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Class number calculation of a sextic field from the elliptic unit

by

KEN NAKAMULA (Tokyo)

Introduction. Let K be a sextic field which contains a real quadratic subfield K_2 and a not totally real cubic subfield K_3. The object of this paper is to give an effective way of calculating the class number and fundamental units of K at a time. Together with [4], it generalizes our previous result in [2] which treats K_3 itself. The method depends on the class number formula of R. Schertz [6] related to the elliptic unit. The main part of the computation is that of the relative class number and relative units of K with respect to K_2 and K_3. A summary of the results has been given in [1, III], see also the errata at the end of this paper.

As the features of our algorithm are explained in the papers [2], [4], [5], they are not repeated here. The idea used in the algorithm is typically seen in the proofs of Theorem 2 of [2], Theorem 2 and Proposition 2 of [4], and in Proposition 7 of [5]. In the case we are going to investigate in this paper, it is necessary for growth of efficiency of the method to do some technical consideration, so the description may become more detailed than in general case.

In § 1, we quote some results in [3] and show preliminary lemmas, introducing notations. Theorems 1 and 2, which assure the effectiveness of the computation, are proved in § 2. The actual calculation will be explained in § 3, § 4, § 5. In § 6, another non-galois sextic subfield of the galois closure of K/Q is studied. Pure sextic fields $Q(\sqrt[6]{n})$ and $Q(\sqrt[6]{-27n})$ are contained in those types of fields we consider. Numerical examples are given in § 7.

1. Notations and preliminaries. All number fields we consider are finite extensions of Q in C. A subgroup of the multiplicative group of C generated by $\alpha_1, \ldots, \alpha_n$ is denoted by $\langle\alpha_1, \ldots, \alpha_n\rangle$. The nth root $\sqrt[n]{\alpha}$ is always positive when α is positive real.

Let K_2 be a quadratic number field with discriminant $d_2 > 0$ and K_3 be a cubic number field with discriminant $-d_3 < 0$. Then the composite field $K = K_2 \cdot K_3$ is a non-galois sextic field with positive discriminant D. We consider K is contained in $\boldsymbol{R}$. According to [3], we choose the fundamental units $\varepsilon_1, \varepsilon_2, \varepsilon_3$ of K as follows. Let E be the group of positive units of K and H be its subgroup which consists of relative units with respect to K_2 and K_3. As the first fundamental unit, take the generator $\varepsilon_1 > 1$ of the infinite cyclic group H. Namely, we let

$$(1)\qquad \langle\varepsilon_1\rangle = H = \{\varepsilon\in E \mid N_{K/K_2}(\varepsilon) = N_{K/K_3}(\varepsilon) = 1\},\qquad \varepsilon_1 > 1.$$

Further let $\eta_i > 1$ be the fundamental unit of K_i for $i = 2, 3$. Then, as the second and the third fundamental units, we can take

$$(2)\qquad \varepsilon_2 \in \{\varepsilon_1\eta_2,\ \sqrt[3]{\varepsilon_1\eta_2^{\pm 1}},\ \sqrt[3]{\eta_2}\},\qquad \varepsilon_2 > 1,$$

$$(3)\qquad \varepsilon_3 \in \{\varepsilon_1\eta_3,\ \sqrt{\varepsilon_1\eta_3}\},\qquad \varepsilon_3 > 1.$$

As is shown in (12)–(17) and Remark 3 of [3], these $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are uniquely decided and generate E. Let h, h_2, h_3 respectively be the class numbers of K, K_2, K_3. Then, by (4.28) of [6],

$$h/h_2h_3 = (E : \langle\eta, \eta_2, \eta_3\rangle)/6$$

holds with the elliptic unit $\eta > 1$, which will be defined by (12) in § 3. Since η belongs to H, we have

$$(4)\qquad \eta = \varepsilon_1^{h'} \quad\text{with}\quad h' = (H : \langle\eta\rangle).$$

Therefore it is obtained by (1)–(3) that

$$(5)\qquad h/h_2h_3 = h'h_0/6 \quad\text{with}\quad h_0 = (E : \langle\varepsilon_1, \eta_2, \eta_3\rangle) = 1, 2, 3 \text{ or } 6.$$

We may assume h_2, h_3 and η_2, η_3 are known as they are computed in the usual manner and by the result in [2]. We shall decide h', h_0 via the determination of the $\varepsilon_1, \varepsilon_2, \varepsilon_3$, which will be given in the form of their minimal polynomials over $\boldsymbol{Q}$.

For $\varepsilon\in H$, $\varepsilon > 1$, the conjugates are $\varepsilon^{\pm 1}$, $\sqrt{\varepsilon^{\pm 1}}\exp(\pm\sqrt{-1}\,\theta)$ with $0 < \theta < \pi$, so its minimal polynomial has the form

$$(6)\qquad X^6 - sX^5 + tX^4 - uX^3 + tX^2 - sX + 1,\qquad u = s^2 + 2s - 2t + 2 \quad\text{with}\quad s, t\in\boldsymbol{Z}.$$

Moreover, we see that

$$(7)\qquad -2 < T := 2\cos(\theta) < 2 < \alpha := \sqrt{\varepsilon} + \sqrt{\varepsilon^{-1}} < \beta := \alpha^2 - 2 = \varepsilon + \varepsilon^{-1} = \mathrm{Tr}_{K/K_3}(\varepsilon).$$

Lemma 1. *The coefficient s in (6) is completely decided by the conditions*

$$|s-\beta| < 2\alpha,\qquad \beta(s+2-\beta) + \alpha^{-2}(s+2)^2 \in \boldsymbol{Z}.$$

And then t in (6) is given by $t = \beta(s+2-\beta) + \alpha^{-2}(s+2)^2 - 2s - 1$.

Proof. Since $s = \beta + \alpha T$, $t = \beta(1+\alpha T) + T^2 + 1$, the lemma immediately follows from (6).

Further let $D(\varepsilon)$ be the discriminant of ε with respect to $K/\boldsymbol{Q}$. Then it is a non-zero multiple of D.

Lemma 2. *The estimation $D \leqslant D(\varepsilon) < 16\big((\varepsilon+\frac{9}{7})^7 - 290\big)^2$ holds.*

Proof. To see the second inequality, we put $g(X) = (\alpha\beta - \alpha - X)^2 \times (\alpha - X)^3(\alpha + X)(4 - X^2)$ as in the proof of Lemma 2 in [4]. Then it follows from (7) that

$$\sqrt{D(\varepsilon)} = (\varepsilon - \varepsilon^{-1})(\beta - 2)\,g(T).$$

We see $g(X) \leqslant g(M)$ if $|X| < 2$, where M is decided by the condition $g'(M) = 0$ ($|M| < 2$) and satisfies

$$-(\beta^2 + 9\beta - 14)\alpha M = 4\beta^2 + 8\beta - (\beta^2 + 3\beta + 14)M^2 - (3\beta - 4)\alpha M^3 + 4M^4,$$
$$-1 < M < -2\alpha^{-1}.$$

Utilizing the equality, we can transform $g(M)$ after a tedious calculation into the form

$$g(M) = A_0 - \sum_{i=1}^{6} A_i(-\alpha M)^i + BM^8.$$

Here the positive numbers $A_0, \ldots, A_6$, B are given by the following expressions with $\gamma = \alpha^2$:

$$\begin{aligned}
A_0 &= 4\gamma^5 + 8\gamma^4 - 28\gamma^3,\\
A_1 &= 80\gamma^3 - 272\gamma^2,\\
A_2 &= \gamma^4 + 2\gamma^3 - 39\gamma^2 + 188\gamma,\\
A_3 &= 2\gamma^3 - 20\gamma^2 + 74\gamma - 108,\\
A_4 &= 6\gamma - 33 + 44\gamma^{-1},\\
A_5 &= 2 - 14\gamma^{-1} + 40\gamma^{-2},\\
A_6 &= 1 - 2\gamma^{-1} + 5\gamma^{-2} - 20\gamma^{-3},\\
B &= 8\gamma - 7 - 2\alpha M^{-1}(3\gamma^2 - 12\gamma + 8).
\end{aligned}$$

Since $-2\alpha M^{-1} < \gamma$, we get $B < 3\gamma^3 - 12\gamma^2 + 16\gamma - 7$. Therefore

$$\begin{aligned}
g(M) &< A_0 - \sum_{i=1}^{6} 2^i A_i + 3\gamma^3 - 12\gamma^2 + 16\gamma - 7\\
&= 4\gamma^5 + 4\gamma^4 - 209\gamma^3 + 848\gamma^2 - 1424\gamma + 1225 - 128\gamma^{-1} - \\
&\quad - 1600\gamma^{-2} - 1280\gamma^{-3},
\end{aligned}$$

and hence

$$(\beta-2)\,g(M) < 4\gamma^6-12\gamma^5-225\gamma^4+1684\gamma^3-4816\gamma^2+6921\gamma-4900.$$

Multiplying this by $\varepsilon-\varepsilon^{-1}$, we easily obtain the expected inequality in the lemma.

For any $\xi \in E$ such that $K = \mathbf{Q}(\xi)$, let

$$X^6-s(\xi)X^5+t(\xi)X^4-u(\xi)X^3+v(\xi)X^2-w(\xi)X+x(\xi) \tag{8}$$

be its minimal polynomial over $\mathbf{Q}$. In particular, we denote

$$\begin{aligned} s_i &= s(\varepsilon_i), & t_i &= t(\varepsilon_i), & u_i &= u(\varepsilon_i), \\ v_i &= v(\varepsilon_i), & w_i &= w(\varepsilon_i), & x_i &= x(\varepsilon_i), \end{aligned} \tag{9}$$

for $i = 1, 2, 3$. Note that $s_1 = w_1$, $t_1 = v_1$, $u_1 = s_1^2+2s_1-2t_1+2$ and $x_1 = 1$ by (6). The minimal polynomials of η_2 and η_3 are respectively denoted by

$$X^2-lX+c \quad \text{and} \quad X^3-yX^2+zX-1. \tag{10}$$

LEMMA 3. (i) *Put* $d_2' = \sqrt{(l^2-4c)(s_1^2+4s_1-4t_1)}$, *then* $d_2' \in N$ *and* $d_2 \mid d_2'$. *For* $\xi = \varepsilon_1\eta_2$, *the coefficients in* (8) *are given by*

$$s(\xi) = w(\xi) = \tfrac{1}{2}(ls_1+d_2'), \qquad t(\xi) = cv(\xi) = l^2 s_1+c(t_1-2s_1)-ls(\xi),$$
$$u(\xi) = l^3+cl(u_1-s_1+t_1-5)-cs_1 s(\xi), \qquad x(\xi) = c.$$

For $\xi = \varepsilon_1\eta_2^{-1}$, *we may only replace* l *by* cl *and* d_2' *by* $-cd_2'$ *in the above.*

(ii) *Put* $\beta = \varepsilon_1+\varepsilon_1^{-1}$ *and let*

$$\varkappa = 3\beta^2-2s_1\beta-s_1^2+4t_1-12, \qquad \lambda = 3\eta_3^{-2}-2z\eta_3^{-1}-z^2+4y,$$

then $\varkappa\lambda > 0$ *and* $\sqrt{\varkappa\lambda} \notin \mathbf{Z}$. *For* $\gamma_\pm = \beta\eta_3^{\pm 1}$, *put the rational integers*

$$y_\pm = \mathrm{Tr}_{K_3/\mathbf{Q}}(\gamma_\pm), \qquad z_\pm = u_0\gamma_\pm^{-1}+\gamma_\pm(y_\pm-\gamma_\pm),$$

where $u_0 = u_1-2s_1 = N_{K_3/\mathbf{Q}}(\beta)$. *Then it follows that*

$$y_- = \gamma_-+\tfrac{1}{2}(s_1-\beta)(z-\eta_3^{-1})\pm\tfrac{1}{2}\sqrt{\varkappa\lambda}$$

for a unique appropriate choice of the sign and that

$$y_+ = \gamma_++(s_1-\beta)(y-\eta_3)-(y_--\gamma_-)\eta_3^{-1}.$$

For $\xi = \varepsilon_1\eta_3$, *the coefficients in* (8) *are given by*

$$s(\xi) = y_+, \quad t(\xi) = y^2-2z+z_+, \quad w(\xi) = y_-, \quad v(\xi) = z^2-2y+z_-,$$
$$u(\xi) = u_0+(yz-1)s_1-yy_--zy_+, \qquad x(\xi) = 1.$$

Proof. Put $\varepsilon = \varepsilon_1$ and use the notation in (7). Observe that $K_2 = \mathbf{Q}(\eta_2) = \mathbf{Q}(\varepsilon+T\sqrt{\varepsilon^{-1}})$. Then the first statement of (i) follows since $\varepsilon^{\pm 1}+T\sqrt{\varepsilon^{\mp 1}}$ are the zeros of $X^2-s_1X+t_1-s_1$. Other assertions in (i) are shown by (10) like Lemma 3 of [4]. To prove (ii), let φ be the argument of the imaginary conjugate of η_3^{-1} which is taken so that

$$y_- = \gamma_-+\tfrac{1}{2}(s_1-\beta)(z-\eta_3^{-1})+\sqrt{(\beta-2)(4-T^2)\eta_3}\,\sin(\varphi).$$

Since $K_3 = \mathbf{Q}(\eta_3^{-1}) = \mathbf{Q}(\beta)$ and β is the real zero of $X^3-s_1X^2+(t_1-3)X-u_0$, we see, on account of (10), that $\varkappa = (\beta-2)(4-T^2) > 0$, $\lambda = 4\eta_3\sin^2(\varphi) > 0$, and that the discriminants $D_3(\beta)$, $D_3(\eta_3^{-1})$ of β, η_3^{-1} are given by

$$\varkappa(\varkappa+s_1^2-3t_1+9)^2 = -D_3(\beta), \qquad \lambda(\lambda+z^2-3y)^2 = -D_3(\eta_3^{-1}).$$

We notice that these are the defining equations of $\varkappa$ and λ. If we suppose that $n := \sqrt{\varkappa\lambda} \in \mathbf{Z}$, then

$$D_3(\beta)\lambda^3 = -n^2\big((s_1^2-3t_1+9)\lambda+n^2\big)^2,$$

therefore $D_3(\eta_3^{-1}) = m^3$ with $m = \frac{1}{2}(z^2-3y)$ by the uniqueness of the defining equation of λ, hence $(\lambda+m)(\lambda^2+3m\lambda+m^2) = 0$, which is a contradiction. This implies that $\sqrt{\varkappa\lambda} \notin \mathbf{Z}$, so one and only one of

$$\gamma_-+\tfrac{1}{2}(s_1-\beta)(z-\eta_3^{-1})\pm\tfrac{1}{2}\sqrt{\varkappa\lambda}$$

belongs to $\mathbf{Z}$ and coincides with y_-. Other assertions in (ii) are checked by direct calculation.

Remark 1. The above lemma gives a way to compute the minimal polynomials of $\varepsilon_1\eta_2^{\pm 1}$, $\varepsilon_1\eta_3$ from those of ε_1, η_2, η_3. For $\xi = \varepsilon_1\eta_3$, we need good approximate values of both β and η_3. The following relations are also useful:

$$y_\pm z+s_1 z_\pm+(t_1-3)\,y_\mp = u_0 m_\pm^2-(3u_1-3s_1-s_1t_1)\,n_\pm,$$

where $m_+ = n_- = y$, $n_+ = m_- = z$. It is shown that $\gamma_\pm$ are primitive elements of K_3, so another method similar to that in § 4 of [2] may be used to decide $y_\pm$, $z_\pm$.

2. An upper bound of h' and a recursive sequence.

2.1. Let $\varepsilon \in H$, $\varepsilon > 1$, then $\varepsilon = \varepsilon_1^n$ with $n \in N$. As it is easy to see $D \geqslant 5^3\,23^2$, the following estimate of the index n is always meaningful.

THEOREM 1. *The notation being as above, the following holds*:

$$n < B(\varepsilon) := \log(\varepsilon)/\log\big(\sqrt[7]{\sqrt{D/16}+290}-(9/7)\big).$$

Proof. If we notice that $K = \mathbf{Q}(\varepsilon)$, the estimation is derived from Lemma 2 in the same way as Theorem 1 of [2].

For the elliptic unit η, we get $h' < B(\eta)$ by (4), so the relative class number is estimated as below by (5).

COROLLARY 1. *The inequality* $h/h_2h_3 < B(\eta)$ *holds.*

2.2. For $i, j \in \mathbf{Z}$, put $k = i^2+2i-2j+2$ and define a recursive sequence $r_n = r_n(i, j)$ $(n = 1, 2, 3, \ldots)$ by the following:

$$r_1 = i,\ r_2 = ir_1-2j,\ r_3 = ir_2-jr_1+3k,\ r_4 = ir_3-jr_2+kr_1-4j,$$

$$r_5 = ir_4-jr_3+kr_2-jr_1+5i,\ r_6 = ir_5-jr_4+kr_3-jr_2+ir_1-6,$$

$$r_n = ir_{n-1}-jr_{n-2}+kr_{n-3}-jr_{n-4}+ir_{n-5}-r_{n-6} \quad (n = 7, 8, 9, \ldots).$$

Let $\xi \in H$, $\xi > 1$, $n \in \mathbf{N}$ and put $\varepsilon = \sqrt[n]{\xi}$, $\alpha = \sqrt{\varepsilon}+\sqrt{\varepsilon^{-1}}$, $\beta = \varepsilon+\varepsilon^{-1}$. For $i \in \mathbf{Z}$, denote by i' the nearest element of $\mathbf{Z}$ to $\beta(i+2-\beta)+\alpha^{-2}(i+2)^2-2i-1$. We may consider only $s(\cdot)$ and $t(\cdot)$ in (8) for units in H on account of (6).

THEOREM 2. *The notation being as above, the real number ε is a unit in H if and only if a certain $i \in \mathbf{Z}$ satisfies that*

$$(11)\qquad |i-\beta| < 2\alpha,\qquad r_n(i, i') = s(\xi),\quad r_n(i'-i-3,\ r_3(i, i')+i'-i) = t(\xi)-s(\xi)-3.$$

Moreover $s(\varepsilon) = i$ and $t(\varepsilon) = i'$ if (11) holds with $i \in \mathbf{Z}$.

Proof. Similar to that of Theorem 2 in [2] or [4] by virtue of Lemma 1.

Remark 2. We have a finite number of possible values of i which may satisfy (11), so we can test effectively whether ξ is a nth root in K or not. The number of possibilities, however, increases as ε becomes large. This differs from the cubic case [2] or the quartic case [4].

3. Calculation of a ring class group. To obtain the elliptic unit of K, we compute a ring class group of an imaginary quadratic field. The principle of this section is the same as in § 3 of [2].

Let $F = \mathbf{Q}(\sqrt{-d_3})$, $\tilde{K}_2 = \mathbf{Q}(\sqrt{-d_2 d_3})$ and $-d$, $-\tilde{d}_2$ respectively be their discriminants. Then $d_2\tilde{d}_2 = df_2^2$, $d_3 = df_3^2$ with $f_2, f_3 \in \mathbf{N}$. Let f be the least common multiple of f_2 and f_3, then $D = dd_2d_3f^2$ holds. For $n = 2, 3$, let U_n^* be a subgroup of index n in the ring class group $R^*(f)$ modulo f in F such that its conductor is exactly f_n. Put $U^* = U_2^* \cap U_3^*$ and let L be the cyclic sextic class field over F corresponding to U^*. Every K we consider is realized as the maximal real subfield of such an L. The class group U^* and a class $r^* \in R^*(f)$, $r^* \notin U_2^*$, $r^* \notin U_3^*$, are obtained systematically by the exact sequence (4) of [2], assuming that d and f are given. It will be complicated to write down the general result exactly, so a special case is given below explicitly.

Suppose the ring I_1 of integers of F is principal and $f = f_2 = f_3$, then

$$d = 3, 4, 7, 8, 11, 19, 43, 67 \text{ or } 163;$$

$$f = f_2 = f_3 = p_1 \ldots p_m, \quad (f, 6) = 1.$$

Here $m > 0$ and $p_1, \ldots, p_m$ are distinct prime numbers such that

$$p_i \equiv \left(\frac{-d}{p_i}\right) \pmod 3 \quad (i = 1, \ldots, m),$$

where $\left(\frac{-d}{\cdot}\right)$ is the Kronecker symbol. For $i = 1, \ldots, m$, put

$$k_i = \frac{1}{6}\left(p_i-\left(\frac{-d}{p_i}\right)\right)$$

and let z_i be a *primitive ring root* modulo p_i, see Remark 2 of [2], such that $z_i \in \mathbf{Q}+(f/p_i)I_1$, $(z_i, f) = 1$, with the additional condition $z_i^{k_i}(1+\sqrt{-3}) \in \mathbf{Q}+p_iI_1$ if $d = 3$. Take $x_i = 1$ or 5 for $i = 1, \ldots, m-1$. In case $d = 3$, we further assume

$$k_1-x_1k_2+x_1x_2k_3- \ldots +(-1)^{m-1}x_1 \ldots x_{m-1}k_m \equiv 0 \pmod 3.$$

PROPOSITION 1. *For every choice of $x_1, \ldots, x_{m-1}$, the following principal ideals of F represent U^* satisfying the conditions above:*

$$\prod_{i=1}^{m-1}(z_i^{x_i}z_{i+1})^{X_i}\prod_{i=1}^{m} z_i^{6Y_i}I_1,$$

where

$$0 \leqslant X_i < 6 \ (i = 1, \ldots, m-1), \quad 0 \leqslant Y_i < k_i \ (i = 1, \ldots, m),$$

$$X_{m-1}+6Y_m < 2k_m \ \text{ if } d = 3, \quad Y_m < \tfrac{1}{2}k_m \ \text{ if } d = 4.$$

The class r^* represented by z_1I_1 generates $R^*(f)$ together with U^*.

Proof. Similar to Proposition 2 of [2].

The representatives of r^* and U^* being obtained as integral ideals of F prime to f, the elliptic unit η of K is given by

$$(12)\qquad \eta = \prod_{k \in U}\frac{\Delta(r^2k)\Delta(r^3k)}{\Delta(k)\Delta(rk)}, \quad \Delta(k) = \sqrt{\mathrm{Im}(Z_k)}|\eta(Z_k)|^2 \quad (k \in R(f)).$$

Here the notation is the same as in § 3 of [2].

4. Calculation of η, ε_1 and h'. For η in (12), we can compute $s(\eta)$, $t(\eta)$ in (8) by Lemma 1 from a good approximate value of η as in § 4 of [2], and then the minimal polynomial of η is obtained by (6). The computation of ε_1 in (1), s_1, t_1 in (9) and h' in (4) goes mostly in the same manner as in § 5 of [2]. The effectiveness of the calculation is now assured by Theorems 1 and 2 above. Although we need many steps of computation, see Remark 2, it is possible to write this algorithm in a program of electric computers. Indeed, numerical examples in § 7 are computed by a BASIC program applied to a pocket computer. The example in § 7.1 will show fully how the computation actually goes.

5. Calculation of ε_2, ε_3 and h_0. After we have obtained ε_1, h', we should decide ε_2, ε_3 in (2), (3) and h_0 in (5). More precisely, we should compute the integers in (9) for $n = 2, 3$, from s_1, t_1, l, c, x, y in (9), (10).

PROPOSITION 2. *Put* $\xi = \varepsilon_1 \eta_2^{\pm 1}$ *and let* $\varepsilon = \sqrt[3]{\xi}$, $\alpha = \varepsilon + c\varepsilon^{-1}$, $\beta = \varepsilon\sqrt{\varepsilon_1^{-1}} + c\varepsilon^{-1}\sqrt{\varepsilon_1}$. *Then it is necessary and sufficient for* $\varepsilon \in E$ *that certain* $i, j, k \in \boldsymbol{Z}$ *satisfy*

$$|i-\alpha| < 2|\beta|, \qquad |j-\beta^2-\alpha(i-\alpha)| < 2,$$
$$s(\xi) = i^3-3m, \qquad t(\xi)-c = j(j^2-3ni-3)+3mn,$$
$$u(\xi) = k(k^2-3i^2-3cj^2+9c)+6n(j+c),$$

where $m = i(j+c)-k$, $n = m-ci$; *and then* $s_2 = w_2 = i$, $t_2 = cv_2 = j+c$, $u_2 = k$, $x_2 = c$ *and* $3 \mid h_0$.

Proof. Since $K = \boldsymbol{Q}(\xi)$, all the assertions are proved in the same way as Proposition 2 of [4].

PROPOSITION 3. *Put* $\xi = \varepsilon_1 \eta_3$, $\alpha = \sqrt{\varepsilon_1} + \sqrt{\varepsilon_1^{-1}}$, $\gamma = \sqrt{\eta_3}$, $\varepsilon = \sqrt{\xi}$, $s_0 = s_1 + 2$ *and let* $y_\pm$, $z_\pm$ *be as in Lemma* 3 (ii). *Then the condition that* $\varepsilon \in E$ *is equivalent to the existence of* $i, j \in \boldsymbol{Z}$ *such that*

$$\text{(13)} \qquad |i-\alpha\gamma| < 2\sqrt{s_0\alpha^{-1}\gamma^{-1}}, \qquad y_+ + 2y = i^2-2j,$$
$$z_+ - 2y_- + 2s_0 z = j^2 - 2s_0 i.$$

When (13) *is satisfied by* $i, j \in \boldsymbol{Z}$, *put* $(i', j') = (3n, 3n^2)$ *if* $(i, j, s_0) = (ny, n^2 z, n^3)$ *with* $n = 1$ *or* 2, *otherwise take* $i', j' \in \boldsymbol{Z}$, *which are uniquely decided by the following condition*:

$$|i'-\alpha\gamma^{-1}| < 2\sqrt{s_0\alpha^{-1}\gamma}, \qquad j' = s_0\alpha^{-1}\gamma + \alpha\gamma^{-1}(i'-\alpha\gamma^{-1}).$$

Then $2 \mid h_0$ *and the coefficients in* (9) *of* ε_3 *are given by*

$$s_3 = i, \qquad t_3 = j+y, \qquad w_3 = i', \qquad v_3 = j'+z,$$
$$u_3 - s_0 = (s_0 z + 2z - y_- + jy + j')/i, \qquad x_3 = 1.$$

Proof. Put $\beta = \varepsilon_1 + \varepsilon_1^{-1}$, then $X^2 - \beta\eta_3 X + \eta_3^2$ is the minimal polynomial of ξ over K_3. Therefore, in the same way as before, the condition that $\varepsilon \in E$ is equivalent to the existence of integers λ, μ of K_3 such that $\beta\eta_3 = \lambda^2 - 2\mu$, $\mu^2 = \eta_3^2$, and so to the condition that $\alpha\gamma = \sqrt{(\beta+2)\eta_3} \in K_3$ since μ should be positive. Under these equivalent conditions, the minimal polynomial of ε over K_3 is

$$X^2 - \alpha\gamma X + \eta_3.$$

Now we apply the same technique to judge whether $(\beta+2)\eta_3$ is a square or not in K_3. Note that $K_3 = \boldsymbol{Q}((\beta+2)\eta_3)$ holds. Because, if not so, the minimal polynomial of $\beta+2$ is written in two ways, namely

$$X^3 - (s_0+4)X^2 + (4s_0+t_1+1)X - s_0^2 = X^3 - zkX^2 + yk^2X - k^3$$

with $k := (\beta+2)\eta_3 > 4$, hence $16 = k(k^2 - z^2k + 8z)$, a contradiction. Since the minimal polynomial of $(\beta+2)\eta_3$ is equal to

$$X^3 - (y_+ + 2y)X^2 + (z_+ - 2y_- + 2s_0 z)X - s_0^2,$$

we see that $\alpha\gamma$ belongs to K_3, or equivalently ε belongs to E, if and only if (13) holds with rational integers i, j. Assume (13) is satisfied by $i, j \in \boldsymbol{Z}$. Then the minimal polynomial of $\alpha\gamma$ is given by

$$X^3 - iX^2 + jX - s_0,$$

and $\alpha\gamma^{-1}$ also belongs to K_3. If we put

$$i' = \mathrm{Tr}_{K_3/\boldsymbol{Q}}(\alpha\gamma^{-1}), \quad j' = s_0 \mathrm{Tr}_{K_3/\boldsymbol{Q}}(\alpha^{-1}\gamma),$$

then $\alpha\gamma^{-1}$ is a real zero of

$$X^3 - i'X^2 + j'X - s_0.$$

When $n := \alpha\gamma^{-1} \in \boldsymbol{N}$, considering the minimal polynomials of $\alpha\gamma$ and $\beta+2$ similarly as above, we get

$$(i, j, s_0) = (ny, n^2z, n^3), \qquad 16 = n^2(n^4 - y^2n^2 + 8y),$$

therefore $n = 1$ or 2, and then $(i', j') = (3n, 3n^2)$. Otherwise i', j' is decided by the condition in the proposition. It is now easy to compute the minimal polynomial of $\varepsilon_3 = \varepsilon$, and the proof ends.

Remark 3. For $\xi = \varepsilon_1\eta_3$, we have another equivalent condition to $\varepsilon_3 = \sqrt{\xi}$ in (3), and that can be shown like Proposition 2. Precisely, the solubility of the following simultaneous diophantine equations with variables s, t, u, v, w is the equivalent condition:

$$s(\xi) = s^2 - 2t, \qquad t(\xi) = t^2 - 2su + 2v,$$
$$w(\xi) = w^2 - 2v, \qquad v(\xi) = v^2 - 2wu + 2t, \qquad u(\xi) = u^2 + 2sw - 2tv - 2.$$

This has more possibilities of generalization, though the minimal polynomial of $-\sqrt{\xi}$ may be obtained. In Proposition 3, the case $\alpha\gamma^{-1} = 1$ or 2 surely occurs, see § 7.2 (i), at most 10 possible fields exist, however. The conditions in Propositions 2 and 3 seem to be complicated. But they are arranged to increase efficiency to go on the numerical computation, see § 7.1.

Let $-d_3'$ be the discriminant of η_3, then

$$27d_3' = m^2 - 4n^3 \quad \text{with} \quad m = 2y^3 - 9yz + 27, \; n = y^2 - 3z.$$

Put $\varepsilon = \sqrt[3]{\eta_2}$ and $a = \sqrt{27d_3'(l^2-4c)}$, then we have a refinement of Proposition 1 in [1, III] as follows.

PROPOSITION 4. *If $\varepsilon\in E$, then $s_2=t_2=v_2=w_2=0$, $u_2=l$, $x_2=c$, $3\mid h_0$ and $K_2=\boldsymbol{Q}(\sqrt{3d_3})$, i.e. $a\in N$. Under the condition that $a\in N$, the case $\varepsilon\in E$ occurs if and only if*

$$b^3-3cnb=\tfrac{1}{2}(lm\pm a)\quad \text{or}\quad (m,n,c)=(b^3 l, b^2, 1)$$

holds with a certain $b\in\boldsymbol{Z}$.

Proof. When $\varepsilon\in E$, the discriminant of $\varepsilon+c\varepsilon^{-1}$ is $-27(l^2-4c)$, hence the first statement is proved easily. Suppose that $a\in N$ and let $\beta>0$ be the larger root of the irreducible equation $X^2-mX+n^3=0$, then $K_2=\boldsymbol{Q}(\beta)$ and $K=K_3(\beta)$. The equation $X^2-(3\eta_3-y)X+n=0$ has distinct positive roots, so let α be the larger one. Then, since $(3\eta_3-y)^3-3n(3\eta_3-y)=m$, we see $\alpha^3=\beta$, therefore $K_3(\beta)\subset K_3(\alpha)$. As α is at most quadratic over K_3, this shows that $\alpha\in K$. Consequently $K=K_2(\alpha)$ because $\alpha\notin K_2$. As is easily seen, this implies that $K=K_2(\varepsilon)$ holds if and only if $\beta\eta_2$ or $c\beta\eta_2^{-1}$ is a cube in K_2. When $\beta\eta_2^{-1}\in\boldsymbol{Z}$, we have $(m,n,c)=(b^3 l, b^2, 1)$ and $\beta\eta_2^{-1}=b^3$ with $b\in\boldsymbol{Z}$. Otherwise $\beta\eta_2$ or $c\beta\eta_2^{-1}$ respectively is a root of the irreducible equation

$$X^2-\tfrac{1}{2}(lm\pm a)X+cn^3=0,$$

whose root is a cube in K_2 if and only if $\frac{1}{2}(lm\pm a)=b^3-3cnb$ with $b\in\boldsymbol{Z}$ as before. Thus we complete the proof.

Remark 4. The original proof of this proposition comes from the fact that the galois closure L of $K/\boldsymbol{Q}$ contains $\zeta=\frac{1}{2}(-1+\sqrt{-3})$, i.e. $\boldsymbol{Q}(\sqrt{-d_2d_3})=\boldsymbol{Q}(\sqrt{-3})$, and that α is the Lagrange resolvent relative to η_3 and ζ. Proposition 3 of [4] is also shown by the same idea.

By these propositions above, we can completely determine ε_2, ε_3, h_0, and thus the class number h is obtained together by (5) provided h_2, h_3 are known. All necessary informations to go on this algorithm are obtained from Lemma 3.

Remark 5. As to (5), we have $6\mid h_0h'(h_2,h_3,3)$ by Satz (3.3) of [7] and $6\mid h_0h'$ unless $K_2=\boldsymbol{Q}(\sqrt{3d_3})$ by the remark there. Using the notation in § 3, we can verify that $\varepsilon_2=\sqrt[3]{\eta_2}$ may occur only when $F=\boldsymbol{Q}(\sqrt{-3d_2})$ and f is a power of 3. In other words, we may test whether η_2 is a cube or not in K only in case $d=3d_2$, $f_3\mid 9$, $f_2=1$ or in case $3d=d_2$, $f_3\mid 9$, $f_2=3$. These facts are useful to make our calculation more efficient.

6. Another sextic subfield $\tilde{K}$. The galois closure $L=K(\sqrt{-d_3})$ contains a totally imaginary non-galois sextic subfield $\tilde{K}=K_3(\sqrt{-d_2d_3})$ and its quadratic subfield $\tilde{K}_2=\boldsymbol{Q}(\sqrt{-d_2d_3})$ with discriminant $-\tilde{d}_2$. As in Remark 3 and (13) of [3], we can take a relative fundamental unit $\tilde{\varepsilon}_0$ of $\tilde{K}/K_3$, i.e. a unit which generates together with roots of unity the subgroup of units of $\tilde{K}$ whose absolute values are equal to 1, as the first fundamental unit of $\tilde{K}$.

Then $\tilde{\varepsilon}_3:=N_{L/\tilde{K}}(\varepsilon_3')$ becomes the second fundamental unit of $\tilde{K}$ by Corollary to Proposition 6 of [3], where ε_3' is any imaginary conjugate of ε_3 in (3). Let $\tilde{h}$, $\tilde{h}_2$ be the class numbers of $\tilde{K}$, $\tilde{K}_2$ respectively. Then the following class number relation is derived from a character relation of the galois group of $L/\boldsymbol{Q}$ by using the Brauer–Kuroda theorem:

$$\frac{h\log\left(N_{K/K_2}(\varepsilon_2)\right)}{h_2\log(\eta_2)}=\frac{\tilde{h}\left|\log\left(N_{\tilde{K}/K_3}(\tilde{\varepsilon}_3)\right)\log\left(|\varepsilon_0'|^2\right)\right|}{\tilde{h}_2\log\left(N_{K/K_3}(\varepsilon_3)\right)\log(\varepsilon_1)}.$$

Here ε_0' is a conjugate of $\tilde{\varepsilon}_0$ such that $\varepsilon_0'\notin\tilde{K}$. Since $|\varepsilon_0'|^2\in H$ by (1) and since $N_{K/K_3}(\varepsilon_3)N_{\tilde{K}/K_3}(\tilde{\varepsilon}_3)=1$, this implies that

$$(2,h_0)h'h_3/2=3h/(3,h_0)h_2=\tilde{h}(H:\langle|\tilde{\varepsilon}_0|^2\rangle)/\tilde{h}_2$$

on account of (2), (5). Applying Proposition 6 of [3] to the above, we get

$$(H:\langle|\tilde{\varepsilon}_0|^2\rangle)=3/\tilde{h}_0$$

with

(14) $$\tilde{h}_0=(\langle\varrho,\tilde{\varepsilon}_0\rangle:\langle\varrho,\tilde{\varepsilon}_1\rangle)=1\ \text{or}\ 3,\qquad \tilde{\varepsilon}_1=N_{L/\tilde{K}}(\varepsilon_1'),$$

where ε_1' is an imaginary conjugate of ε_1 and $\varrho=\frac{1}{2}(-1+\sqrt{-3})$ if $\tilde{d}_2=3$, otherwise $\varrho=1$. Hence we have

(15) $$\tilde{h}/\tilde{h}_2h_3=(2,h_0)h'\tilde{h}_0/6=\tilde{h}_0h/(3,h_0)h_2h_3.$$

Thus the computation of $\tilde{h}$ is reduced to that of $\tilde{h}_2$, which is easily known, and of $\tilde{h}_0$, which is determined by testing whether $\tilde{\varepsilon}_1$ is a cube in $\tilde{K}$ modulo $\langle\varrho\rangle$ or not so, see (14).

Under the notation in (9), we determine the minimal polynomial

(16) $$X^6-\tilde{s}_0X^5+\tilde{t}_0X^4-\tilde{u}_0X^3+\tilde{t}_0X^2-\tilde{s}_0X+1$$

of a relative fundamental unit $\tilde{\varepsilon}_0$, and the index $\tilde{h}_0$.

PROPOSITION 5. *If a certain $i\in\boldsymbol{Z}$ satisfies*

(17) $$i(i^2-3s_1-9)=t_1+5s_1+9,$$

then $\tilde{h}_0=3$ and $\tilde{s}_0=i$, $\tilde{t}_0=i+s_1+3$, $\tilde{u}_0=i^2-2s_1-4$. In case $\tilde{d}_2=3$, if certain $i,j\in\boldsymbol{Z}$ satisfy

(17′) $$i^2-ij+j^2=s_1+3,\qquad 9i^2j-(i+j)^3=t_1+5s_1+9,$$

then $\tilde{h}_0=3$ and

$$\tilde{s}_0=2i-j,\qquad \tilde{t}_0=s_1+3-i+2j,\qquad \tilde{u}_0=4ij-i^2-j^2-1.$$

Otherwise $\tilde{h}_0=1$ and

$$\tilde{s}_0=t_1-s_1-3,\qquad \tilde{t}_0=s_1^3-3s_1t_1+3u_1+t_1-s_1,\qquad \tilde{u}_0=\tilde{s}_0^2+2\tilde{s}_0-2\tilde{t}_0+2.$$

Proof. By the definition, the minimal polynomial of $\tilde{\varepsilon}_1$ over $\tilde{K}_2$ is given by $X^3-\alpha X^2+\bar{\alpha}X-1$ with

$$\alpha+\bar{\alpha} = t_1-s_1-3, \qquad \alpha\bar{\alpha} = s_1^3-3s_1t_1+3u_1+3.$$

Therefore, for $\omega\in\tilde{K}_2$, $\omega^3=1$, the unit $\omega\tilde{\varepsilon}_1$ is a root of the irreducible polynomial $X^3-\omega\alpha X^2+\overline{\omega\alpha}\,X-1$ over $\tilde{K}_2$. This implies that $\omega\tilde{\varepsilon}_1=\varepsilon^3$ with $\varepsilon\in\tilde{K}$ if and only if

$$\omega\alpha = \beta^3-3\beta\gamma+3\delta, \qquad \overline{\omega\alpha} = \gamma^3-3\beta\gamma\delta+3\delta^2, \qquad \delta^3=1$$

hold with certain integers β, γ, δ of K_2, and then $X^3-\beta X^2+\gamma X-\delta$ is the minimal polynomial of such an ε over $\tilde{K}_2$. By Lemma 2 of [3], we observe that $\delta=\omega$, $\gamma=\omega\bar{\beta}$ and $\beta\bar{\beta}=s_1+3$ should hold then. Hence $\tilde{h}_0=3$ occurs if and only if $\alpha+3s_1+6=\bar{\omega}\beta^3$ with certain β, $\omega\in\tilde{K}_2$ such that $\beta\bar{\beta}=s_1+3$, $\omega^3=1$; and $X^3-\beta X^2+\omega\bar{\beta}X-\omega$ is the minimal polynomial of a relative fundamental unit $\tilde{\varepsilon}_0$ over K_2 then. Otherwise $X^3-\alpha X^2+\bar{\alpha}X-1$ is that of $\tilde{\varepsilon}_0=\tilde{\varepsilon}_1$. The proposition is now easily shown as an interpretation of this fact.

Proposition 5 completes the computation of $\tilde{h}$ on account of (15). It is an easy matter to see that the minimal polynomial

$$X^6-\tilde{s}_3X^5+\tilde{t}_3X^4-\tilde{u}_3X^3+\tilde{v}_3X^2-\tilde{w}_3X+1 \tag{18}$$

of the second fundamental unit $\tilde{\varepsilon}_3$ above is given by

$$\begin{aligned} &\tilde{s}_3 = t_3-w_3-y_0, && \tilde{t}_3 = v_3+(s_3+z_0)s_0-w_3(t_3-y_0),\\ (19)\quad &\tilde{u}_3 = s_3w_3-u_3+t_0-s_0-3, &&\\ &\tilde{w}_3 = v_3-s_3-z_0, && \tilde{v}_3 = t_3+(w_3+y_0)s_0-s_3(v_3-z_0). \end{aligned}$$

Here, when $h_0=1$,

$$y_0=y^2-2z, \quad z_0=z^2-2y, \quad s_0=s_1^2-2t_1+2, \quad t_0=t_1^2-2s_1u_1+2t_1, \tag{20}$$

and, when $h_0=3$,

$$y_0=y, \quad z_0=z, \quad s_0=s_1+2, \quad t_0=t_1. \tag{21}$$

Proposition 6 of [3] and (15) above prove an analogous result to Corollary 2 of [4].

COROLLARY 2. *If* $\varepsilon_2=\sqrt[3]{\varepsilon_1\eta_2^{\pm1}}$ *in* (2), *we have* $\tilde{h}_0=(h_0,3)=3$, *so* $h/h_2h_3=\tilde{h}/\tilde{h}_2h_3$.

Remark 6. By Satz (3.3) of [7], it is known that $\tilde{h}_2h_3\mid\tilde{h}$. This is also utilized to go on the actual calculation together with Remark 5. In particular, if h' is prime to 6, the conclusion in Corollary 2 is valid unless $\tilde{d}_2=3$, and then the integer $h/h_2h_3=\tilde{h}/\tilde{h}_2h_3$ is prime to 6.

7. Examples. We keep the notation in § 1 and § 6 for the class numbers and the indices of groups of units. Recall the symbols in (8)–(10), (16) and (18) for units. When the meaning is clear, we simply denote $s=s(\xi)$, $t=t(\xi)$, $u=u(\xi)$, $v=v(\xi)$, $w=w(\xi)$ and $x=x(\xi)$ for $\xi\in E$.

7.1. This example is given to describe our algorithm clearly. Let $d=3$, $f=10$ in § 3, then $R^*(10)$ is cyclic of order 6 and K is uniquely decided as the maximal real subfield of the ring class field modulo 10 over $F=Q(\sqrt{-3})$. The ring ideals

$$2Z+\alpha Z, \quad 6Z+\alpha Z, \quad 8Z+(\alpha\pm2)Z, \quad 14Z+(\alpha\pm6)Z$$

represent the class group of ring ideals modulo 10, where $\alpha=10\sqrt{-3}$. We also see that $f_2=5$, $f_3=10$, $d_2=5$, $d_3=300$, $\tilde{d}_2=15$, $D=450000$ and

$$K=Q(\sqrt[6]{80}), \quad K_2=Q(\sqrt{5}), \quad K_3=Q(\sqrt[3]{10}).$$

Computing η in (12) approximately, see Lemma 3 of [1, I] for example, we get

$$\eta\sim 9.649476639.$$

Let $\varepsilon=\eta$ in Lemma 1, then $3\leqslant s(\eta)\leqslant 16$. Computing

$$\delta_i=(i+2-\beta)\beta+(i+2)^2\alpha^{-2}-2i-1 \qquad (i=3,4,\ldots,16)$$

approximately, see Table 1, we find that δ_{16} is the nearest to Z, and hence

$$s(\eta)=16, \quad t(\eta)=75, \quad u(\eta)=140.$$

Table 1

i	δ_i	i	δ_i	i	δ_i	i	δ_i
3	−51.23049638	7	−15.45336292	11	23.04645433	15	64.26895538
4	−42.54146462	8	−6.083660219	12	33.09682799	16	74.99999999
5	−33.68226513	9	3.456210229	13	43.31736938		
6	−24.65289789	10	13.16624841	14	53.70807851		

Since $B(\eta)\sim 21.072$, Theorem 1 claims that $h'\leqslant 21$ in (4). We start the divisibility test of h'. In Theorem 2, let $\xi=\eta$, $n=2$. Then, by the inequality in (11), we may calculate the recursive sequence for $(i,i')=(-1,-7)$, $(0,-5)$, $(1,-3)$, $(2,0)$, $(3,3)$, $(4,6)$, $(5,10)$, $(6,14)$, $(7,19)$, $(8,24)$. Among these, only $r_2(8,24)$ coincides with $s(\eta)$, but $r_2(24-8-3, r_3(8,24)+24-8)\neq t(\eta)-s(\eta)-3$. Therefore $\sqrt{\eta}\notin H$ and $2\nmid h'$. Next let $\xi=\eta$, $n=3$ in Theorem 2. Then it is verified that (11) holds for $(i,i')=(1,0)$, and then $\sqrt[3]{\eta}\in H$, $h'':=h'/3\in Z$, $s(\sqrt[3]{\eta})=1$, $t(\sqrt[3]{\eta})=0$ follow. Put now $\xi=\sqrt[3]{\eta}$ and repeat the same procedure for $n=3,5,7$, and we see that h'' is prime to 3, 5, 7, so $h''=1$ since $h''\leqslant 7.024$. Consequently

$$h'=3, \quad \varepsilon_1=\sqrt[3]{\eta}, \quad (s_1,t_1,u_1)=(1,0,5).$$

We know that $\eta_2 = \frac{1}{2}(1+\sqrt{5})$, $(l, c) = (1, -1)$ and, by [2], that $\eta_3 \sim 23.30224192$, $(y, z) = (23, -7)$. Since $d_2' = 5$ in Lemma 3 (i), it follows that

$$(s, t, u, v, w, x) = \begin{cases} (3, 0, 5, 0, 3, -1) & \text{for} \quad \xi = \varepsilon_1 \eta_2, \\ (2, 5, 0, -5, 2, -1) & \text{for} \quad \xi = \varepsilon_1 \eta_2^{-1}. \end{cases}$$

By an approximate computation in Lemma 3 (ii), we see that

$$\gamma_- + \tfrac{1}{2}(s_1 - \beta)(z - \eta_3^{-1}) \begin{cases} +\frac{1}{2}\sqrt{\varkappa\lambda} \sim 10.48236833, \\ -\frac{1}{2}\sqrt{\varkappa\lambda} \sim 0.999999999, \end{cases}$$

and thus $(y_+, z_+, y_-, z_-) = (61, 27, 1, 27)$ is obtained. Therefore

$$(s, t, u, v, w, x) = (61, 570, 245, 30, 1, 1) \quad \text{for} \quad \xi = \varepsilon_1 \eta_3.$$

Putting $\xi = \varepsilon_1 \eta_2$ in Proposition 2, the first inequality shows $\varepsilon \notin E$. For $\xi = \varepsilon_1 \eta_2^{-1}$, only $i = 1$ satisfies the first inequality, which is impossible by the first equality, however. Since $\tilde{d}_2 \neq 3$, we have shown that

$$3 \nmid h_0, \quad \varepsilon_2 = \varepsilon_1 \eta_2, \quad (s_2, t_2, u_2, v_2, w_2, x_2) = (3, 0, 5, 0, 3, -1)$$

by virtue of Proposition 4. In Proposition 3, we may check the condition only for $i = 10, 11$ by the inequality in (13), and $(i, j) = (11, 7)$ actually satisfies (13). And then $(i', j') = (1, 7)$, hence

$$2 \mid h_0, \quad \varepsilon_3 = \sqrt{\varepsilon_1 \eta_3}, \quad (s_3, t_3, u_3, v_3, w_3, x_3) = (11, 30, 15, 0, 1, 1).$$

This result is an expected one by Remark 5. Of course, these integers satisfy the equations in Remark 3.

Since we know $h_2 = h_3 = 1$, we have computed by (5) that

$$h = h_2 h_3 = 1, \quad h_0 = 2.$$

For the totally imaginary fields

$$\tilde{K} = \boldsymbol{Q}(\sqrt[6]{-2160}), \quad \tilde{K}_2 = \boldsymbol{Q}(\sqrt{-15}),$$

the insolubility of (17) is clear. Therefore Proposition 5 tells us that

$$\tilde{h}_0 = 1, \quad (\tilde{s}_0, \tilde{t}_0, \tilde{u}_0) = (-4, 15, -20),$$

and (15) shows that

$$\tilde{h} = \tilde{h}_2 h_3 = 2$$

since $\tilde{h}_2 = 2$. Moreover, from (19) and (21), follows

$$(\tilde{s}_3, \tilde{t}_3, \tilde{u}_3, \tilde{v}_3, \tilde{w}_3) = (6, 5, -10, 25, -4).$$

7.2. (i) In the same way as above, for

$$K = \boldsymbol{Q}(\sqrt[6]{756}), \quad K_2 = \boldsymbol{Q}(\sqrt{21}), \quad K_3 = \boldsymbol{Q}(\sqrt[3]{28}),$$

we can compute from the approximate value $\eta \sim 23.90624455$ that $s(\eta) = 20$, $t(\eta) = -69$, $u(\eta) = 580$ and that

$$h' = 3, \quad (s_1, t_1, u_1) = (-1, -6, 13).$$

Utilizing $\eta_2 = \frac{1}{2}(5+\sqrt{21})$, $(l, c) = (5, 1)$, $\eta_3 \sim 5.227871412$, $(y, z) = (5, -1)$, we obtain by Lemma 3 (i), Propositions 2, 4 that

$$3 \nmid h_0, \quad (s_2, t_2, u_2, v_2, w_2, x_2) = (8, -69, 148, -69, 8, 1),$$

and by Lemma 3 (ii) that $(y_+, z_+, y_-, z_-) = (17, 3, 5, 27)$. We find that $(i, j) = (5, -1)$ satisfies (13), so, putting $i' = j' = 3$, obtain by Proposition 3 that

$$2 \mid h_0, \quad (s_3, t_3, u_3, v_3, w_3, x_3) = (5, 4, -1, 4, 3, 1).$$

This is an example of the case $\alpha\gamma^{-1} \in \boldsymbol{Z}$ pointed out in Remark 3. Consequently

$$h = h_2 h_3 = 3, \quad h_0 = 2; \quad h_2 = 1, \ h_3 = 3.$$

For $\tilde{K} = \boldsymbol{Q}(\sqrt[6]{-28})$, $\tilde{K}_2 = \boldsymbol{Q}(\sqrt{-7})$, the results of computation are

$$(\tilde{s}_0, \tilde{t}_0, \tilde{u}_0) = (-8, 15, 20), \quad (\tilde{s}_3, \tilde{t}_3, \tilde{u}_3, \tilde{v}_3, \tilde{w}_3) = (-4, 9, 6, -3, -2),$$

$$\tilde{h} = \tilde{h}_2 h_3 = 3, \quad \tilde{h}_0 = 1; \quad \tilde{h}_2 = 1.$$

(ii) Let $K = \boldsymbol{Q}(\sqrt[6]{5})$, then $\eta \sim 161.1852493$ and $B(\eta) \sim 23.078$. In this case, for every prime $p \leqslant 23$, the divisibility test fails. Namely, from Lemma 1, Theorems 1 and 2, it follows that

$$h' = 1, \quad (s_1, t_1, u_1) = (141, -3090, 26345).$$

As is expected by Remark 5, indeed we have

$$h_0 = 6, \quad h = h_2 h_3 = 1.$$

More precisely, the conditions in Proposition 2 are satisfied by $(i, j, k) = (3, 1, 35)$ for $\xi = \varepsilon_1 \eta_2^{-1}$, and those in Proposition 3 hold for $(i, j, i', j') = (141, -93, 21, 147)$, hence we have

$$(s_2, t_2, u_2, v_2, w_2, x_2) = (3, 0, 35, 0, 3, -1),$$

$$(s_3, t_3, u_3, v_3, w_3, x_3) = (141, 30, 65, 150, 21, 1).$$

For $\tilde{K} = \boldsymbol{Q}(\sqrt[6]{-135})$, it follows from Remark 6 or Corollary 2 that

$$\tilde{h}_0 = 3, \quad \tilde{h} = \tilde{h}_2 h_3 = 2.$$

Really (17) is satisfied by $i = 6$ and

$$(\tilde{s}_0, \tilde{t}_0, \tilde{u}_0) = (6, 150, -250)$$

by Proposition 5. By (19) and (21), we obtain

$$(\tilde{s}_3, \tilde{t}_3, \tilde{u}_3, \tilde{v}_3, \tilde{w}_3) = (-114, 22695, -340, -105, 6).$$

(iii) Let $K_2 = Q(\sqrt{5})$ and K_3 be the maximal real subfield of the absolute class field of $F = Q(\sqrt{-23})$, i.e. $d_2 = 5$, $d_3 = 23$. Then $D = 66125$ and $K = K_2 K_3$ is the field with the smallest discriminant among those we consider. In the same way, we obtain $\eta \sim 760.7114218$, $B(\eta) \sim 245.593$ and $(s(\eta), t(\eta), u(\eta)) = (806, 35215, 580820)$. Since the conditions in Theorem 2 hold for $\xi = \eta$, $n = 2$ with $(i, i') = (38, 319)$, we have $h'' := h'/2 \in Z$. The indivisibility of h'' by any prime less than $B(\sqrt{\eta}) \sim 122.796$ is verified then. Thus we have

$$h' = 2, \quad (s_1, t_1, u_1) = (38, 319, 884).$$

Since (13) is not satisfied by any $i, j \in Z$, we see that

$$h_0 = 3, \quad h = h_2 h_3 = 1.$$

Other results are computed similarly as follows:

$$(s_2, t_2, u_2, v_2, w_2, x_2) = (4, 6, 3, -6, 4, -1),$$

$$(s_3, t_3, u_3, v_3, w_3, x_3) = (28, -290, 798, -103, 14, 1).$$

$$\tilde{h}_0 = 3, \quad \tilde{h} = \tilde{h}_2 h_3 = 2,$$

$$(\tilde{s}_0, \tilde{t}_0, \tilde{u}_0) = (-7, 34, -31),$$

$$(\tilde{s}_3, \tilde{t}_3, \tilde{u}_3, \tilde{v}_3, \tilde{w}_3) = (-306, 27417, 33998, 15550, -132).$$

(iv) Let K be the maximal real subfield of the absolute class field of $F = Q(\sqrt{-87})$. Then we get

$$h' = 1, \quad (s_1, t_1, u_1) = (1, -6, 17).$$

Since $d_2 = 29$, $d_3 = 87$, we should test whether η_2 is a cube in K by Proposition 4. Actually, we see $m = 61$, $n = 7$, $d_3' = 87$, $a = 261$ under the notation there, so $b = 1$ satisfies the equation for minus case, therefore

$$(s_2, t_2, u_2, v_2, w_2, x_2) = (0, 0, 5, 0, 0, -1).$$

Other results computed are as follows:

$$(s_3, t_3, u_3, v_3, w_3, x_3) = (4, 3, 0, -3, -1, 1),$$

$$h_0 = 6, \quad h = h_2 h_3 = 1.$$

$$\tilde{h}_0 = 3, \quad \tilde{h} = \tilde{h}_2 h_3 = 1,$$

$$(\tilde{s}_0, \tilde{t}_0, \tilde{u}_0) = (2, 6, 7), \quad (\tilde{s}_3, \tilde{t}_3, \tilde{u}_3, \tilde{v}_3, \tilde{w}_3) = (2, 7, -16, 14, -6).$$

The equation (17') is satisfied by $i = j = 2$ in this case.

7.3. In the following, we give a few numerical results for fields with small discriminants. Table 2 contains some pure sextic fields $K = Q(\sqrt[6]{n})$ and $\tilde{K} = Q(\sqrt[6]{-27n})$. Table 3 contains K and $\tilde{K}$ whose galois closure L is an unramified cyclic sextic extension over $F = Q(\sqrt{-d})$ with $d \leqslant 500$. Since

$$u_1 = s_1^2 + 2s_1 - 2t_1 + 2, \quad (v_2, w_2, x_2) = (ct_2, s_2, c), \quad x_3 = 1,$$

are always true, they are not listed in the tables. The class numbers and the fundamental units of K_3 are computed by the method in [2].

Errata for [1, III]

page	line	for	read
364	22	$2(s-t+2)$	$2(s-t+1)$
	31	$t(\xi)$	$t(\xi)-s(\xi)-3$
	33	$r_3(s, t)+t_0-3$	$r_3(s, t)+t-s$
365	27	if $\gamma\eta_2^2$ is	if $\gamma\eta_2$ or $\gamma\eta_2^2$ is
366	3	$\mathfrak{r}\mathfrak{f}$	$\mathfrak{r}^2\mathfrak{f}$
	3	$\mathfrak{r}^2\mathfrak{f}$	$\mathfrak{r}\mathfrak{f}$

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DEPARTMENT OF MATHEMATICS, TOKYO METROPOLITAN UNIVERSITY
2-1-1 Fukazawa, Setagaya, Tokyo, 158 Japan

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Table 2

n	s_1	t_1	h'	s_2	t_2	u_2	c	s_3	t_3	u_3	v_3	w_3
60	1	0	3	3	0	5	−1	11	30	15	0	1
108	42	375	1	0	3	16	1	12	15	38	21	6
756	−1	−6	3	8	−69	148	1	5	4	−1	2	3
5	141	−3090	1	3	0	35	−1	141	30	65	150	21
17	142	2735	1	10	2	12	−1	346	1392	1346	474	40
513	158	2751	1	32	300	550	1	42	−82	134	10	−10
2700	2	−9	3	16	−45	80	1	12	11	−22	1	2
2160	1666	59015	1	4	35	280	1	196	215	1630	5	−14
189	1581	−111810	1	15	−90	155	1	138	60	1658	282	24
1188	28	191	3	908	17615	100340	1	292	3821	−2510	521	−38

Table 3

d	s_1	t_1	h'	s_2	t_2	u_2	c	s_3	t_3	u_3	v_3	w_3
87	1	−6	1	0	0	5	−1	4	3	0	−3	−1
104	−1	−4	1	−1	2	1	−1	5	6	3	4	3
116	−1	8	1	3	2	1	−1	5	2	−5	0	3
152	2	−5	1	0	2	2	−1	6	−1	−4	7	−2
212	5	−2	1	2	−6	0	−1	11	12	1	4	−5
231	22	−193	1	4	14	25	1	16	41	28	−4	−2
231	−1	−9	1	7	33	65	1	5	1	−7	−1	3
231	25	162	1	0	0	9	1	11	15	24	15	4
244	1	−14	1	7	13	34	−1	15	40	25	0	−1
247	22	−65	1	−2	−2	13	−1	14	36	28	1	0
255	22	63	1	4	2	−3	−1	10	−24	12	9	−4
255	1	−3	1	3	3	7	−1	5	0	−3	6	−2
255	49	458	1	0	0	9	−1	14	−43	38	5	−7
327	81	−486	1	0	0	261	−1	25	−85	52	29	4
335	86	−65	1	0	−10	19	−1	20	16	82	−21	−4
339	10	−78	1	0	0	1552	−1	26	60	54	30	4
356	110	−2561	1	56	285	912	−1	74	−53	76	75	2
411	22	6	1	0	0	3488	−1	37	45	0	33	11
424	19	96	1	2	8	0	−1	45	190	207	88	15
436	7	−8	1	20	106	212	−1	39	124	−49	42	−11
440	23	124	1	3	2	5	−1	41	58	−61	40	−1
440	266	−9893	1	2	−3	60	−1	170	475	452	91	−22
440	2	−37	1	4	2	2	−1	30	71	−68	−1	6
451	30	−401	1	14	46	−44	−1	38	−8	2	38	0
472	0	−2	3	4	−6	10	−1	32	220	18	−14	−4

h_0	h_2	h_3	h	$\tilde{s}_0$	$\tilde{t}_0$	$\tilde{u}_0$	$\tilde{h}_0$	$\tilde{h}_2$	$\tilde{h}$	$\tilde{s}_3$	$\tilde{t}_3$	$\tilde{u}_3$	$\tilde{v}_3$	$\tilde{w}_3$
2	1	1	1	−4	15	−20	1	2	2	6	5	−10	25	−4
6	1	1	1	−6	39	−52	3	1	1	6	345	362	123	12
2	1	3	3	−8	15	20	1	1	3	−4	9	6	−3	−2
6	1	1	1	6	150	−250	3	2	2	−114	22695	−340	−105	6
6	1	1	1	−11	134	−167	3	2	2	380	41274	15082	1800	74
6	1	3	3	25	186	305	3	1	3	−86	6090	2034	222	−34
2	1	1	1	−14	135	−100	1	1	1	−14	45	30	15	−4
6	2	1	2	−14	1655	−3140	3	2	2	206	317945	52970	12875	−184
6	1	3	3	−78	1506	2918	3	1	3	24	227082	−111742	18960	138
2	1	1	1	−5	26	−35	3	1	3	−148	5603	−8428	3755	116

h_0	h_2	h_3	h	$\tilde{s}_0$	$\tilde{t}_0$	$\tilde{u}_0$	$\tilde{h}_0$	$\tilde{h}_2$	$\tilde{h}$	$\tilde{s}_3$	$\tilde{t}_3$	$\tilde{u}_3$	$\tilde{v}_3$	$\tilde{w}_3$
6	1	1	1	2	6	7	3	1	1	2	7	−16	14	6
6	1	1	1	0	2	−2	3	1	1	−2	7	4	−1	−2
6	1	1	1	2	4	2	3	1	1	−6	11	8	−5	−2
6	1	1	1	−1	4	−7	3	1	1	−8	31	−20	15	−4
6	1	1	1	−4	4	2	3	1	1	2	59	−68	27	−6
6	1	1	1	1	26	−47	3	1	1	38	356	−280	113	−16
6	1	1	1	1	3	−1	3	1	1	−7	12	9	−6	−2
6	1	1	1	8	30	43	3	1	1	6	164	152	49	8
6	1	1	1	0	4	−6	3	1	1	14	43	−60	43	−10
6	1	1	1	9	34	33	3	1	1	32	265	−120	76	−10
6	1	1	1	−7	18	1	3	2	2	−28	241	−16	32	−6
6	1	1	1	−1	3	−5	3	2	2	−6	20	−16	13	−4
6	2	1	2	10	56	91	3	1	1	−44	617	268	8	−14
6	1	1	1	6	96	155	3	1	1	−98	2480	−524	269	4
6	1	1	1	17	106	113	3	1	1	16	1699	−318	416	−40
6	1	1	1	2	18	22	3	1	1	21	374	−43	34	−7
6	1	1	1	14	127	−28	3	1	1	−98	9787	−2604	251	−10
6	1	1	1	5	30	49	3	1	1	−22	802	386	62	6
6	1	1	1	−4	18	−26	3	1	1	46	559	540	199	22
6	1	1	1	6	16	18	3	1	1	−6	323	−400	163	−10
6	1	1	1	−4	22	−34	3	2	2	−34	655	116	103	14
6	1	1	1	14	283	−340	3	4	4	404	50035	−14356	1483	−64
6	2	1	2	3	8	1	3	1	1	−28	191	204	47	−16
6	1	1	1	−11	22	57	3	1	1	−44	1574	−438	80	−10
2	1	1	1	−5	16	−15	1	3	3	7	8	−153	230	−19