### The divisor problem for arithmetic progressions

by  
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1. **Introduction.** Let \( n \geq 1 \) and \( r \geq 2 \) be integers and let \( d_r(n) \) denote the number of ordered \( r \)-tuples \((n_1, \ldots, n_r)\) of positive integers for which

\[
\prod_{1 \leq i < j \leq r} n_i = n.
\]

For \((a, q) = 1\) define

\[
D_r(X, q, a) = \sum_{n \leq X \atop \gcd(n, q) = 1} d_r(n).
\]

We are interested here in finding real numbers \( \theta_r \), as large as possible, such that the following statement holds.

(S) For each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that

\[
D_r(X, q, a) \ll \frac{X}{q^{\delta}} P_r(\log X) X^{\frac{1}{r} - \delta},
\]

provided that \( q < X^{1/r} \).

Here \( P_r(\log X) \) is the residue at \( s = 1 \) of \( s^{-1} L(s, \chi_0) X^{s-1} \), where \( \chi_0 \) is the principal character of modulus \( q \).

It was discovered independently by Selberg and by Hooley that Weil's estimate for the Kloosterman sum yielded the above statement with \( \theta_2 = 2/3 \). The authors [2] recently proved that one may take \( \theta_3 = 1/2 + 1/230 \). The result with \( \theta_4 = 1/2 \) seems harder to trace but was known to Linnik. In this paper we are able to improve the results \( \theta_r = 8/(3r+4) \) for \( r \geq 5 \) which are due to Layvik [5].

**Theorem.** The statement (S) holds in the following cases:

(1) \( \theta_2 = 9/20 \),

(II) \( r \geq 6 \) and \( \theta_r = \min \{8/3r, 5/12\} \),

(III) \( q \) is restricted to cube-free integers, \( r \geq 7 \), and \( \theta_r = \min \{4/r, 5/12\} \).

Although the proof of this result involves some fairly deep arguments, these are for the most part already recorded in the literature and we shall quote liberally therefrom.

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The main ingredients in the proof are the Burgess estimates [1] for character sums and the recent work by several authors [3] on the difference between consecutive primes. Indeed, it is no coincidence that $1-\gamma_2$ and $1-\gamma_3$ have been numbers of significance in this latter area.

2. Lemmata.

**Lemma 1** (Burgess [1]). For $q$ a positive integer, let

$$\gamma = \gamma(q) = \begin{cases} \frac{1}{2} & \text{if } q \text{ is cube-free,} \\ \frac{1}{3} & \text{otherwise.} \end{cases}$$

Let $\varepsilon > 0$. There exists $\delta(q) > 0$ such that, for all non-principal characters $\chi \mod q$ and all $L \geq q^{\varepsilon}$, we have

$$\sum_{i \leq L} \chi(i) \ll_{\varepsilon} Lq^{-\delta}.$$

**Lemma 2.** For $T \geq 2$ we have

$$\sum_{\chi \mod q} |\sum_{n=1}^{T} \chi(n)| \ll qT \log q T.$$

Lemma 2 follows at once from Theorem 10.1 of [6].

**Lemma 3.**

$$\sum_{\chi \mod q} \left| \sum_{n \leq N} \chi(n) \right|^4 \ll N^2 q \log q.$$

**Proof.** We may assume that $N$ is half an odd integer and is less than $q$, whence

$$\sum_{n \leq N} \chi(n) = \frac{1}{2\alpha} \int_{1/2-iT}^{1/2+iT} L(s, \chi) N^s ds + O(NT^{-1} \log N),$$

where $U = 1 + \log^{-1} N$, $T = (qN)^{1/2}$.

Since

$$|L(s, \chi)| \ll (qT)^{(1-n)/2}$$

for $\frac{1}{2} \leq n \leq U$,

we have

$$S = \sum_{n \leq N} \chi(n) \ll N^{1/2} \int_{1/2-iT}^{1/2+iT} |L(s, \chi)| N^s ds + O(N^{1/2}).$$

Hölder's inequality gives

$$S^4 \ll N^{2 \log q} \int_{1/2-iT}^{1/2+iT} |L(s, \chi)|^4 N^s ds + O(N^2).$$

We sum over $\chi$, then apply Lemma 2 and partial summation. This completes the proof.

**Lemma 4.** Let $L, M, N \geq 1$, $LMN = X$, $(a, q) = 1$, $\varepsilon > 0$, and $\gamma = \gamma(q)$ as in Lemma 1. Assume

1. $q \ll X^{9/20} \varepsilon$,
2. $X^\varepsilon q \ll L$,
3. $M, N \ll X^{1-\varepsilon} q^{-6/5}$.

Let $[\alpha_{mn}]_{m \leq M}$, $[\beta_{mn}]_{n \leq N}$ be sequences of complex numbers with $|\alpha_{mn}| \ll X$, $|\beta_{mn}| \ll X^\varepsilon$, for all $m, n$. Then, there exists $\delta > 0$ such that

$$\sum_{l \leq L} \sum_{x \mod q} |\sum_{m \leq M} \alpha_{m} \chi(m)| \sum_{n \leq N} \beta_{m} \chi(n) \ll_{\varepsilon} X^{1-\delta}.$$

We omit the proof of Lemma 4 as it is an almost verbatim duplicate of the argument on pages 102–104 of [4], the only difference being that there one dealt with Dirichlet polynomials on $\sigma = 1/2$ rather than $\sigma = 0$ as here. It should be mentioned however that this proof is the heart of the matter. In addition to using Lemmata 1 and 3 it appeals to the large sieve and the Halász–Montgomery–Huxley method. It is perhaps best described as an analogue for arithmetic progressions of the lemma of Heath-Brown used in [3] to estimate the difference of consecutive primes.

3. Proof of the theorem. Let $N_1, \ldots, N_r$ satisfy

$$N_1 \geq N_2 \geq \ldots \geq N_r \geq 1, \quad \prod_{j=1}^{r} N_j = X.$$

Let $\Delta = X^{-\eta}$ where $\eta > 0$ is fixed. Let

$$E(N_1, \ldots, N_r) = \sum_{1 \leq n \leq N_r} \sum_{1 \leq n \leq N_r} \sum_{1 \leq n \leq N_r} \frac{1}{\phi(q)} \sum_{\alpha(a) = \chi(a)} \sum_{\beta(a) = \beta_0(a)} \sum_{\gamma(a) = \gamma_0(a)} \sum_{a \in \phi(r)}.$$

By an elementary argument (see, for example, the first part of the proof of Theorem 5 of [2]), the proof may be reduced to the demonstration that, for an arbitrary box $N_1, \ldots, N_r$ satisfying (4), we have

$$E(N_1, \ldots, N_r) \ll \frac{1}{\phi(q)} X^{1-\delta}.$$

Now,

$$E(N_1, \ldots, N_r) = \frac{1}{\phi(q)} \sum_{\chi \equiv \chi_0} \sum_{\neq \chi_0} \sum_{\neq \chi_0} \sum_{\neq \chi_0} \chi(n_1 \ldots n_r)$$

$$\ll \frac{1}{\phi(q)} \sum_{\chi \equiv \chi_0} |\sum_{\chi_0} \sum_{\neq \chi_0} \sum_{\neq \chi_0} \sum_{\neq \chi_0} \chi(n_1 \ldots n_r)|.$$
\[ M = \prod_{j \in \mathcal{M}} N_j, \quad N = \prod_{j \in \mathcal{N}} N_j, \] with \( \alpha_m \) being the number of representations of \( m \) in the form

\[ \prod_{j \in \mathcal{M}} \prod_{j \in \mathcal{N}} n_j = m, \quad (1 - a) n_j < n_j \in N_j \]

and \( \beta_n \) defined similarly. Note that

\[ \alpha_m \leq d_{r-1}(m), \quad \beta_n \leq d_{r-1}(n). \]

By Lemma 4, it remains to show that, given any \( N_1, \ldots, N_r \), satisfying (4), we have \( N_1 > q^{\frac{1}{2}} x^r \) and there exists a partition \( \mathcal{M}, \mathcal{N} \) for which (3) holds. Since \( N_1 \geq X^{1/r} \) by (4), the first requirement follows easily. Moreover, by choosing \( \mathcal{M} = [2, 3, \ldots, k] \) where \( M = \prod_{j \in \mathcal{M}} N_j \geq X^{1/2} \) with \( k \) maximal, we have \( M \geq X^{1/2} N_1^{-1} \) so \( N \leq X^{1/2} \). Combining this with (3) we get the theorem for \( r \geq 6 \) (as well as the statement with \( \theta_3 = 5/12 \)). The remaining estimate with \( \theta_3 = 9/20 \) requires a little more effort.

**Case I:** If \( N_2 \leq q^{1/2+\epsilon} \), then \( \mathcal{M} = [2, 3] \), \( \mathcal{N} = [4, 5] \) gives a decomposition with \( M, N \leq N_2^2 \leq q^{1+2\epsilon} \). Since \( q < X^{1/2} \) if \( q < X^{5/11} < X^{1/2-\epsilon} \), (3) holds and Lemma 4 gives the result in this case if \( \delta(\epsilon) \) is sufficiently small.

**Case II:** If \( N_2 > q^{1/2+\epsilon} \) we abandon Lemma 4 and write

\[ E(N_1, \ldots, N_5) \leq \frac{1}{\varphi(q)} \sum_{x \leq X} \left| \sum_{n_1} \chi(n_1) \right| \left| \sum_{n_2} \chi(n_2) \right| \left| \sum_{m \in M} \alpha_m \chi(m) \right| \]

where \( m = n_3 n_4 n_5 \) and \( \alpha_m \) is defined by the same prescription as before. By Hölder's inequality,

\[ E(N_1, \ldots, N_5) \leq \frac{1}{\varphi(q)} \left( \sum_{x \leq X} \left| \sum_{n_1} \chi(n_1) \right|^4 \right)^{1/4} \left( \sum_{x \leq X} \left| \sum_{n_2} \chi(n_2) \right|^4 \right)^{1/4} \left( \sum_{x} \left| \sum_{m \in M} \alpha_m \chi(m) \right|^2 \right)^{1/2}, \]

and, for any \( \epsilon > 0 \), this is

\[ \leq \epsilon \frac{1}{\varphi(q)} q^{1/2} (M + q)^{1/2} X^{1/2+\epsilon} \]

by Lemma 3 and the large sieve inequality. Since \( q M = q X N_1^{-1} N_2^{-1} < X^{1/2} \) (we may assume \( q > X^{1/2} \)), and since \( q < X^{1/2-\epsilon} \), the result follows.

**Remark.** The estimate \( \theta_3 = 5/12 \) for \( r \geq 6 \) cannot be improved by this method as can be seen by considering the case \( N_1 = \ldots = N_5 = X^{1/6} \), \( q = X^{3/12} \). Here, for any partition, either \( M \geq X^{1/2} \) or \( N \geq X^{1/2} \) so (3) fails to hold. A similar phenomenon has already been observed by several people in connection with the problem of consecutive primes.

References


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