Note on an index formula of elliptic units in a ring class field

by

Heima Hayashi (Kumamoto, Japan)

1. Introduction. Let $Q$ be the field of rational numbers, $Z$ the ring of rational integers and $\Sigma = Q(\sqrt{D})$ an imaginary quadratic number field with discriminant $D$. For a natural number $f$, we denote by $K_f$ the ring class field over $\Sigma$ with conductor $f$, by $O_f$ the order in $\Sigma$ with conductor $f$ ($O_1$ means the maximal order in $\Sigma$) and by $\text{Cl}(K_f/\Sigma)$ the ring ideal class group modulo $f$ in $\Sigma$, i.e. the group of the equivalent classes of proper $O_f$-ideals in $\Sigma$ by the usual equivalence relation. We denote by $\sigma : \text{Cl}(K_f/\Sigma) \to \text{Gal}(K_f/\Sigma)$ the isomorphism from $\text{Cl}(K_f/\Sigma)$ to the galois group of $K_f$ over $\Sigma$ via Artin's reciprocity law. For any intermediate field $K$ of $K_f/\Sigma$ with conductor $f$, we define $\text{Cl}(K_f/K)$ to be the subgroup of $\text{Cl}(K_f/\Sigma)$ whose image by $\sigma$ is equal to the galois group of $K_f$ over $K$, and $\text{Cl}(K/\Sigma)$ to be the factor group of $\text{Cl}(K_f/\Sigma)$ by $\text{Cl}(K_f/K)$.

Let $C$ be a class in $\text{Cl}(K_f/\Sigma)$ and let $a_f$ be an $O_f$-ideal in $C^{-1}$. Let $a$ be an element in $\Sigma$ such that $(a_fO_1)^h = (a)$ as an $O_1$-ideal, where $h$ is the class number of $\Sigma$. We define $\delta_{K_f}(C)$ by

$$\delta_{K_f}(C) = a^{12} \left( \frac{A(a_f)}{A(O_f)} \right)^h.$$ 

Herein $\Delta(\ )$ means the usual lattice function expressed by using the Dedekind eta-function as follows:

$$\Delta(m) = \left( \frac{2\pi}{\omega_1} \right)^{12} \eta \left( \frac{\omega_1}{\omega_2} \right)^{24},$$

where $m = [\omega_1, \omega_2]$ is a 2-dimensional complex lattice with $Z$-basis $\{\omega_1, \omega_2\}$, $\text{Im}(\omega_1/\omega_2) > 0$, $\delta_{K_f}(C)$ depends only on the class $C$, not on the choice of $a_f$, and is a unit in $K_f$. As is well known,

$$\delta_{K_f}(C^{m(C)}) = \delta_{K_f}(CC_1)/\delta_{K_f}(C_1)$$

for any $C, C_1$ in $\text{Cl}(K_f/\Sigma)$.

For each $c$ in $\text{Cl}(K/\Sigma)$ we define $\delta_K(c)$ as the relative norm of $\delta_{K_f}(C)$ w.r.t.
where $C$ is a class in $\text{Cl}(K_f; \Sigma)$ contained in $c$. Of course $\delta_K(c)$ depends only on $c$ and is a unit in $K$.

For typographical reasons, we shall sometimes write

$$\delta_K(c) = \delta_f(c),$$

if $K = K_f$.

For each $K$, we denote by $h_K$, $E_K$, $\mu_K$ and $w_K$ respectively the class number of $K$, the group of all units in $K$, the torsion part of $E_K$ and the number of elements in $\mu_K$. (When $K = \Sigma$, the subscript $K$ is omitted from these notations.) Let $\Delta_K$ be the subgroup of $E_K$ generated by $\mu_K$ and the $\delta_K(c)$'s. Then the following index formula holds (cf. [77]):

$$(E_K; \Delta_K) = (12w)^k n^{-1} \frac{w}{w_K} \prod_{\chi} a(\chi)^{h_K,h},$$

where $\chi$ ranges over all non-principal characters associated with the extension $K/\Sigma$, and for each $\chi$, $a(\chi)$ is given by

$$a(\chi) = \frac{\chi}{f} \sum_{p} \left( \frac{K_f; K_f}{p} \right) \prod_{v} \left( 1 - \chi(p) \right)^{h_K,h}.$$}

Herein $f_p$ means the conductor of $\chi$, $f_1$ ranges over all positive divisors of $f$ such that $f_f/f_1$, $p$ ranges over all prime ideals $(O_f, f)$ in $\Sigma$ containing $f_1$ and $np$ denotes the absolute norm of $p$. We note here that $a(\chi)$ is another expression of $\prod_{p} G(p^{n^{2k}}, \chi)$ in the paper of Schertz [6], and $a(\chi) = 1$ for any $\chi$ if $f = 1$.

Recently Kersey proved a refined index formula for the unramified case as follows ([E2]; we referred it to [3], ch. 9):

**Theorem 1.** In the case where $f = 1$, there exists a subgroup $\delta_K$ of $E_K$ such that $\delta_K^{2k} = \Delta_K$, for which the following index formula holds:

$$(E_K; \delta_K) = \left[ K; \Sigma \right] \frac{h_K}{h}.$$}

The most important part of Kersey’s proof is the proof of Proposition 1 ($\S$ 3) in the case where $f = 1$. He proved it by making use of the distribution of the Siegel function and many complicated tools including the quadratic relation of Klein forms ([E3], ch. 12).

Our main purpose in this note is to give another simple proof for Proposition 1. Our proof is based only on the classical facts in the theory of complex multiplication, and can be accomplished also in the case where $f \neq 1$. Moreover by following the same procedures as those of Kersey, we have the following:

**Theorem 2.** There exists a subgroup $\delta_K$ of $E_K$ such that $\delta_K^{2k} = \Delta_K$, for which the following index formula holds:

$$(E_K; \delta_K) = \frac{1}{e} \left[ K; \Sigma \right] \prod_{\chi} a(\chi)^{w_h} h_K,$$

where $e = 2$ or $1$ according to whether $w_K$ is divisible by $8$ or not.

**Remark 1.** In the cases where $D = -3, -4$ the assertion of Theorem 1 is trivial, and hence Theorem 2 is a generalization of Theorem 1.

2. Lemmas and terminologies. In this section we shall provide several lemmas. Lemmas 1 and 2 are classical facts in the theory of modular forms, and Lemmas 3 and 4 are fundamental results in the theory of complex multiplication.

Let the notations be the same as in Section 1. Let $\eta(t)$ and $j(t)$ respectively be the Dedekind eta-function and $j$-function in the theory of modular functions as usual. We denote by $Q(X, Y)$ the field of rational functions on $X$ and $Y$ with $Q$-coefficients.

**Lemma 1** (Weber [8]). Let $s$ be a natural number prime to $6$. Then

$$\left( \frac{\eta(s)}{\eta(t)} \right)^{2n} \in Q(j(t), j(s))$$

where $n = \begin{cases} 2 & \text{for } s \equiv 1 (\text{mod} 3), \\ 3 & \text{for } s \equiv 1 (\text{mod} 4), \\ 6 & \text{otherwise}. \end{cases}$

**Lemma 2** (Newman [4]). Let $s$ be a natural number and $\{r_n\}$ a family of rational integers, whose indices $n$ are positive divisors of $s$, satisfying the following conditions:

1. $r_1 = 0$,
2. $\sum_{n} (n-1) r_n = 0 \mod(24)$,
3. $\sum_{n} \left( \frac{2-s}{n} \right) r_n = 0 \mod(24)$ and
4. $\prod_{n} n^{r_n}$ is a square of a rational integer.

Then $g(t) = \prod_{n} \left( \frac{\eta(n)}{\eta(t)} \right)^{r_n}$ is contained in $Q(j(t), j(s))$.

By Satz (3,4) in [5], the statement in Lemma 1 can be translated into the statement on the special value ($\eta(s)/\eta(t)$), where $\omega$ is a number in $\Sigma$ such that $\Im(\omega) > 0$. In order to translate the function-theoretic result such as in Lemma 2 into the result on the special value of the same function, we need the following consideration:

Let the primitive matrix of determinant $s$ mean the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a, b, c, d \in Z$, $(a, b, c, d) = 1$ and $ad - bc = s$. 

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
For a primitive matrix \( S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) we denote \( \frac{ax + b}{cy + d} \) by \( S(t) \). Two primitive matrices \( S_1 \) and \( S_2 \) of the same determinant will be called equivalent if and only if there exists a unimodular matrix \( M \) such that \( S_1 = MS_2 \). As is well known, \( j(t) \) is algebraic over \( Q(j(t)) \) and its irreducible polynomial over \( Q(j(t)) \) is given by

\[
I_s(X, j(t)) = \prod_s (X - j(S(t))),
\]

where \( S \) ranges over a complete representative system of the inequivalent primitive matrices of determinant \( s \). Now for any complex number \( \omega \) such that \( \text{Im}(\omega) > 0 \) and for any function \( g(\tau) \) in \( Q(j(\tau), j(\tau)) \), the value \( g(\omega) \) is contained in \( Q(j(\omega), j(\omega)) \) whenever \( I_j(j(\omega), j(\omega)) \neq 0 \), where \( I_j(j(\omega), j(\omega)) \) means the differential quotient of \( I_j(X, j(\omega)) \) at \( X = j(\omega) \) ([1]). A necessary and sufficient condition for \( I_{s}(j(s\omega), j(\omega)) \neq 0 \) is that \( s\omega \) is not modular equivalent to \( s(\omega) \) for any primitive matrix \( S \) of determinant \( s \) such that \( S \) is not equivalent to \( \begin{bmatrix} s & 0 \\ 0 & 1 \end{bmatrix} \).

**Lemma 3.** Let \( C \) be a class in \( \text{Cl}(K/F) \), and let \( f \neq 1 \) when \( D = -3 \) or \(-4 \). Then for any prime number \( q \),

1. \( \delta_f(C)^{q^{+1}} = \delta_f(C)q^{n_C}N_{f/k}(\delta_{sf}(\overline{C})) \) if \( q \nmid a \) in \( \Sigma \) and \( q \nmid f \).
2. \( \delta_f(C)^{q^{+1}} = \delta_f(C)q^{n_C}N_{f/k}(\delta_{sf}(\overline{C})) \) if \( q \nmid a \) in \( \Sigma \) and \( q \nmid f \).
3. \( \delta_f(C)^{q^{+1}} = N_{f/k}(\delta_{sf}(\overline{C})) \) if \( q \nmid a \) in \( \Sigma \) and \( q \mid f \).
4. \( \delta_f(C)^{q^{+1}} = \delta_{sf}(\overline{C})N_{f/k}(\delta_{sf}(\overline{C})) \) if \( q = f \).

Herein \( C \) (resp. \( C_f \)) is the class in \( \text{Cl}(K/F) \) which contains the \( O_f \)-ideal \( q \cap O_f \) (resp. \( q \cap O_{f} \)), \( \overline{C} \) is any one class in \( \text{Cl}(K/F) \) such that \( \sigma(C) \) is an extension of \( \sigma(C) \) to \( \text{Gal}(K/F) \), \( \overline{C} \) is the class in \( \text{Cl}(K_{f+1}/F) \) such that \( \sigma(\overline{C}) \) is the restriction of \( \sigma(C) \) to \( \text{Gal}(K_{f+1}/F) \), and \( N_{f/k} \) means the norm map from \( K_f \) to \( K_f \).

**Proof.** A proof can be accomplished, using the following identity:

\[
q^{1/2} = \prod_q q^{1/2} \cdot \frac{\Delta(S)}{\Delta(S)} \cdot \frac{a_1}{a_2},
\]

where \( \{a_1, a_2\} \) is a \( Z \)-basis of a complex lattice, and \( S \) ranges a complete representative system of the inequivalent primitive matrices of determinant \( q \).

**Lemma 4.** For any \( C, C_1 \) in \( \text{Cl}(K/F) \), \( \delta_f(C_1)^{1-a(C)} \) is in \( E_k^{24} \).

**Proof.** From the definition, we have

\[
\delta_{sf}(C_1)^{a(C)} = \frac{A(a_1)}{A(O_f)},
\]

where \( a(C) \) is an \( O_f \)-ideal in \( C_{1}^{-1} \) and \( a \) is an element in \( \Sigma \) such that \( (a(C)) \) is a principal \( O_f \)-ideal. By a result of Deuring ([1], p. 41), \( A(a(C))/A(O_f) \) is contained in \( K^{24}_f \).

Let \( \mathfrak{U}_f \) be the multiplicative group of the fractional ideals in \( \Sigma \) relatively prime to \( f \). (Hereinafter ideal means \( O_f \)-ideal.) Let \( \mathfrak{U} \) be the subgroup of \( \mathfrak{U}_f \) associated with the extension \( K/F \) and \( \text{Cl}(K/F) \) the factor group of \( \mathfrak{U}_f \) by \( \mathfrak{U} \). Of course \( \text{Cl}(K/F) \cong \text{Cl}(K/F)^* \) and an explicit correspondence between \( \text{Cl}(K/F) \) and \( \text{Cl}(K/F)^* \) has been given in [1]. For each class \( C \) in \( \text{Cl}(K/F) \), we denote by \( c(C) \) the class in \( \text{Cl}(K/F)^* \) which corresponds to \( C \). Now since \( \Sigma(\mu_k) \) is an abelian extension over \( \mathfrak{U} \), \( \Sigma(\mu_k)/\mathfrak{U} \) is an extension of type \( (2, 2, \ldots) \) (cf. [1]), and hence \( w_K \) must be at most a divisor of 24. Moreover for any ideal \( a \) in \( \Sigma \) prime to \( w_f \), the value of the absolute norm \( Na \bmod w_f \) depends only on the class \( c \) to which \( a \) belongs.

**Definition.** An ideal \( a \) in \( \Sigma \) will be said to be \( K \)-admissible if it is prime to \( w_f \) and satisfies the following conditions:

- If \( w_K = 2 \), then \( Na \equiv 1 \bmod 24 \).
- If \( w_K = 4 \), then
  \[
  Na \equiv \begin{cases} 1 \bmod 24 & \text{for } \Sigma = \mathfrak{U}(\sqrt{-4}), \\ 1 \text{ or } 7 \bmod 24 & \text{otherwise} \end{cases}
  \]
- If \( w_K = 6 \), then
  \[
  Na \equiv \begin{cases} 1 \bmod 24 & \text{for } \Sigma = \mathfrak{U}(\sqrt{-3}), \\ 1 \text{ or } 17 \bmod 24 & \text{otherwise} \end{cases}
  \]
- If \( w_K = 8 \), then
  \[
  Na \equiv \begin{cases} 1 \text{ or } 13 \bmod 24 & \text{for } \Sigma = \mathfrak{U}(\sqrt{-4}), \\ 1, 7, 13 \text{ or } 19 \bmod 24 & \text{otherwise} \end{cases}
  \]
- If \( w_K = 12 \), then
  \[
  Na \equiv \begin{cases} 1 \text{ or } 17 \bmod 24 & \text{for } \Sigma = \mathfrak{U}(\sqrt{-4}), \\ 1 \text{ or } 7 \bmod 24 & \text{for } \Sigma = \mathfrak{U}(\sqrt{-3}), \\ 1, 7, 17 \text{ or } 23 \bmod 24 & \text{otherwise} \end{cases}
  \]
- If \( w_K = 24 \), then
  \[
  Na \equiv \begin{cases} 1, 5, 13 \text{ or } 17 \bmod 24 & \text{for } \Sigma = \mathfrak{U}(\sqrt{-4}), \\ 1, 7, 13 \text{ or } 19 \bmod 24 & \text{for } \Sigma = \mathfrak{U}(\sqrt{-3}), \\ 1, 5, 7, 11, 13, 17, 19 \text{ or } 23 \bmod 24 & \text{otherwise} \end{cases}
  \]
For any K-admissible ideal a the value of Na(mod 24) depends only on the class e to which a belongs. In each class of Cl(K/Σ)* there are infinitely many K-admissible prime ideals of degree 1. We define a homomorphism \( l: Cl(K/Σ) → (Z/24Z)^* \) by \( l(c) = Na \mod 24 \), where a is a K-admissible ideal in the associated class e. Plainly the kernel of l contains Cl(K/Σ)^2.

Remark 2. Our definition of K-admissibility is essentially equal to that of Kersey, but has been slightly modified.

3. Proof of main result. Let the notations be the same as in Sections 1 and 2. In this section we shall prove the following:

Proposition 1. Let \( c_1 \) and \( c_2 \) be any two classes in Cl(K/Σ) and \( n \) the least positive rational integer such that

\[ n(\ell(c_1) - 1)(\ell(c_2) - 1) \equiv 0 \mod 24. \]

Then

\[ \left( \frac{\delta_k(c_1,c_2)}{\delta_k(c_1)\delta_k(c_2)} \right)^n \in E_k^{24n}. \]

Proof. It suffices to prove this only for the case where \( K = K_f \). Since in the cases where \( (D,f) = (-3,1), (-3,2), (-3,3), (-4,1) \) or \( (-4,2) \) the assertion is trivial, we shall exclude these cases throughout this proof.

Let \( p_1 \) and \( p_2 \) be \( K_f \)-admissible prime ideals of degree 1 in \( C_f^{\ast -1} \) and \( C_f^{\ast -1} \) respectively, and let \( \ell = Np_i \) (i = 1, 2). Then \( n \) is determined by the congruence \( n(p_1-1)(p_2-1) \equiv 0 \mod 24 \). Let \( v \) be a rational integer such that

\[ p_1, \frac{\sqrt{D+v}}{2} = p_1 \quad \text{and} \quad \frac{\sqrt{D+v}}{2} = p_2. \]

Indeed \( v \) is determined by the congruence \( v^2 \equiv D \mod 4p_2p_2 \). For our later arguments, we will choose \( v \) as \( v^2 \equiv D \mod 4p_1p_2 \). Clearly \( \frac{1}{2} + \frac{\sqrt{D+v}}{2} = O_f \), and the following three \( O_f \)-

\[ \left[ \frac{p_1, f \sqrt{D+v}}{2} \right], \left[ \frac{p_2, f \sqrt{D+v}}{2} \right] \quad \text{and} \quad \left[ \frac{p_1p_2, f \sqrt{D+v}}{2} \right] \]

represent the classes \( C_1^{\ast -1} \), \( C_2^{\ast -1} \) and \( (C_1 C_2)^{-1} \) respectively ([1]). From the definition we have

\[ \frac{\delta_C(c_1,c_2)}{\delta_C(c_1)\delta_C(c_2)} = \left( \frac{\eta(f(\sqrt{D+v})}{\eta(\sqrt{D+v})} \right)^{24n} \left( \frac{\eta(\sqrt{D+v})}{\eta(\sqrt{D+v})} \right)^{24n} \]

where \( \omega = \frac{f(\sqrt{D+v})}{2p_1p_2} \). Let \( \eta(p_1,p_2,v) = \eta(p_1p_2,\omega) / \eta(p_1,\omega) \eta(p_2,\omega) \). By using Lemma 1, it can be easily confirmed that

\[ \eta(p_1,p_2,v)^{2n} \in K_f. \]
Herein, \( C_0, C_1 \) and \( C_2 \) are the same as in Lemma 3. By the formula (3.1) the left-hand side of the equation (3.3) is contained in \( K_3^{24} \). Since there exists a class \( B \) of order \( q \) in \( \text{Cl}(K_\ell/\Sigma) \), \( \{\delta_f(C_1, C_2), \delta_f(C_1, C_2), \delta_f(C_2, C_2)\}^n \) is contained in \( K_4^{24} \), and hence \( \{\delta_f(C_1, C_2), \delta_f(C_1, C_2), \delta_f(C_2, C_2)\}^n \) is also contained in \( K_7^{24} \).

Now we know the following facts:

(I) When \( D = -4 \) and \( f \) is divisible by an odd prime number other than \( \{3, 7, 17, 31\} \) or by \( 16 \), then there exists a class \( B \) in \( \text{Cl}(K_\ell/\Sigma) \) whose order is divisible by an odd prime number or by \( 16 \). Especially when \( (D, f) = (-4, 17) \) or \( (-4, 31) \), then \( w_k = w_4 = 4 \), and there exists a class \( B \) of order 8 in \( \text{Cl}(K_\ell/\Sigma) \).

(II) When \( D = -8 \) and \( f \) is divisible by an odd prime number other than \( \{3, 7\} \) or by \( 16 \), then there exists a class \( B \) in \( \text{Cl}(K_\ell/\Sigma) \) whose order is divisible by an odd prime number or by \( 16 \). Especially when \( (D, f) = (8, 7) \), then \( w_k = w_2 = 2 \) and \( \text{Cl}(K_\ell/\Sigma) \) is a cyclic group of order 8.

Therefore for the complete proof of Proposition 1 we have only to check the finite number of cases listed up in Table I. In Table I, TYPE indicates the type of abelian group \( \text{Cl}(K_\ell/\Sigma) \). It is not so difficult to verify that the assertion of Proposition 1 is true for all cases in Table I, but this is very tedious. We shall here omit the precise verification and point out only a few essential facts.

Verification for all cases except for the three cases where \( (D, f) = (-4, 12), (-8, 6) \) and \( (-8, 12) \) can be carried out only by using the tools employed in Steps 1 and 2. Verification for the cases where \( (D, f) = (-4, 12) \) and \( (-8, 6) \) can be done by using (3) of Lemma 3. Verification for the case where \( (D, f) = (-8, 6) \) can be done by numerical computations. Therein we may use the following representative \( O_{\ell, k} \)-ideals:

\[
(1, 6\sqrt{-2}], [2, 3\sqrt{-2} + 1], [3, 2\sqrt{-2}] \quad \text{and} \quad [6, \sqrt{-2} + 1].
\]

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Remark 3. In the unramified case, \( w_k \) is at most a divisor of 12, and \( w_k > 2 \) if and only if \( D = -3 \) or \(-4 \). Hence for any pair \((c_1, c_2)\) of the
classes in $\text{Cl}(K/\mathbb{Q})$, $n((c_1)-1)((c_2)-1)$ is divisible by 24 if and only if it is divisible by $w_K$, and hence Proposition 1 is a generalization of Kersey’s result ([3], ch. 11).

Let $\Delta_K(c_0)$ be the subgroup of $\Delta_K$ generated by $\mu_K$ and all units of the following form:

$$\left(\frac{c_1 c_2}{\delta_k(c_1) \delta_k(c_2)}\right)^n,$$

with a rational integer $n$ such that $n((c_1)-1)((c_2)-1) \equiv 0 \pmod{24}$. Then by Proposition 1, $\Delta_K(c_0)$ is contained in $\mu_K E_2^{24k}$. Let $n_0$ be the least positive rational integer such that $n_0((c_1)-1)((c_2)-1) \equiv 0 \pmod{24}$ for any pair $(c_1, c_2)$ of the classes in $\text{Cl}(K/\mathbb{Q})$. As can be easily confirmed, $n_0 = 1, 2, 3$ or 6, and $n_0 = \frac{1}{2}(w_k/w)$ or $(w_k/w)$ according to whether $w_k$ is divisible by 8 or not. Moreover by following a procedure similar to one of Kersey ([3], ch. 9, 5), we have $(\Delta_K; \Delta_K(c_0)) = n_0[K; 2]$, and hence Theorem 2.

References


DEPARTMENT OF MATHEMATICS
KYUSHU-TOUKAI UNIVERSITY
223-1, Oroshi, Kitamoto-cho, Kitamoto 862 Japan

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