equality of Theorem 2 by $f(\beta)$ and integrate w.r.t. β in the range $-1/2 \leqslant \beta \leqslant 1/2$. This yields

$$\frac{1}{H} \int_{T}^{T+H} |F(it)|^2 \, dt \int_{-1/2}^{1/2} f(\beta) \, d\beta \gg \sum_{n \leqslant H/200} |a_n|^2 \int_{-1/2}^{1/2} \frac{|\beta| (\log(H/n))^\beta}{|(\log H)^\beta - 1|} f(\beta) \, d\beta$$

$$\gg \sum_{n \leqslant H/200} |a_n|^2 \int_{1/2 \geqslant |\beta| \geqslant 1/\log\log H} \frac{(\log(H/n))^\beta}{|(\log H)^\beta - 1|} \, d\beta \gg \sum_{n \leqslant H/200} |a_n|^2 \int_{-4/\log N}^{-2/\log N} + \int_{2/\log M}^{4/\log M}$$

$$\gg \sum_{n \leqslant H/200} |a_n|^2 \int_{-4/\log N}^{-2/\log N} (\log(H/n))^\beta \, d\beta + \sum_{n \leqslant H/200} |a_n|^2 \int_{2/\log M}^{4/\log M} \left(\frac{\log H}{\log(H/n)}\right)^{-\beta} d\beta$$

$$\gg \sum_{n \leqslant H/200} |a_n|^2 \left(\frac{1}{\log N} + \frac{1}{\log M}\right)$$

and this yields (iii).

Multiplying both sides of the equality of Theorem 2 by $1/(|\beta| \log^2 |1/\beta|)$ and integrating w.r.t. β in the range $-1/2 \leqslant \beta \leqslant 1/2$, we get (iv). This completes the proof of the theorem. We remark that slightly different choices of β gives marginally better results like $1/\log M \log\log M (\log\log\log M)^2$. Still the proof of the conjecture is open.

References

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On positive definite quadratic polynomials

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1. Introduction. It is well known that an indefinite quadratic form in 21 or more variables takes on arbitrarily small values at integer points (see Davenport and Ridout [8] for a full list of references). An analogous problem for positive definite quadratic forms has been considered by Davenport and Lewis in their interesting paper [7] which contains the following

Theorem 1 (Davenport and Lewis). *There exists an integer n_0 (absolute) with the following property:*

Let $Q(x) = Q(x_1, \ldots, x_n)$ be a positive definite quadratic form with real coefficients and suppose that $n \geq n_0$. Then, if $x_1^*, \ldots, x_n^*$ are integers with $\max|x_i^*|$ *sufficiently large, there exist integers $x_1, \ldots, x_n$, not all zero, such that*

$$|Q(x+x^*) - Q(x^*)| < 1. \tag{1}$$

In the course of their proof, Davenport and Lewis have however overlooked the (trivial) solution $x = -2x^*$ for (1); indeed, their proof of Theorem 1 assumes that (1) has no nonzero solutions and proceeds then to obtain a contradiction. The object of this note is to show that the very same analytic arguments used by them not only remove this lacuna but can also be adapted to yield many more integer solutions x of (1).

In (1), the term 1 can be replaced by an arbitrary $\varepsilon > 0$, and the result can then be regarded as a recurrence theorem. The quadratic form Q returns to the neighbourhood of values it has taken. Examples such as $\theta(x_1^2 + \ldots + x_n^2)$ show that it is not possible to obtain a theorem of the form "Q takes values close to all sufficiently large real numbers X" without some additional condition, such as incommensurability of the coefficients of Q.

Theorem 2. *There exists an integer $n_0 \leq 995$ and a constant $\tau > 0$ with the following property:*

Let $F(x)$ be a positive definite quadratic form with real coefficients and suppose that $n \geq n_0$. Then, if $x_1^*, \ldots, x_n^*$ are integers with $\max|x_i^*|$ *sufficiently large, then there exist at least $[|x^*|^\tau]$ integer points $x \in Z^n$ such that*

$$|F(x+x^*) - F(x^*)| < 1. \tag{2}$$