| $m$ | $A$ | $B$ | $C$ | $p$ | $\left(\frac{A+B\sqrt{m}}{p}\right)$ | $(-1)^{(p-1)/8}$, if $p \equiv 1\,(8)$; $(-1)^{(p+m-1)/8}$, if $p \equiv 7\,(8)$ | $\left(\frac{p}{m/2}\right)_4$ |
|---|---|---|---|---|---|---|---|
| 10 | 50 | 9 | 13 | 31 | −1 | −1 | 1 |
| 10 | 50 | 13 | 9 | 31 | −1 | −1 | 1 |
| 26 | 130 | 11 | 23 | 17 | −1 | 1 | −1 |
| 26 | 130 | 17 | 19 | 23 | −1 | 1 | −1 |
| 34 | 34 | 3 | 5 | 47 | 1 | 1 | 1 |
| 34 | 170 | 3 | 29 | 47 | 1 | 1 | 1 |
| 58 | 58 | 3 | 7 | 23 | 1 | 1 | 1 |
| 58 | 58 | 7 | 3 | 23 | 1 | 1 | 1 |
| 74 | 74 | 5 | 7 | 41 | 1 | −1 | −1 |
| 74 | 74 | 7 | 5 | 41 | 1 | −1 | −1 |
| 82 | 82 | 1 | 9 | 23 | −1 | −1 | 1 |
| 82 | 82 | 9 | 1 | 23 | −1 | −1 | 1 |

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# The divisor problem for arithmetic progressions

by

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**1. Introduction.** Let $n \geqslant 1$ and $r \geqslant 2$ be integers and let $d_r(n)$ denote the number of ordered $r$-tuples $(n_1, \ldots, n_r)$ of positive integers for which $\prod_{1 \leqslant j \leqslant r} n_j = n$.

For $(a, q) = 1$ define

$$D_r(X, q, a) = \sum_{\substack{n \leqslant X \\ n \equiv a \,(\mathrm{mod}\, q)}} d_r(n).$$

We are interested here in finding real numbers $\theta_r$, as large as possible, such that the following statement holds.

(S) *For each $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$D_r(X, q, a) - \frac{X}{\varphi(q)} P_r(\log X) \ll_\varepsilon \frac{X^{1-\delta}}{\varphi(q)},$$

*provided that $q < X^{\theta_r - \varepsilon}$.*

Here $P_r(\log X)$ is the residue at $s = 1$ of $s^{-1} L^r(s, \chi_0) X^{s-1}$, where $\chi_0$ is the principal character of modulus $q$.

It was discovered independently by Selberg and by Hooley that Weil's estimate for the Kloosterman sum yielded the above statement with $\theta_2 = 2/3$. The authors [2] recently proved that one may take $\theta_3 = 1/2 + 1/230$. The result with $\theta_4 = 1/2$ seems harder to trace but was known to Linnik. In this paper we are able to improve the results $\theta_r = 8/(3r+4)$ for $r \geqslant 5$ which are due to Lavrik [5].

Theorem. *The statement* (S) *holds in the following cases*:

(I) $\theta_5 = 9/20$,

(II) $r \geqslant 6$ *and* $\theta_r = \min\{8/3r,\ 5/12\}$,

(III) *$q$ is restricted to cube-free integers, $r \geqslant 7$, and* $\theta_r = \min\{4/r,\ 5/12\}$.

Although the proof of this result involves some fairly deep arguments, these are for the most part already recorded in the literature and we shall quote liberally therefrom.

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The main ingredients in the proof are the Burgess estimates [1] for character sums and the recent work by several authors [3] on the difference between consecutive primes. Indeed, it is no coincidence that $1-\frac{5}{12}$ and $1-\frac{9}{20}$ have been numbers of significance in this latter area.

**2. Lemmata.**

LEMMA 1 (Burgess [1]). *For $q$ a positive integer, let*

$$\gamma=\gamma(q)=\begin{cases}\frac{1}{4} & \text{if } q \text{ is cube-free},\\ \frac{3}{8} & \text{otherwise}.\end{cases}$$

*Let $\varepsilon>0$. There exists $\delta(\varepsilon)>0$ such that, for all non-principal characters $\chi \bmod q$ and all $L\geqslant q^{\gamma+\varepsilon}$, we have*

$$\sum_{l\leqslant L}\chi(l)\ll_\varepsilon Lq^{-\delta}.$$

LEMMA 2. *For $T\geqslant 2$ we have*

$$\sum_{\chi(\bmod q)}\int_{-T}^{T}|L(\tfrac{1}{2}+it,\chi)|^4\,dt\ll qT\log^4 qT.$$

Lemma 2 follows at once from Theorem 10.1 of [6].

LEMMA 3.

$$\sum_{\chi\neq\chi_0}\Big|\sum_{n\leqslant N}\chi(n)\Big|^4\ll N^2q\log^6 q.$$

Proof. We may assume that $N$ is half an odd integer and is less than $q$, whence

$$\sum_{n\leqslant N}\chi(n)=\frac{1}{2\pi i}\int_{U-iT}^{U+iT}L(s,\chi)N^s\frac{ds}{s}+O(NT^{-1}\log N),$$

where $U=1+\log^{-1}N$, $T=(qN)^{1/2}$.

Since

$$|L(\sigma\pm iT)|\ll_\varepsilon(qT)^{(1-\sigma)/2+\varepsilon}\quad\text{for}\quad \tfrac{1}{2}\leqslant\sigma\leqslant U,$$

we have

$$S=\Big|\sum_{n\leqslant N}\chi(n)\Big|\ll N^{1/2}\int_{1/2-iT}^{1/2+iT}|L(s,\chi)|\left|\frac{ds}{s}\right|+O(N^{1/2}).$$

Hölder's inequality gives

$$S^4\ll N^2\log q\int_{1/2-iT}^{1/2+iT}|L(s,\chi)|^4\left|\frac{ds}{s}\right|+O(N^2).$$

We sum over $\chi$, then apply Lemma 2 and partial summation. This completes the proof.



LEMMA 4. *Let $L, M, N\geqslant 1$, $LMN=X$, $(a,q)=1$, $\varepsilon>0$, and $\gamma=\gamma(q)$ as in Lemma 1. Assume*

(1) $$q<X^{9/20-\varepsilon},$$

(2) $$X^\varepsilon q^\gamma<L,$$

(3) $$M,N<X^{1-\varepsilon}q^{-6/5}.$$

*Let $\{\alpha_m|\,m\leqslant M\}$, $\{\beta_n|\,n\leqslant N\}$ be sequences of complex numbers with $|\alpha_m|<X^\varepsilon$, $|\beta_n|<X^\varepsilon$, for all $m$, $n$. Then, there exists $\delta(\varepsilon)>0$ such that*

$$\sum_{\chi\neq\chi_0}\Big|\sum_{l\leqslant L}\chi(l)\Big|\Big|\sum_{m\leqslant M}\alpha_m\chi(m)\Big|\Big|\sum_{n\leqslant N}\beta_n\chi(n)\Big|\ll_\varepsilon X^{1-\delta}.$$

We omit the proof of Lemma 4 as it is an almost *verbatim* duplicate of the argument on pages 102–104 of [4], the only difference being that there one dealt with Dirichlet polynomials on $\sigma=1/2$ rather than $\sigma=0$ as here. It should be mentioned however that this proof is the heart of the matter. In addition to using Lemmata 1 and 3 it appeals to the large sieve and the Halász–Montgomery–Huxley method. It is perhaps best described as an analogue for arithmetic progressions of the lemma of Heath-Brown used in [3] to estimate the difference of consecutive primes.

**3. Proof of the theorem.** Let $N_1,\ldots,N_r$ satisfy

(4) $$N_1\geqslant N_2\geqslant\ldots\geqslant N_r\geqslant 1,\qquad \prod_{j=1}^{r}N_j=X.$$

Let $\Delta=X^{-\eta}$ where $\eta>0$ is fixed. Let

$$E(N_1,\ldots,N_r)=\sum_{\substack{(1-\Delta)N_j<n_j\leqslant N_j\\ n_1\ldots n_r\equiv a(\bmod q)}}\cdots\sum 1-\frac{1}{\varphi(q)}\sum_{\substack{(1-\Delta)N_j<n_j\leqslant N_j\\ (n_1\ldots n_r,q)=1}}\cdots\sum 1.$$

By an elementary argument (see, for example, the first part of the proof of Theorem 5 of [2]), the proof may be reduced to the demonstration that, for an arbitrary box $N_1,\ldots,N_r$ satisfying (4), we have

$$E(N_1,\ldots,N_r)\ll\frac{1}{\varphi(q)}X^{1-\delta}.$$

Now,

$$E(N_1,\ldots,N_r)=\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0}\bar\chi(a)\sum_{(1-\Delta)N_j<n_j\leqslant N_j}\cdots\sum\chi(n_1\ldots n_r)$$
$$\ll\frac{1}{\varphi(q)}\sum_{\chi\neq\chi_0}\Big|\sum_{n_1}\chi(n_1)\Big|\Big|\sum_{m\leqslant M}\alpha_m\chi(m)\Big|\Big|\sum_{n\leqslant N}\beta_n\chi(n)\Big|.$$

Here we have fixed a partition $\mathcal{M}$, $\mathcal{N}$ of the set $\{2,\ldots,r\}$, let

$M=\prod_{j\in\mathcal{M}} N_j$, $N=\prod_{j\in\mathcal{N}} N_j$, with $\alpha_m$ being the number of representations of $m$ in the form

$$\prod_{\substack{(1-\Delta)N_j<n_j\leqslant N_j\\ j\in\mathcal{M}}} n_j = m,$$

and $\beta_n$ defined similarly. Note that

$$\alpha_m \leqslant d_{r-1}(m), \qquad \beta_n \leqslant d_{r-1}(n).$$

By Lemma 4, it remains to show that, given any $N_1, \ldots, N_r$ satisfying (4), we have $N_1 > q^{\gamma} x^{\varepsilon}$ and there exists a partition $\mathcal{M}$, $\mathcal{N}$ for which (3) holds. Since $N_1 \geqslant X^{1/r}$ by (4), the first requirement follows easily. Moreover, by choosing say $\mathcal{M}=\{2, 3, \ldots, k\}$ where $M=\prod_{j=2}^{k} N_j \leqslant X^{1/2}$ with $k$ maximal, we have $M \geqslant X^{1/2} N_1^{-1}$ so $N \leqslant X^{1/2}$. Combining this with (3) we get the theorem for $r \geqslant 6$ (as well as the statement with $\theta_5 = 5/12$). The remaining estimate with $\theta_5 = 9/20$ requires a little more effort.

Case I: If $N_2 \leqslant q^{1/2+\delta}$, then $\mathcal{M}=\{2, 3\}$, $\mathcal{N}=\{4, 5\}$ gives a decomposition with $M$, $N \leqslant N_2^2 \leqslant q^{1+2\delta}$. Since $q < Xq^{-6/5}$ if $q < X^{5/11} < X^{9/20-\varepsilon}$, (3) holds and Lemma 4 gives the result in this case if $\delta(\varepsilon)$ is sufficiently small.

Case II: If $N_2 > q^{1/2+\delta}$ we abandon Lemma 4 and write

$$E(N_1, \ldots, N_5) \ll \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} \Big|\sum_{n_1} \chi(n_1)\Big| \Big|\sum_{n_2} \chi(n_2)\Big| \Big|\sum_{m\leqslant M} \alpha_m \chi(m)\Big|$$

where $m = n_3 n_4 n_5$ and $\alpha_m$ is defined by the same prescription as before. By Hölder's inequality,

$$E(N_1, \ldots, N_5) \ll \frac{1}{\varphi(q)} \Big(\sum_{\chi\neq\chi_0} \Big|\sum_{n_1} \chi(n_1)\Big|^4\Big)^{1/4} \Big(\sum_{\chi\neq\chi_0} \Big|\sum_{n_2} \chi(n_2)\Big|^4\Big)^{1/4} \Big(\sum_{\chi} \Big|\sum_{m} \alpha_m \chi(m)\Big|^2\Big)^{1/2},$$

and, for any $\varepsilon > 0$, this is

$$\ll_{\varepsilon} \frac{1}{\varphi(q)} q^{1/2} (M+q)^{1/2} X^{1/2+\varepsilon}$$

by Lemma 3 and the large sieve inequality. Since $qM = qXN_1^{-1}N_2^{-1} < Xq^{-2\delta} < X^{1-\delta_1}$ (we may assume $q > X^{1/3}$), and since $q < X^{1/2-\delta_2}$, the result follows.

Remark. The estimate $\theta_r = 5/12$ for $r \geqslant 6$ cannot be improved by this method as can be seen by considering the case $N_1 = \ldots = N_6 = X^{1/6}$, $q = X^{5/12}$. Here, for any partition, either $M \geqslant X^{1/2}$ or $N \geqslant X^{1/2}$ so (3) fails to hold. A similar phenomenon has already been observed by several people in connection with the problem of consecutive primes.



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