ACTA ARITHMETICA
XLV (1985)

On the method of exponent pairs

by

G. KOLESNIK (Los Angeles, Cal.)

1. Introduction. The method of exponent pairs (henceforth abbreviated to m.e.p.) was introduced by van der Corput [2] for estimating certain one-dimensional exponential sums. We write, as is customary, $e(x)$ for $\exp(2\pi ix)$. The sums in question have the form

$$\sum_{X<x<X+Y} e(f(x)),$$

where $Y \leqslant X$ and $f(x)$ behaves rather like a power of x (not a positive integer power). We shall introduce notation below to make explicit the similarity required between $f(x)$ and the power of x (for one- and several-dimensional functions $f(x)$).

As is well known, such exponential sums are of great importance in analytic number theory. Let us suppose that $f(x)$ is similar to $FX^{-\alpha}x^{\alpha}$, then, in the refined form given by E. Phillips, the m.e.p. leads to an estimate of the form $\Big|\sum_{X\leqslant x<X+Y} e(f(x))\Big| \ll F^k X^l$, where (k, l) is an exponent pair (e.p.). (We have modified Phillips' definition slightly for the purpose of this paper; in his definition $f' \cong F$.) It is trivial to see that $(0, 1)$ is an e.p. All other e.p. are produced by two processes A and B. Process A is essentially an application of Cauchy–Schwarz inequality, and process B is essentially the combination of the Poisson summation formula with the method of stationary phase. Rankin [6] proved that $\inf_{(k,l)\in E}(2k+l) = 0.829021356...$ (in fact, $(2k+l)$ corresponds to $k+l$ in his definition), where E denotes the set of all e.p. obtained by the one dimensional m.e.p. Consequently the one dimensional m.e.p., when applied to the problem of bounding $\zeta(\frac{1}{2}+it)$, cannot give a better result than $\zeta(\frac{1}{2}+it) \ll t^{\theta}$ for $\theta = 0.16451067...$ As Rankin observed, this is inferior to what may be proved by using two dimensional exponential sums. Srinivasan [7] developed a m.e.p. for the estimation of many dimensional exponential sums $\sum_{x\in D} e(f(x))$ where $D \in \mathscr{D}$, $f \in \mathscr{F}$ (see Section 2), but he fails to take full advantage of the Cauchy–Schwarz inequality, and so it leads

to weaker results than one would like. The m.e.p. gives the same estimate for all $f \in \mathscr{F}$, $D \in \mathscr{D}$:

$$S \equiv \Big|\sum_{x \in \mathscr{D}} e(f(x))\Big| \ll F^{\lambda_0} X_1^{\lambda_1} \ldots X_n^{\lambda_n}.$$

The vector $\lambda = (\lambda_0, \ldots, \lambda_n)$ is called an *exponent* $(n+1)$*-tuple*, or we write $\lambda \in E_n$ (note that the above definition differs from the corresponding definitions in [5]–[7]). As in the one dimensional case, m.e.p. consists of repeated applications of processes $A \in \mathscr{A}$ and $B \in \mathscr{B}$, defined respectively by Lemmas 2 and 1 (in [5]–[7] Lemma 2 is used with $k = 1$ only). After applying processes A and B several times and estimating the last sum trivially, we obtain an exponent pair (which depends on the number of times we apply each of processes A and B).

The transformation B is applied whenever $F^{\varepsilon_0 - 1/3} \ll A_k^{-1} \cong H_k(f(x)) \ll 1$ and the remainder $R = N^{1+\varepsilon}\sqrt{F^{k-1} A_k N_k^{-2}}$ in Lemma 1 is smaller than the principal sum. If we could ignore the error term, then, denoting by $S(F; X)$ the best estimate of the exponential sum $\big|\sum_{x \in D} e(f(x))\big|$ which can be obtained by m.e.p. for $f \in \mathscr{F}(F, X, \Delta)$, we could in such a case write Lemma 1 in the following form:

(1) $\quad S(F; X) \lll X_1 \ldots X_k F^{-k/2} S(F; F/X_1, \ldots, F/X_k, X_{k+1}, \ldots, X_n)$

for any $k \in [1, n]$.

If we estimate the last sum trivially, then we get

(2) $\quad S(F; X) \lll N/F^k (X_1 \ldots X_k)^{-2} \cong N|H_k(f(x))|^{1/2}$

which gives a non-trivial estimate for $S(F; X)$ if $|H_k(f(x))|$ is small. If $|H_k(f(x))|$ is not small, one applies a transformation A sufficiently many times to make it small. Lemma 2 reduces the estimation of S to the estimation of another exponential sum, $\big|\sum_{x,h} e(g(x, h))\big|$, where $g(x, h) = h_1 f_{x_1}(x) + \ldots + h_m f_{x_m}(x) + g_1(x, h)$, where $g_1(x, h)$ is usually small so that we have

$$|H_l(g(x, h))| \ll |H_l(f(x))|\, \varrho^l, \quad \text{where} \quad \varrho = |h_1/X_1| + \ldots + |h_n/X_n|.$$

If we take $m = 1$ (or $A \in \mathscr{A}_1$), then $g(x, h_1) \in \mathscr{F}(Fh_1/X_1; X, \Delta + h_1/X_1)$; also, after applying Lemma 1 to the last sum we get a new sum with a function, say, $\varphi(x, h_1)$ such that for $h_1 \in [H_1, 2H_1]$, $\varphi \in \mathscr{F}(FH_1/X_1; X, H_1, \Delta + H_1/X_1)$. So, we can write transformation A for $A \in \mathscr{A}_1$ as follows:

(3) $$|S(F; X)|^2 \lll N^2/q + Nq^{-1} \max_{1 \leqslant H \leqslant q} S(FH/X_1; X, H) = N^2/q + Nq^{-1} S(Fq/X_1; X; q)$$

for any $q \leqslant X_1$. If $m > 1$, then $g(x, h) \notin \mathscr{F}$. Also, while applying Lemma 2 with $m > 1$ allows us to reduce the values of $|H_l(g(x, h))|$ much faster than if we take $m = 1$ and in a generic case $|H_l(g(x, h))| \cong |H_l(f(x))|\, \varrho^l$, it can be too small at some points of $\mathscr{D}_1$ and we cannot use Lemma 1. Suppose, however, that $\mathscr{F}_0 \subset \mathscr{F}$, $\mathscr{A}_0 \subset \mathscr{A}$, $\mathscr{B}_0 \subset \mathscr{B}$ are such that for all $f \in \mathscr{F}_0$, $A \in \mathscr{A}_0$, $B \in \mathscr{B}_0$ the inequality (1) holds and suppose that we can ignore the points at which $H_l(g(x, h))$ is very small so that the proceses A can be written as

(4) $\quad S(F; X)^2 \ll N^2 Q^{-1} + NQ^{-1} S(F\varrho, X, Q),$

where $\varrho = q_1/X_1 = \ldots = q_m/X_m = \sqrt[m]{Q/N_m}$, $Q = q_1 \ldots q_m$, $N_m = X_1 \ldots X_m$, $1 \leqslant m \leqslant m(n, \mathscr{A}_0) \leqslant n$. One can show that the conjecture that both (1) and (4) are true is equivalent to one conjecture:

$$S(F; X_1, \ldots, X_n) \ll S(F; X_1) \ldots S(F; X_n).$$

In particular, (1) and (4) hold if we take $\mathscr{A}_0 = \mathscr{A}_1$, $\mathscr{B}_0 = \mathscr{B}$, $\mathscr{F}_0 = \mathscr{F}$. Using (1) and (3), one can develop an improvement of the m.e.p. of [7] for $A \in \mathscr{A}_1$. One can, however, obtain a better estimate by using our Theorems 1–3:

THEOREM 1. *Let k_1, k_2 be positive integers with $1 \leqslant k_1, k_2 \leqslant 3$, and let α, β be real numbers such that $\alpha, \beta \neq$ integers, $\alpha + \beta \neq k_1$, $\alpha + \beta \neq k_1 + 1$, $(\alpha - k_1 - 1)^2 + \beta(\alpha - k_1) \neq 0$, $\alpha + \beta \neq k_1 + 1 + 1/k_2$. Let Δ be small and let $f(x, y)$ be a real C^∞ function in*

$$D \subset \{(x, y)|\ X \leqslant x \leqslant 2X,\ Y \leqslant y \leqslant 2Y\}$$

such that

$$f(x, y) \underset{\Delta}{\sim} FX^{-\alpha} Y^{-\beta} x^\alpha y^\beta, \quad X \geqslant Y,\ XY = N.$$

Then

$$S \equiv \Big|\sum_{(x,y) \in \mathscr{D}} e(f(x, y))\Big| \lll N/\sqrt{Q_1},$$

where Q_1 is the largest number satisfying the following inequalities:

(5)

$$Q_1 \leqslant \min\{(N^{2k_1K_1 + 4K_1 - k_1k_2 - 2k_1 - k_2 - 4} F^{2k_2 + 4 - 4K_1})^{1/(4K_1K_2 - 2K_2 - k_2K_1 - 3K_1 + k_2 + 2)};\ (N^{3k_1K_2 + 6K_2 - 2k_1 - 2} F^{4 - 6K_2})^{1/(6K_1K_2 - 2K_1 - 3K_2 + 2)}\},$$

where $K_1 = 2^{k_1}$, $K_2 = 2^{k_2}$; also for $k_1 = 3$,

$$\sqrt[4]{N\Delta^2} \leqslant Q_1 \leqslant \min\{(F^2 N^{-3})^{11/115};\ (N^5 F^{-2})^{1/13};\ (N^{47} F^{-18})^{1/151}\};$$

for $k_1 = 2$,

$$\sqrt{N\Delta^2} \leqslant Q_1 \leqslant \min\{(FN^{-1})^{2/5};\ (N^2 F^{-1})^{9/29}\};$$

for $k_1 = 1$, $Q_1 \leqslant \min\{\Delta^{-1}; N^{2/7}\}$.

Note that the restrictions $1 \leqslant k_1,\ k_2 \leqslant 3$ and $(\alpha-k_1-1)^2+\beta(\alpha-k_1) \neq 0$ can be removed and are introduced to simplify the proof. Also, it seems that using the idea of the proof of the theorem one can develop the method of exponent pairs for double and possibly multiple sums. Note that $O(N^{\varepsilon})$ can be replaced with $O(\log^4 N)$. Using Theorem 1, one can similarly to [4] show that $\zeta(\frac{1}{2}+it) \lll t^{139/858}$, $\Delta(R) \ll R^{139/429}$, and many other constants can be improved.

THEOREM 2. *Let* $\alpha,\ \beta,\ \gamma$ *be real numbers,* $\alpha+\beta+\gamma \neq 1,\ \alpha+\beta+\gamma \neq 2$, $\alpha\beta\gamma(\alpha-1)(\beta-1)(\gamma-1) \neq 0$, $\Delta \leqslant \varepsilon_0$, *and let* $f(x, y, z)$ *be a real* C^{∞} *function such that* $f(x, y, z) \underset{\Delta}{\sim} FX^{-\alpha}Y^{-\beta}Z^{-\gamma}x^{\alpha}y^{\beta}z^{\gamma}$ *throughout*

$$D \subset \{(x, y, z)|\ X \leqslant x \leqslant 2X,\ Y \leqslant y \leqslant 2Y,\ Z \leqslant z \leqslant 2Z\},$$
$$X \geqslant Y \geqslant Z, \qquad XYZ = N.$$

Then

$$S \equiv \Big|\sum_{(x,y,z)\in\mathscr{D}} e(f(x, y, z))\Big|$$
$$\ll N[FN^{-1}+Z^{-1}+N^{-1/4}+\sqrt[5]{F^3X^4N^{-5}}+\sqrt[8]{F^6\Delta^3X^6N^{-8}}+$$
$$+\sqrt[6]{\Delta Y^{-2}Z^{-2}}+F^{-1/2}+Y^{-2/5}Z^{-2/5}+\sqrt[8]{F^{-1}Y^{-2}Z^{-2}}]^{1/2}\log^4 N.$$

THEOREM 3. *Let* $\alpha_1, \ldots, \alpha_n$ *be real numbers such that* $\alpha_1\ldots\alpha_n \neq 0$, $\alpha_1+\ldots+\alpha_n \neq 2$, $\alpha_1+\alpha_2+\alpha_3 \neq 2$, $\alpha_{j_1}+\ldots+\alpha_{j_i} \neq 1$ *for any* $\{j_1, \ldots, j_i\} \subset \{1, \ldots, n\}$. *Let* $n \geqslant 4$, $\Delta \leqslant \varepsilon_0$, $F \geqslant X_1^2$, $X_1 \geqslant X_2 \geqslant \ldots \geqslant X_n$, $X_1X_2\ldots X_n = N$, *and let* $f(x)$ *be a real* C^{∞} *function such that for* $x \in D \subset \{x|\ X_i \leqslant x_i \leqslant 2X_i,\ i = 1, \ldots, n\}$

$$f(x) \underset{\Delta}{\sim} FX_1^{-\alpha_1}\ldots X_n^{-\alpha_n}x_1^{\alpha_1}\ldots x_n^{\alpha_n}.$$

Then

$$S \equiv \Big|\sum_{x\in D} e(f(x))\Big| \ll N^{1+\varepsilon}[(F^{3n}X_1^{-n}X_2^{-n}X_3^{-n}N^{-6})^{1/(n+6)}+$$
$$+X_3^{-1}\sqrt[19]{F^3X_1^{-1}X_2^{-1}}+X_2^{-1}X_3^{-1}\sqrt[23]{F^{15}/X_1^5}+$$
$$+X_3^{-1}\sqrt[7]{F^3\Delta^6X_1^{-1}X_2^{-1}}+X_2^{-1}X_3^{-1}\sqrt[8]{F^6\Delta^3X_1^{-2}}+N^{-1/(n+1)}+$$
$$+X_3^{-3/4}+\sqrt{\Delta/X_3}]^{1/2}\log^2 N.$$

Theorems 1–3 are applied in the case when using (3) once we reduce the value of $H(f(x))$ so that (2) gives a non-trivial estimate of $S(F;\ X)$. If the determinant is still "large", then one can apply (3) several times and the corresponding Theorem 1, 2 or 3 after that. We can improve the above theorems under the restriction that we apply the transformations A and $B \leqslant K$ times. Then similarly to [4] one can show that there exists a polynomial P, depending on K only, such that for all $f \in \mathscr{F}$ with $P(\alpha) \neq 0$ one can use (1) and (4) $\leqslant K$ times.

Unfortunately, it is difficult to find P explicitly. We think however (see Section 4) that (1) and (4) hold for all $f \in \mathscr{F}$ and all m. Assuming this, one can develop a general m.e.p.

THEOREM 4. *Let* $m \in [1, n]$, $(\lambda_0, \lambda_1, \ldots, \lambda_n) \in E_n$. *Then* $(\tilde{\lambda}_0, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) \in E_n$, *where* $\tilde{\lambda}_0 = \lambda_0+\lambda_1+\ldots+\lambda_m-m/2$, $\tilde{\lambda}_j = 1-\lambda_j$ $(j = 1, \ldots, m)$, $\tilde{\lambda}_j = \lambda_j$ $(j = m+1, \ldots, n)$.

THEOREM 5. *Let* $\lambda_j \geqslant 0$ $(j = 0, \ldots, n)$, $m \in [1, n]$, $\lambda_j^* \geqslant 0$ $(j = 1, \ldots, m)$, $\lambda_0+|\lambda^*| \equiv \lambda_0+\lambda_1^*+\ldots+\lambda_m^* \geqslant m$, $(\lambda_0, \lambda_1, \ldots, \lambda_n, \lambda_1^*, \ldots, \lambda_m^*) \in E_{m+n}$. *Then* $(\tilde{\lambda}_0, \tilde{\lambda}_1, \ldots, \tilde{\lambda}_n) \in E_n$, *where* $\tilde{\lambda}_0 = \lambda_0 m/2(\lambda_0+|\lambda_n^*|)$, $\tilde{\lambda}_j = (\lambda_j+\lambda_j^*-1)m/2(\lambda_0+|\lambda^*|)+ +1/2$ *for* $j = 1, \ldots, m$, $\tilde{\lambda}_j = (\lambda_j-1)m/2(\lambda_0+|\lambda^*|)+1$ *for* $j = m+1, \ldots, n$.

While these theorems are conditional upon the truth of (1) and (4) for all $f \in \mathscr{F}$, they allow to estimate the upper bound of the estimates one can hope to obtain by using the general m.e.p.

Using the conditional Theorems 5 and 6, one can write a program so that a computer can find various possible exponent pairs (the algorithm is due to Enrico Bombieri). Indeed, we start with e.p. $(0|1, \ldots, 1)$ and apply Theorem 4, then we apply Theorem 5 several times, etc.

Ending the process at any time, we obtain an exponent pair. Another algorithm allows to estimate the limits of the m.e.p. Suppose, we know the sizes of $F,\ X_1, \ldots, X_n$ depending on some parameter T. We apply (1) with some k such that $X_1^2\ldots X_k^2 \geqslant F^k$. If such k do not exist, then we take $Q = N^2T^{-2\alpha}$ and if $Q \geqslant N$, then it is impossible to prove that $S(F;\ X) \ll T^{\alpha}$ by m.e.p. Otherwise we apply (4), etc. Using the algorithm, we discovered that one cannot show that $\zeta(\frac{1}{2}+it) \lll t^{0.1618}$ by using the m.e.p. only. Under the condition that Theorems 5 and 6 are true, we found that

$$\left(\frac{6966}{117930}\Big|\frac{99727}{117930},\ \frac{99727}{117930}\right) \in E_2,$$

$$\left(\frac{47728}{274910}\Big|\frac{1}{2}\right) \in E_1, \quad \text{and} \quad \zeta(\tfrac{1}{2}+it) \lll t^{47728/274910},$$

where $47728/274910 = 0.16183920$. For comparison, the best published unconditional result is $\zeta(\frac{1}{2}+it) \lll t^{35/216}$, [4], where $35/216 = 0.162037\ldots$; Theorem 1 of this paper implies a little better constant: $\zeta(\frac{1}{2}+it) \lll t^{139/858}$, where $139/858 = 0.162004\ldots$ Similar calculations can be done for other problems.

The author would like to express his gratitude to Professor Enrico Bombieri for generously sharing his ideas.

2. Notation. Throughout the paper we suppose that all considered domains $D, D_1, \ldots$ belong to the set $\mathscr{D} = \mathscr{D}(x)$ of all subdomains of $\{x|\ X_i \leqslant x_i \leqslant 2X_i,\ i = 1, \ldots, n\}$ such that the boundary of each of them consists of $\leqslant C$ algebraic curves of degrees $\leqslant C$, $x=(x_1, \ldots, x_n)$, $i=(i_1, \ldots, i_n)$, etc. $X_1, \ldots, X_n$ are sufficiently large numbers such that $X_1 \geqslant X_2 \geqslant \ldots \geqslant X_n$, $X_1 \ldots X_n = N$; $X_1 \ldots X_m = N_m$.

$f(x) \ll g(x)$ means that $f(x) = O(g(x))$.

$f(x) \lll g(x)$ means that $f(x) \ll N^\varepsilon g(x)$.

$f(x) \cong g(x)$ means that $f(x) \ll g(x) \ll f(x)$.

$f(x) \approx g(x)$ means that $f(x) - g(x) = o(g(x))$.

$$f^i(x) = \frac{\partial^{i_1+\ldots+i_n} f(x)}{\partial x_1^{i_1} \ldots \partial x_n^{i_n}}.$$

$f(x) \underset{\Delta}{\sim} g(x)$ means that $f^i(x) = g^i(x) + O(\Delta g^i(x))$ for all x and i for which it makes sense.

$\mathscr{F} \equiv \mathscr{F}(F; X; \Delta)$ is the set of all sufficiently many times differentiable real functions $f(x)$ in D such that $f(x) \underset{\Delta}{\sim} F X_1^{-\alpha_1} \ldots X_n^{-\alpha_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, where $\alpha_{j_1} + \ldots + \alpha_{j_i} \neq$ integer for any $\{j_1, \ldots, j_i\} \subset \{1, \ldots, n\}$, $X_1 \lll F$.

$R(P_1(x_1, \ldots, x_k), \ldots, P_k(x_1, \ldots, x_k))$ — the resultant of polynomials $P_1, \ldots, P_k$.

$H_j(f(x))$ — the determinant of the jth principal minor of the Hessian.

$H(f(x)) = H_n(f(x))$.

$C, C_1, \ldots$ — appropriate constants.

$D(m; f) = \{(y_1, \ldots, y_m, x_{m+1}, \ldots, x_n) |\ y_j = f'_{x_j}(x),\ j = 1, \ldots, m;\ x \in \mathscr{D}\}$.

$D(B_{11}, B_{12}, \ldots, B_{n1}, B_{n2}) = \{x |\ x \in D,\ B_{j1} \leqslant x_j \leqslant B_{j2},\ j = 1, \ldots, n\}$.

$e(f) = \exp(2\pi i f)$.

3. Transformation B. Transformation B is a combination of Poisson summation formula (or some variation) with the method of stationary phase:

LEMMA 1. *Let $f(x)$ be a real C^∞ function such that for all $x \in \mathscr{D}$ we have:*

(i) *for all integral $a_{i_l} \in [-c, c]$ the functions $\sum_i \prod_{l \leqslant c} a_{i_l} f^{i_l}(x)$ do not change sign, where $i = (i_1, \ldots, i_c)$, $i_l = (i_{1l}, \ldots, i_{nl})$, $i_{jl} \leqslant c$.*

(ii) $|f^i(x)| \ll F X_1^{-i_1} \ldots X_n^{-i_n}$ *for some F.*

(iii) $|H_k(f(x))| \cong (A_k)^{-1}$ *for some $k \in [1, n]$ and some $A_k \leqslant (X_1 \ldots X_k)^2 \times F^{1/6-\varepsilon-k}$. Then there exist real numbers B_{ij} such that for $D_1 = D_2(B_{11}, B_{12}, \ldots, B_{n1}, B_{n2})$, $D_2 = D_3(k; f)$, $D_3 = D(B_{13}, B_{14}, \ldots, B_{n3}, B_{n4})$ we have*

$$S \equiv \Big|\sum_{x \in D} e(f(x))\Big| \ll \sqrt{A_k} \Big| \sum_{(m, x_{k+1}, \ldots, x_n) \in D_1} e(f_1(m, x_{k+1}, \ldots, x_n))\Big| + O(N^\varepsilon) + O(N^{1+\varepsilon} \sqrt{F^{-k-1} A_k N_k^{-2}})$$
$$\ll |\mathscr{D}| A_k^{-1/2} + N^{1+\varepsilon} \sqrt{F^{k-1} A_k N_k^{-2}} + N^\varepsilon + N X_k^{-1} A_k^{-1/2} + X_{k+1} \ldots X_n (F/X_2 + 1) \ldots (F/X_k + 1) \sqrt{A_k},$$

where $m = (m_1, \ldots, m_j)$, $f_1(m, x_{k+1}, \ldots, x_n) = f(\varphi_1, \ldots, \varphi_k, x_{k+1}, \ldots, x_n) - \varphi_1 m_1 - \ldots - \varphi_k m_k$, $\varphi_1, \varphi_2, \ldots, \varphi_k$ are the solutions of the system $f'_{x_j}(\varphi_1, \ldots, \varphi_k, x_{k+1}, \ldots, x_n) = m_j$ $(j = 1, 2, \ldots, k)$.

Proof. We suppose that $k = n$, the general case can be proved similarly. Using the Poisson summation formula, we write

$$S = \sum_{m_1, \ldots, m_n} \int_D e(f(x) - m \cdot x)\, dx + O(N^\varepsilon),$$

where the last sum is over an n-dimensional rectangle containing $D(n; f)$. Denoting

$$\varphi(x) \equiv \varphi_m(x) = F^{-1}(f(x) - m \cdot x)$$

and

$$g(x) = \int_{R^n} \chi_D(x_1 - y_1 \delta_1, \ldots, x_n - y_n \delta_n) U(y)\, dy,$$

where $\delta_j = X_j \sqrt{F^{n-1} A_n N^{-2}}$, $U(y) = U_1(y_1) \ldots U_1(y_n)$, $\chi_D(x)$ is the characteristic function of D,

$$U_1(y) = \begin{cases} c^{-1} e^{(y^2-1)^{-1}} & \text{if } |y| < 1, \\ 0 & \text{if } |y| \geqslant 1, \end{cases}$$

$$c = \int_{-1}^{1} e^{(y^2-1)^{-1}}\, dy,$$

we obtain:

$$S = \sum_m \int_{R^n} g(x) e(F\varphi(x))\, dx + O(N^\varepsilon) + \sum_x (\chi_D(x) - g(x)) e(f(x))$$
$$= \sum_m \int_{R^n} g(x) e(F\varphi(x))\, dx + O(N^{1+\varepsilon} \sqrt{F^{k-1} A_k N_k^{-2}}) + O(N^\varepsilon).$$

Denoting

$$g_1(x) = g(X_1 x_1, \ldots, X_n x_n), \quad \varphi_1(x) = \varphi(x_1 X_1, \ldots, X_n x_n),$$

we obtain:

$$S = N \sum_m \int_{R^n} g_1(x) e(F\varphi_1(x))\, dx + O(N^{1+\varepsilon} \sqrt{F^{k-1} A_k N_k^{-2}}). \tag{6}$$

Now we use (3.17) of [1] to find an asymptotic expansion for the last integral:

$$I(F) \equiv \int_{R^n} g_1(x) e(F\varphi_1(x))\, dx$$
$$= F^{-n/2} |H(\varphi_1(x^0))^{-1/2} e(n/8 - \operatorname{Ind} H(x^0)/4 + F\varphi_1(x^0)).$$

$$\{g_1(x^0) + \sum_{k=1}^{K} (k!)^{-1} [(-i/2F \langle H^{-1}(x^0) D_x, D_x \rangle)^k \times (g_1(x) e(Fh(x)))]_{x=x^0}\} + e(F\varphi_1(x^0)) P_k(F, x^0) \tag{7}$$

where $P_k(F, x^0)$ is defined by (3.30), (3.27), (3.28), (3.22) of [1] (the expression for $P_k(F;x^0)$ is too long; since it will be shown to be small later, we do not include it here); $x^0=(x_1^0,\ldots,x_n^0)$ is a stationary point of $\varphi_1(x)$;

$$\langle H^{-1}(x^0)D_x, D_x\rangle\times B(x)=\sum_{i,j}h'_{ij}B_{x_ix_j}(x);$$

h'_{ij} is the (i,j) entry of the inverse matrix for the Hessian $H(x)$ of $\varphi_1(x)$ at $x=x^0$; Ind $H(x)$ is the number of negative eigenvalues of $H(x)$;

$$h(x)=\varphi_1(x)-\varphi_1(x^0)-\tfrac{1}{2}\sum_{i,j}\big(\varphi_{1x_ix_j}(x^0)(x_i-x_i^0)(x_j-x_j^0)\big).$$

For $|i|\leqslant 4n/\varepsilon_0=2K$ we have

$$\left|\frac{\partial^{|i|}}{\partial x_i^{i_1}\ldots\partial x_n^{i_n}}\big(g_1(x)e(Fh(x))\big)\right|_{x=x^0}\ll F^{|i|/3}+\delta_1^{-i_1}X_1^{i_1}\ldots\delta_n^{-i_n}X_n^{i_n};$$

also, if for all $x\in D$

$$|H(f(x))|\cong\delta F^nN^{-2}\quad\text{with}\quad\delta=F^{-n}N^2A_n^{-1}>F^{\varepsilon_0-1/3},$$

then

$$h'_{ij}\ll 1/\delta\ll F^{1/3-\varepsilon_0}$$

so that each term in (7) is piecewise monotone as a function of each variable and bounded by an absolute constant; as in the proof of (3.40) of [1], one can show that

$$P_K(F,x^0)\ll(\delta F^{-1/3})^{-K-1}\leqslant F^{-\varepsilon_0(K+1)}\ll F^{-n}.$$

Using this together with (6) and (7) and applying Abel's summation formula, we obtain

$$S\ll N\delta^{-1/2}F^{-n/2}\Big|\sum_{m\in\mathscr{D}_2}e(f_1(m))\Big|+N^{1+\varepsilon}\sqrt{F^{n-1}A_nN^{-2}},$$

where $f_1(m)=F\varphi_1(x^0)=f(x(m))-m\cdot x(m)$ and $x(m)$ is the solution of the system $\nabla f(x)=m$, where ∇ is the gradient vector.

This lemma is in applications better than Lemma 1 of [3], because the remainder contains only one A_K, while the remainder in Lemma 1 of [3] contains $A_1,\ldots,A_K$. However, the error term is too large in some cases and needs to be improved. If

$$f(x)\underset{\Delta}{\sim}Ax_1^{\alpha_1}\ldots x_n^{\alpha_n},$$

where

$$A=FX_1^{-\alpha_1}\ldots X_n^{-\alpha_n},$$

then

$$f_1(m)\underset{\Delta}{\sim}Bm_1^{\beta_1}\ldots m_n^{\beta_n},$$

where

$$B=(1-\alpha_1-\ldots-\alpha_n)(\alpha_1^{\alpha_1}\ldots\alpha_n^{\alpha_n}/A)^{1/(\alpha_1+\ldots+\alpha_n-1)},$$

$$\beta_j=\alpha_j/(\alpha_1+\ldots+\alpha_n-1)\quad(j=1,\ldots,n),$$

$$m_j\cong F/X_j=M_j\quad(j=1,\ldots,n),\quad f_1(m)\cong F.$$

So, if $f\in\mathscr{F}(F;X,\Delta)$, then $f_1\in\mathscr{F}(F;M,\Delta)$, and Lemma 1 implies that

$$S(F;X)\ll N_kF^{-k/2}S(F;F/X_k,\ldots,F/X_1,X_{k+1},\ldots,X_n)+NF^{-1/2}.$$

While this inequality differs from (1) by a summand $NF^{-1/2}$, in the applications this term is usually smaller than the principal term.

It may also be possible to improve it. Indeed, the error term comes from the estimate for $R=\big|\sum_x g_1(x)e(f(x))\big|$, where $g_1(x)=\chi_D(x)-g(x)$, $g_1(x)\neq 0$ in a domain with the volume $\ll\sqrt{A_nF^{n-1}}$, $0\leqslant g_1(x)\leqslant 1$. In the case where D is such that it can be divided into $\leqslant C$ subdomains in each of which

$$\int_{-1}^{1}\ldots\int_{-1}^{1}\frac{\partial^n}{\partial t_1\ldots\partial t_n}\big(g_1(x_1+t_1,\ldots,x_n+t_n)\big)dt$$

does not change sign (this is true if D is a parallelpiped; using some devices, one can very often reduce the problem of estimating S to such case), then we can apply Abel's summation formula, and obtain

$$R\ll\Big|\sum_{x\in D_0}e(f(x))\Big|,\quad\text{where}\quad|D_0|\leqslant\sqrt{A_nF^{n-1}},$$

and it is possible to estimate it non-trivially. Because the last sum is of the same type as S but over a much smaller domain, it is natural to expect $R\leqslant\varepsilon_0 S$. In such case we get (1). If $A_k\gg N_k^2F^{1/3-\varepsilon_0-k}$ (this does not happen if $f\in\mathscr{F}$), then one needs to estimate the sum differently. One expects that it is possible to find a subsum $\big|\sum_{x_{i_1},\ldots,x_{i_m}}e(f(x))\big|$ such that $H_{i_1,\ldots,i_m}(f(x))$ is not small. Applying Lemma 1 to the above subsum, we can find a nontrivial estimate of S.

4. Transformation A.

LEMMA 2. *Let $f(x)$ be a real function;*

$$f_1(x,h)=\int_0^1\frac{\partial}{\partial t}\big(f(x+th)\big)dt.$$

Let $q_1,\ldots,q_m$ be positive numbers such that

$$q_1/X_1=\ldots=q_m/X_m=\sqrt[m]{Q/N_m},\quad\textit{where}\quad Q=q_1\ldots q_m,\ N_m=X_1\ldots X_m;$$

$$D_1=\{(x,h)|\ x\in D,\ (x+h)\in D,\ h\neq 0,\ |h_j|\leqslant q_j\};$$

$$|D|\geqslant\sum_{x\in D}1;\quad q_1/X_1\leqslant|D|/N.$$

Then

$$|S|^2 \equiv \Big|\sum_{x\in D} e(f(x))\Big|^2 \ll |D|^2 Q^{-1} + |D|\, Q^{-1} \Big|\sum_{(x,h)\in D_1} e(f_1(h, h))\Big|.$$

In applications the terms in the above sum with some of $h_j = 0$ are easier to estimate so that we will ignore them in the future.

While we do not know how to prove our conjecture (4) at this moment, we can try to "justify" it. If we apply Lemma 2 k times, we obtain

$$|S|^{2^k} \ll |D|^{2^k} Q^{-2^{k-1}} + |D|^{2^k-1} Q^{1-2^k} \Big|\sum_{(x,h)\in D_2} e(f_2(x, h))\Big|,$$

where

$$h = (h^{(1)}, \ldots, h^{(k)}), \quad h^{(j)} = (h_{1j}, \ldots, h_{m_j,j}) \neq 0, \quad |h_{ij}| \leqslant q_{ij},$$
$$q_{1j}\ldots q_{m_j,j} = Q_j = Q^{2^{j-1}} \quad (j = 1, \ldots, k),$$
$$D_2 = \{(x, h) |\ (x + h^{(1)} + \ldots + h^{(j)}) \in D \text{ for } j = 0, \ldots, k\},$$
$$f_2(x, h) = \int_0^1 \ldots \int_0^1 \frac{\partial^k}{\partial t_1 \ldots \partial t_k} \big(f(x + t_1 h^{(1)} + \ldots + t_k h^{(k)})\big)\, dt \ll F\varrho_1 \ldots \varrho_k \equiv F_2,$$

where

$$\varrho_j = \frac{|h_{1j}|}{X_1} + \ldots + \frac{|h_{m_j,j}|}{X_j}.$$

We divide D_2 into $O(N^\varepsilon)$ subdomains of the type:

$$D_3 = \{(x, h) |\ H_{ij} \leqslant h_{ij} \leqslant (1+\varepsilon_0) H_{ij},\ X_i \leqslant x_i \leqslant (1+\varepsilon_0) X_i,$$
$$H_n(f_2(x, h)) \cong A_n^{-1} = \delta F_2^n N^{-2},\ (x, h) \in D_2\}.$$

If $\delta \geqslant F_2^{\varepsilon - 1/3}$, then we use Lemma 1; otherwise, in a generic case, $|D_3| \ll \delta N \prod_{i,j} H_{ij}$ and there is a j such that $\frac{\partial^2}{\partial x_j^2}(f_2(x, h)) \cong F_2 X_j^{-2}$, and we can use Lemma 1 with $n = 1$. In both cases we obtain

$$S_1 \equiv \Big|\sum_{(x,h)\in D_3} e(f_2(x, h))\Big| \ll N F_2^{-n/2} \delta^{-1/2} \Big|\sum_{(m,h)\in D_4} e(f_3(m, h))\Big| +$$
$$+ (N^{1+\varepsilon} F_2^{-1/2} + N^{1+\varepsilon} F_2^{-1/3} X_n^{-2/3}) \prod_{i,j} H_{ij},$$

where $D_4 \subset D_3(n, f_2)$, $|D_4| \ll \delta N \prod_{i,j} H_{ij}$. So,

$$|S|^{2^k} \ll N^{2^k} Q^{-2^{k-1}} + N^{2^k} Q^{1-2^k} \max_{H_{ij}} \max_{\delta \geqslant F_2^{\varepsilon - 1/3}} \delta^{-1/2} F_2^{-n/2} \Big|\sum_{m,h} e(f_3(m, h))\Big| +$$
$$+ N^{1+\varepsilon} F_2^{-1/2} + N^{1+\varepsilon} F_2^{-1/3} X_n^{-2/3}.$$

The remainder in the above inequality is usually smaller than the first term, and it can also be improved. If we can ignore the remainder, then

$$|S|^{2^k} \ll N^{2^k} Q^{-2^{k-1}} + N^{2^k} Q^{1-2^k} \max_{H_{ij}} \max_{\delta \geqslant F_2^{\varepsilon - 1/3}} \delta^{-1/2} F_2^{-n/2} \Big|\sum_{m,h} e(f_3(m, h))\Big|.$$

We can apply this to the last sum, etc. We can see that the maximum is attained when $\delta = 1$ and that we get similar inequalities as if (taking $k = 1$) we had $f_2(x, h) \in \mathscr{F}(F\varrho;\ X, H)$ with $H = (H_1, \ldots, H_m)$, $H_j \leqslant q_j$, $\varrho = H_1/X_1 + \ldots + H_m/X_m$. This gives some justification to our conjecture that

$$|S(F;\ X)|^2 \lll N^2 Q^{-1} + NQ^{-1} \max_{H_j \leqslant q_j} S(F\varrho;\ X, H).$$

Because the entries of any exponent n-tuple are non-negative, the maximum is attained at $H_j = q_j$, and the last inequality is the same as (4).

5. Estimates. In this section we will prove Theorems 1 and 3 and, assuming (1) and (4), Theorems 4 and 5. To prove Theorem 1, we need the following results:

LEMMA A. *Let $f(x, y)$ be a real C^∞ function such that for any c, $f_{x^i y^j} = c$ has $O(N^\varepsilon)$ solutions; $|f_{x^i y^j}| \ll_{i,j} F X^{-i} Y^{-j}$ for some F and all $(x, y) \in \mathscr{D} \subset \{(x, y) |\ X \leqslant x \leqslant 2X,\ Y \leqslant y \leqslant 2Y\}$; $|f_{x^2}| \cong M_1^{-1} \geqslant F^{2/3+\varepsilon_0} X^{-2}$; $|f_{x^2} f_{y^2} - (f_{xy})^2| \cong M_2^{-1}$; $N = XY$. Then*

$$\Big|\sum_{(x,y)\in\mathscr{D}} e(f(x, y))\Big| \lll \mathscr{D}/\sqrt{M_2} + Y\sqrt{M_1} + F\sqrt{M_2}/X.$$

LEMMA B. *Let $f(x, y)$ be a real C^∞ function such that $|f_{x^k}| \cong aM^{-1} X^{-2}$, $|f_{x^{k-2} y^2}| \cong bM^{-1} Y^{-2}$ for some M and all $(x, y) \in \mathscr{D} \subset \{(x, y) |\ X \leqslant x \leqslant 2X,\ Y \leqslant y \leqslant 2Y\}$; $K = 2^k$, $k \geqslant 2$. Then*

$$\Big|\sum_{(x,y)\in\mathscr{D}} e(f(x, y))\Big| \lll \min\{|\mathscr{D}|(aM^{-1} X^{-2})^{1/K-2} + |\mathscr{D}|^{1-2/K} Y^{2/K}(k-2) +$$
$$+ |\mathscr{D}|^{1-(2k/K)} N^{4/K} (M/a)^{2/K};\ |\mathscr{D}|(bM^{-1} Y^{-2})^{1/(K-2)} +$$
$$+ |\mathscr{D}|^{1-(2/K)} X^{2/K}(k-2) + |\mathscr{D}|^{1-(2k/K)} N^{4/K} (M/b)^{2/K}\}.$$

Lemma A is obtained by applying Lemma 1 to the sum over x and using van der Corput's estimate to the sum over y after that. Lemma B can be proved by induction on k and applying Lemma 2 with $m = 1$ (for $k = 2$, Lemma B follows from van der Corput's estimate).

We denote $K = 2^{k_1}$, $K_1 = 2^{k_2}$, $F_1 = F\sqrt{Q_1^{k-1} N^{-k_1}}$. If $Y \leqslant N^{3/7}$ and $k_1 = 1$, then we use Lemma B with $k = 2$ and $k = 3$ and get:

$$S \lll N \cdot \min\{(F^7 N^{-8})^{1/14} + F^{-1/2};\ (F^7 N^{-12})^{1/42} + N^{-1/7} + F^{-1/4}\}$$

which proves Theorem 1 in the considered case. If $k_1 \geqslant 2$ and Y^{2k_1-2}

$\ll N^{2k_1+1}F^{-2}Q^{2-5K_1/2}$, then we apply Lemma 2 (k_1-1) times with $n=1$ and Q_1 as in (5) and, using Theorem 1 of [3] (which, after some obvious changes connected with application of Lemma 1 of this paper can be stated as follows:

$$S \lll \sqrt[6]{F^2N^3}+N^{5/6}+\sqrt[19]{\Delta^4F^2N^9X^{-4}}),$$

we obtain:

$$(S/N)^{K/2} \lll Q_1^{-K/4}+N^{-1}Q_1^{1-K/2}\sum_h\Big|\sum_{x,y}e\big(f_0(x,y)\big)\Big|$$
$$\ll Q_1^{-K/4}+N^{-1}\big(\sqrt[6]{F^2Q_1^{K-2}X^{2-2k_1}N^3}+N^{5/6}+$$
$$+\sqrt[19]{\Delta^4F^2Q_1^{K-2}X^{-2k_1-2}N^9}\big) \ll Q_1^{-K/4}.$$

In the above we have written $f_0(x, y)$ for the integral

$$\int_0^1\ldots\int_0^1\frac{\partial^{k_1-1}}{\partial t_1\ldots\partial t_{k_1-1}}\big(f(x+h_1t_1+\ldots+h_{k_1-1}t_{k_1-1}\,y)\big)\,dt.$$

The obtained estimates prove Theorem 1 in the considered cases. Now we assume $N^{3/7}\leqslant Y\leqslant\sqrt{N}$ and $Y^{2k_1-2}\geqslant N^{2k_1+1}F^{-2}Q_1^{2-5K_1/2}$ for $k_1\geqslant 2$, and apply Lemma 2 with $n=2$ and Q_1 as in (5).

We obtain:

$$(S/N)^2 \ll Q_1^{-1}+N^{-1}Q_1^{-1}\sum_{h_1,h_2}\Big|\sum_{x,y}e\big(f_1(x,y)\big)\Big|,$$

where

$$f_1(x,y)=\int_0^1\frac{\partial}{\partial t}\big(f(x+h_1t,\ y+h_2t)\big)\,dt.$$

For a fixed (h_1, h_2) with $h_1h_2\neq 0$ (otherwise the sum is easier to estimate and the estimate is smaller) we divide the domain D into subdomains with

$$\delta_0a_1\leqslant|C_{30}\theta+C_{21}\sigma_{11}|\leqslant 2\delta_0a_1,$$
$$\delta_0b_1\leqslant|C_{12}\theta+C_{03}\sigma_{11}|\leqslant 2\delta_0b_1,$$
$$\delta_0^2c_1\leqslant|(C_{30}\theta+C_{21}\sigma_{11})(C_{12}\theta+C_{03}\sigma_{11})-(C_{21}\theta+C_{12}\sigma_{11})^2|\leqslant 2\delta_0^2c_1,$$

where $\theta=y/x$, $\sigma_{11}=h_2/h_1$, $\delta_0=y/x+|\sigma_{11}|$, $C_{ij}=\dfrac{\partial^{i+j}}{\partial x^i\,\partial y^j}(x^\alpha y^\beta)|_{x=y=1}$, and one domain which contains all remaining ($O(\sqrt{N})$, say) points. We assume that a subdomain D_1, corresponding to the largest subsum, is defined by the above inequalities, and consider the following distinct cases (in each case we assume that the conditions of all previous cases are not satisfied):

I. $c_1\ll Q_1^{-1}$. Then we have

$$|D_1|\ll\sqrt{N}+Nc_1\ll N/Q_1,\quad\text{and}\quad S_1\ll N/\sqrt{Q_1}.$$

II. $c_1\ll Q_1^{-1}[N^{k_1+2}F^{-2}(X/Y)^{k_1-1}]^{1/(3K_1-3)}$.

We take $Q_2=[N^{k_1+2}F^{-2}(X/Y)^{k_1-1}]^{2/(3K_1-3)}$ and apply Lemma 2 (k_1-1) times with $n=1$:

$$(S/N)^{K_1}\ll Q_1^{-K/2}+\Big(N^{-1}Q_1^{-1}\sum_h|D_1|\Big)^{K_1/2}Q_2^{-K_1/4}+$$
$$+\Big(N^{-1}Q_1^{-1}\sum_h|D_1|\Big)^{K_1/2-1}N^{-1}Q_1^{-1}Q_2^{1-K_1/2}\sum_h\Big|\sum_{x,y}e\big(f_2(x,y)\big)\Big|,$$

where $h=(h_1,h_2,\ldots,h_{k_1+1})$, $|h_3|\leqslant Q_2,\ldots,|h_{k_1+1}|\leqslant Q_2^{K_1/4}$. Here $|D_1|\ll Nc_1+\sqrt{N}\ll Nc_1$; also since $C_{30}C_{03}\neq C_{21}C_{12}$, $\max\{a_1;b_1\}\geqslant\varepsilon_0$. To estimate the last sum, we use Lemma A and get:

$$(S/N)^{K_1}\ll Q_1^{-K_1/2}+c_1^{K_1/2}\sqrt{F^2Q_1c_1Q_2^{K_1-2}N^{-k_1-2}(X/Y)^{k_1-1}}+$$
$$+c_1^{K_1/2-1}[F^2Q_1Q_2^{K_1-2}N^{-k_1}(X/Y)^{k_1-1}]^{-1/4}+c_1^{(K_1-3)/2}Y^{-1}\ll Q_1^{-K_1/2}.$$

III. $c_1\gg Q_1^{-1}[N^{k_1+2}F^{-2}(X/Y)^{k_1-1}]^{1/(3K_1-3)}$.

If $k_1=1$, then we continue as in 7) below. Otherwise we apply Lemma 2 with $n=2$ and $Q=Q_1^2$:

$$(S/N)^4\ll Q_1^{-2}+N^{-1}Q_1^{-3}\sum_h\Big|\sum_{x,y}e\big(f_3(x,y)\big)\Big|,$$

where

$$f_3(x,y)=\int_0^1\int_0^1\frac{\partial^2}{\partial t_1\,\partial t_2}f(x+h_1t_1+h_3t_2,\ y+h_2t_1+h_4t_2)\,dt_1\,dt_2.$$

Denoting

$$\sigma_{12}=h_2/h_1+h_4/h_3,\quad \sigma_{22}=h_2h_4/h_1h_3,\quad \delta_1=(y/x+|h_2/h_1|)(y/x+|h_4/h_3|),$$

we divide D_1 into subdomains with

$$\delta_1a_2\leqslant|C_{40}\theta^2+C_{31}\sigma_{12}\theta+C_{22}\sigma_{22}|\equiv|P_1(\theta)|\leqslant 2\delta_1a_2,$$
$$\delta_1b_2\leqslant|C_{22}\theta^2+C_{13}\sigma_{12}\theta+C_{04}\sigma_{22}|\equiv|P_2(\theta)|\ll 2\delta_1b_2,$$
$$\delta_1^2c_2\ll|P_1(\theta)P_2(\theta)-(C_{31}\theta^2+C_{22}\sigma_{12}\theta+C_{13}\sigma_{22})^2|\equiv|P_0(\theta)|\ll 2\delta_1^2c_2$$

and one domain which contains $O(\sqrt{N})$ integral points. We suppose that the domain D_2, defined by the above inequalities, corresponds to the largest subsum, and consider the following distinct cases:

A. $c_2 \ll \min\{(N^2 Y^4 F^{-2} Q_1^{-11})^{1/4}; (N^2 X^4 F^{-2} Q_1^{-11})^{9/20}\}$ if $k_1 = 2$ and

$c_2 \ll \min\{(N^2 F^{-2} Q_1^{-27} X^6)^{3/20}; (YQ_1^{-8})^{1/3}\}$ if $k_1 = 3$.

For a fixed (x, y, h_1, h_2) we divide D_2 into subdomains with $P_0'(\theta) \cong a\delta_1^{3/2}$; in such subdomain $\theta = \theta_0(h) + O(c_2/a)$; also, finding the resultant $R(\theta)$ of $P_0(\theta)$ and $P_0'(\theta)$ as functions of h_4/h_3, we get $|R(\theta)| \ll (a+c_2)\delta_1^5$, where $R(\theta)$ is divisible by θ^2 so that $\theta = \theta_0(h_2/h_1) + O(\sqrt[8]{a+c_2})$; also, $h_4/h_3 = \varphi(h_2/h_1, \theta) + O(\sqrt{c_2})$, and we obtain

$$N^{-1} Q_1^{-3} \sum_h |\mathscr{D}_2| \lll \max_a \min\{c_2/a; \sqrt[8]{a+c_2}\,(\sqrt{c_2+Q^{-1}})\}$$
$$\ll c_2^{5/9} + Q_1^{-1} \min\{1; \sqrt[9]{c_2 Q_1}\} \equiv \varrho.$$

Also, since $P_1(\theta)$ and $P_2(\theta)$ do not have common divisors, and, for $(\alpha-k_1-1)^2 + \beta(\alpha-k_1) \neq 0$, $P_1(\theta)$, $P_0(\theta)$ and $P_0'(\theta)$ do not have common divisors, we get $\max\{a_2; b_2\} \gg 1$ and, if a_2 is small, then $N^{-1} Q_1^{-3} \sum_h |\mathscr{D}_2| \ll c_2$.

Applying Lemma B with $k = k_1$ and $M^{-1} = FX^{2-k_1}\sqrt{Q_1^3 N^{-2}}$, we obtain:

$$(S/N)^4 \ll Q_1^{-2} + \varrho(F^2 Q_1^3 N^{-2} X^{-2k_1})^{1/(2K_1-4)} +$$
$$+ \varrho^{1-2/K_1} X^{-2/K_1} |k_1 - 2| + \varrho^{1-4/K_1} X^{2/K_1 - 1/2} (N^2 X^{2k_1-4} F^{-2} Q_1^{-3})^{1/K_1} +$$
$$+ c_2^{1-2/K_1} Y^{-2/K_1} |k_1 - 2| + c_2 (F^2 Q_1^3 N^{-2} X^{4-2k_1} Y^{-4})^{1/(2K_1-4)} \ll Q_1^{-2}.$$

B. $c_2 \ll (N^4 X^{2k_1-4} F^{-2} Q_1^{5-3K_1})^{18/(15K_1-22)}$.

As in A, $Q_1^{-3} \sum_h |D_2| \ll N\varrho$ and, if $a_2 \ll 1$, $Q_1^{-3} \sum_h |D_2| \ll Nc_2$; applying Lemma 2 (k_1-2) times with $n = 1$ and appropriate Q, we obtain:

$$(\Sigma_1)^{K_1/4} \equiv \Big(N^{-1} Q_1^{-3} \sum_h \Big|\sum_{x,y} e\big(f_3(x, y)\big)\Big|\Big)^{K_1/4}$$
$$\lll \varrho^{K_1/4} Q_2^{-K_1/8} + \varrho^{K_1/4-1} Q_2^{1-K_1/4} \sum_h \sum_{h^{(1)}} \Big|\sum_{x,y} e\big(f_4(x, y)\big)\Big|,$$

where

$$Q_2 = (N^4 F^{-2} Q_1^{3/5} X^{2k_1-4})^{20/(15K_1-22)};$$

$$\sum_{h^{(1)}} = \sum_{h_1^{(1)}=1}^{Q_2} \ldots \sum_{h_{k_1-2}^{(1)}=1}^{Q_2^{K_1/8}};$$

$$f_4(x, y) = \int_0^1 \ldots \int_0^1 \frac{\partial^{k_1-2}}{\partial t_1 \ldots \partial t_{k_1-2}} \big(f_3(x + h_1^{(1)} t_1 + \ldots + h_{k_1-2}^{(1)} t_{k_1-2}, y)\big)\,dt.$$

Now we apply Lemma 1 and get

$$(\Sigma_1)^{K_1/4} \ll Q_1^{-K_1/2} + \varrho^{K_1/4-1} [(F^2 Q_1^3 Q_2^{K_1/2-2} c_2 N^{-2} X^{4-2k_1})^{1/2} + 1/X\sqrt{c_2}] +$$
$$+ c_2^{(K_1/4)-3/2} Y^{-1} \ll Q_1^{-K_1/2}.$$

C. $c_2 \gg (N^4 F^{-2} Q_1^{5-3K_1} X^{2k_1-4})^{18/(15K_1-22)}$.

If $k_1 = 2$, we continue as in 7) below. If $k_1 = 3$, we apply Lemma 2 once more with $n = 2$ and $Q = Q_1^4$ and get

$$(S/N)^8 \ll Q_1^{-4} + N^{-1} Q_1^{-4} \sum_h \Big|\sum_{x,y} e\big(f_5(x, y)\big)\Big|,$$

where

$$f_5(x, y) = \int_0^1 \int_0^1 \int_0^1 \frac{\partial^3}{\partial t_1\, \partial t_2\, \partial t_3} \big(f(x + h_1 t_1 + h_3 t_2 + h_5 t_3,\ y + h_2 t_1 + h_4 t_2 + h_6 t_3)\big)\,dt.$$

We denote

$$\varphi_{ij}(\theta) = C_{i+3,j}\theta^3 + C_{i+2,j+1}\theta^2\sigma_1 + C_{i+1,j+2}\theta\sigma_2 + C_{i,j+3}\sigma_3,$$

where σ_1, σ_2, σ_3 are the elementary symmetric functions of h_2/h_1, h_4/h_3, h_6/h_5;

$$\varphi_1(\theta) = \varphi_{20}(\theta)\varphi_{02}(\theta) - (\varphi_{11}(\theta))^2; \qquad \delta_2 = \prod_{j=1}^{3} (Y/X + |h_{2j}/h_{2j+1}|).$$

For a fixed $h = (h_1, \ldots, h_6)$ we divide the domain D_2 into one subdomain containing $O(\sqrt{N})$ integral points and domains defined by inequalities of the type

$$\delta_2 a_3 \ll |\varphi_{20}(\theta)| \ll 2\delta_2 a_3, \quad \delta_2 b_3 \ll |\varphi_{02}(\theta)| \ll 2\delta_2 b_3, \quad \delta_2^2 c_3 \ll |\varphi_1(\theta)| \ll 2c_3\delta_2^2.$$

We suppose that a D_3, corresponding to the largest subsum, is defined by the above inequalities, and consider the following cases:

1) $\sqrt[4]{Q^4/N} + Q_1^8 Y^{-2} \ll c_3 \ll N^5 Q_1^{-15} F^{-2}$, $a_3 \ll \sqrt{N^3 F^{-2} Q_1^9} \ll b_3$, or

$\sqrt[4]{Q^4/N} \ll c_3 \ll N^5 Q_1^{-15} F^{-2}$, $a_3 \gg \sqrt{N^3 F^{-2} Q_1^9}$.

Using Lemma A, we obtain

$$(S/N)^8 \ll Q_1^{-4} + N^{-1} Q_1^{-7} \sum_h \Big|\sum_{x,y} e\big(f_5(x, y)\big)\Big|$$
$$\lll Q_1^{-4} + \sqrt{c_3 F^2 Q_1^7 N^{-5}} + \min\{(b_3 F\sqrt{Q_1^7 N^{-3}})^{-1/2} + (Y\sqrt{c_3})^{-1/2};$$
$$(a_3 F\sqrt{Q_1^7 N^{-3}})^{-1/2} + (X\sqrt{c_3})^{-1/2}\} \ll Q_1^{-4}.$$

2) $\sqrt{N^3 F^{-2} Q_1^9} \ll c_3 \ll N^5 Q_1^{-15} F^{-2}$, $c_3 \ggg (F^2 Q_1^7 N^{-3})^{-1/6}$.

Applying Lemma 1, we obtain

$$N^{-1}Q_1^{-7}\sum_h\Big|\sum_{x,y}e\big(f_5(x,y)\big)\Big| \lll F\sqrt{Q_1^7 c_3 N^{-5}}+(Fc_3\sqrt{Q_1^7N^{-3}})^{-1/2}\ll Q_1^{-4}.$$

3) $\max\{a_3, b_3, c_3\}\ll\sqrt{N^3F^{-2}Q_1^9},\ c_3\gg\sqrt{Q_1^4/N},$

$\max\{a_3, b_3\}\gg(c_3^2+c_3/Q_1^2)\sqrt{N^3F^{-2}Q_1^9}.$

We denote $\gamma_1=\max\{a_3, b_3, \sqrt{c_3}\}$, $\gamma_2=\min\{a_3, b_3, \sqrt{c_3}\}$. If γ_1 is small, then, since in $\mathscr{D}_3$ we have

$$\max\{|\varphi_{2,0}(\theta)|, |\varphi_{0,2}(\theta)|, |\varphi_{1,1}(\theta)|\}\ll\gamma_1\delta_2,$$

we denote $u=\sigma_1/\theta$, $v=\sigma_2/\theta^2$, $w=\sigma_3/\theta^3$ and solving the above system for u, v, w, we obtain that $\sigma_1=d_1\theta+O(\gamma_1\theta)$, $\sigma_2=d_2\theta^2+O(\gamma_1\theta^2)$, $\sigma_3=d_3\theta^3+O(\gamma_1\theta^3)$;

$$h_4/h_3=d_4h_2/h_1+O(\gamma_1\theta),\qquad h_6/h_5=d_5h_2/h_1+O(\gamma_1\theta),$$

$$N^{-1}Q_1^{-7}\sum_h|D_3|\ll\gamma_2(\gamma_1+Q_1^{-2})(\gamma_1+Q_1^{-1}),\qquad Q_1^{-7}\sum_h 1\ll(\gamma_1+Q_1^{-2})(\gamma_1+Q_1^{-1}).$$

Applying Lemma A, we obtain

$$N^{-1}Q_1^{-7}\sum_h\Big|\sum_{x,y}e\big(f_5(x,y)\big)\Big|\lll N^{-1}Q_1^{-7}\sum_h\sum_{x,y}\sqrt{F^2Q_1^7N^{-5}c_3}+$$
$$+Q_1^{-7}\sum_h[(F^2Q_1^7N^{-3})^{-1/4}\max\{a_3^{-1/2}; b_3^{-1/2}\}+Y^{-1}c_3^{-1/2}\ll Q_1^{-4}.$$

4) $\sqrt{Q_1^4/N}\ll c_3\ll\sqrt{N^3F^{-2}Q_1^9},$

$\max\{a_3; b_3\}\ll(c_3^2+c_3Q_1^{-2})\sqrt{N^3F^{-2}Q_1^9}.$

Acting similarly to 3), one can show that if $\max\{a_3, b_3, c_3\}\ll 1$, then

$$\frac{\partial^3}{\partial x^3}\big(f_5(x,y)\big)\gg F_1X^{-3}$$

and, using Lemma B with $k=3$, we obtain

$$(S/N)^8\lll Q_1^{-4}+N^{-1}Q_1^{-7}\sum_h\sum_{x,y}(F_1X^{-3})^{1/6}+X^{-1/4}Q_1^{-7}\sum_h\Big(\sum_{x,y}1/N\Big)^{3/4}+$$
$$+(F_1)^{-1/4}Q_1^{-7}\sum_h\Big(\sum_{x,y}1/N\Big)^{1/2}$$
$$\ll Q_1^{-4}+(F^2Q_1^7N^{-6})^{1/12}(N^3F^{-2}Q_1^9)^2+X^{-1/4}\gamma_2^{3/4}(\gamma_1+Q_1^{-1})(\gamma_1+Q_1^{-2})+$$
$$+(F_1)^{1/4}\gamma_2^{1/2}(\gamma_1+Q_1^{-1})(\gamma_1+Q_1^{-2})\ll Q_1^{-4}.$$

5) $c_3\ll\sqrt{Q_1^4/N},\ \max\{a_3, b_3\}\ll\sqrt{N^3F^{-2}Q_1^9}.$

If $\max\{a_3, b_3\}\ll(c_3^2+c_3Q^{-2})\sqrt{N^3F^{-2}Q_1^9}$, then $N^{-1}Q_1^{-7}\sum_h|D_3|\ll Q_1^{-4}$, and $(S/N)^8\ll Q_1^{-4}$. Otherwise we use Lemma B with $k=2$ to get (if, say, $b_3\geqslant a_3$, which is the worst case):

$$N^{-1}Q_1^{-7}\sum_h\Big|\sum_{x,y}e\big(f_5(x,y)\big)\Big|$$
$$\ll Q_1^{-4}+N^{-1}Q_1^{-7}\sum_h\Big(\sum_{x,y}\sqrt{F_1b_3Y^{-2}}+N\sqrt{F_1b_3}\Big)\ll Q_1^{-4}.$$

6) $a_3\gg\sqrt{N^3F^{-2}Q_1^3},\ c_3\ll\sqrt{Q_1^4N^{-1}}$ or

$a_3\ll\sqrt{N^3F^{-2}Q_1^9}\ll b_3,\ c_3\ll Q_1^8y^{-2}+\sqrt{Q_1^4N^{-1}}.$

Dividing D_3 into subdomains where $|(\varphi_1(\theta)^1)|\cong a\cdot(y/x)^5$, we obtain

$$N^{-1}Q_1^{-7}\sum_h|D_2|$$
$$\lll\max_a\big(\min\{c_3/a; (\sqrt{c_3/c_2+Q_1^{-2}})\cdot\min\{c_2^{5/9}+Q_1^{-1}; a^{1/8}+Q_1^{-1/2}\}\}\big)$$
$$\ll\sqrt{c_3Q_1^{-1}C_2^{-1}}+Q_1^{-5/2}+C_3^{41/81}+Q_1^{-2}(C_3Q_1^2)^{1/9},$$

and if $a_3\gg\sqrt{N^3F^{-2}Q_1^9}$, then we apply Lemma B with $k=2$ to get

$$N^{-1}Q_1^{-7}\Big|\sum_{x,y}e\big(f_5(x,y)\big)\Big|\lll\sqrt[4]{F^2Q_1^7N^{-5}}\cdot N^{-1}Q_1^{-7}\cdot\sum_h|D_2|+Q_1^{-4}$$
$$\ll Q_1^{-4}+\sqrt[4]{F^2Q_1^7N^{-5}}\,[Q_1^{-5/2}+(Q_1^4N^{-1})^{41/162}+$$
$$+(NQ_1^{28})^{-1/18}+(F^{36}Q_1^{440}N^{-139})^{1/196}]\ll Q_1^{-4}.$$

If $a_3\ll\sqrt{N^3Q_1^9F^{-2}}$, then we divide D_3 into subdomains with $|(\varphi_1(\theta))^1|\cong a\cdot(y/x)^5$, and if $a\gg\sqrt[4]{F^2Q_1^{31}N^{-5}Y^{-4}}$, then $|D_3|\ll Nc_3/a$ and, as above, we get

$$(S/N)^8\lll Q_1^{-4}+\sqrt[4]{F^2Q_1^7N^{-3}Y^{-4}}\cdot N^{-1}|D_3|\ll Q_1^{-4}.$$

If $a\ll\sqrt[4]{F^2Q_1^7N^{-5}Y^{-4}}$, then, as in 3), one can show that

$$Q_1^{-7}\sum_h 1\ll(\sqrt{a_3+c_3}+a+Q_1^{-2})\cdot(\sqrt{a_3+c_3}+Q_1^{-1}+a)\quad\text{and}\quad|D_3|\ll\sqrt{a_3}\,|D_2|.$$

If $a_3\gg(c_3+a^2+Q_1^{-4})\sqrt{N^3F^{-2}Q_1^9}$, then as above we obtain

$$(S/N)^8\ll Q_1^{-4}+Q_1^{-7}(F^2Q_1^7N^{-3}a_3^2)^{-1/4}\sum_h 1\ll Q_1^{-4};$$

otherwise use Lemma B to get

$$(S/N)^8\ll Q_1^{-4}+\sqrt[4]{F^2Q_1^7N^{-3}Y^{-4}}\,N^{-1}Q_1^{-7}\sum_h\sum_{x,y}1$$
$$\ll Q_1^{-4}+\sqrt[4]{F^2Q_1^7N^{-3}Y^{-4}}\,(\sqrt[4]{N^3F^{-2}Q_1^9}\cdot\sqrt{F^2Q_1^{31}N^{-5}Y^{-4}})\times$$
$$\times(\sqrt[4]{F^2Q_1^{31}N^{-5}Y^{-4}}+Q_1^{-1})\ll Q_1^{-4}.$$

7) $c_3 \gg N^5 Q_1^{-15} F^{-2}$. We define

$$\Phi_{2,0}(f) = -f_{y^2};\quad \Phi_{1,1}(f) = f_{xy};\quad \Phi_{0,2}(f) = -f_{x^2};\quad M(f) = f_{x^2} f_{y^2} - (f_{xy})^2;$$

$$\Phi_{i,j+1}(f) = [H(f)]^{2i+2j-1}\Big[-\frac{\partial}{\partial x}\big(\Phi_{i,j}(f)(H(f))^{3-2i-2j}\big)f_{xy} + \frac{\partial}{\partial y}\big(\Phi_{i,j}(f)(H(f))^{3-2i-2j}\big)f_{x^2}\Big]$$

and

$$\Phi_{i+1,j}(f) = [H(f)]^{2i+2j-1}\Big[\frac{\partial}{\partial x}\big(\Phi_{i,j}(f)\cdot(M(f))^{3-2i-2j}\big)f_{y^2} - \frac{\partial}{\partial y}\big(\Phi_{i,j}(f)(H(f))^{3-2i-2j}\big)f_{xy}\Big]$$

for $i+j \geq 2$; $\varphi_2(\theta), \ldots, \varphi_{k_2+4}(\theta)$ are the polynomials obtained from $\Phi_{k_2+2,0}(f(x, y)), \ldots, \Phi_{0,k_2+2}(f(x, y))$ respectively by replacing $f_{x^i y^j}$ with $\varphi_{ij}(\theta)$ (they all are polynomials of degree $k_1(3k_2+1)$ in θ and of degree $(3k_2+1)$ in $\sigma_1, \sigma_2, \sigma_3$);

$$\varphi_{k_2+5} = [\varphi_{k_2+4}\,\varphi_{k_2+2} - (\varphi_{k_2+3})^2]/(\varphi_1)^2,$$
$$\varphi_{k_2+6} = (\varphi_{k_2+4}\,\varphi_{k_2+1} - \varphi_{k_2+3}\,\varphi_{k_2+2})/(\varphi_1)^2$$

are both polynomials of degree $k_1(6k_2-2)$ in θ and $6k_2-2$ in $\sigma_1, \sigma_2, \sigma_3$; for $m = 0, 1, \ldots, k_2$ and $l = 0, 1, \ldots, k_2-m+2$,

$$\Psi_{l,m}(\theta) = \sum_{j=0}^{m} \varphi_{l+j+2}(\theta)\,\theta^{m-j}\sigma_{j,m};$$

$\sigma_{j,m}$ is the jth elementary symmetric function of $\alpha_1 = h_{21}/h_{11}, \ldots, \alpha_m = h_{2m,1}/h_{2m-1,1}$;

$$\Psi_{j+1}^1 = [\Psi_{j,k_2-j}\,\Psi_{j+2,k_2-j} - (\Psi_{j+1,k_2-j})^2] \qquad (j = 0, 1, \ldots, k_2);$$
$$\Psi_{j+1}^2 = [\Psi_{j,k_2-j-1}\,\Psi_{j+3,k_2-j-1} - \Psi_{j+1,k_2-j-1}\,\Psi_{j+2,k_2-j-1}] \qquad (j = 0, \ldots, k_2-1).$$

We divide the domain D_3 into $O(N^\varepsilon)$ subdomains and take a subdomain D_4, corresponding to the largest subsum, where, say,

$$d_j(\delta_2)^{9k_2+3} \leq |\varphi_j(\theta)| \leq 2d_j(\delta_2)^{9k_2+3} \qquad (j = 2, \ldots, k_2+4),$$
$$d_j(\delta_2)^{3k_1k_2-2k_1} \leq |\varphi_j(\theta)| \leq 2d_j(\delta_2)^{3k_1k_2-3k_1} \qquad (j = k_2+5,\ k_2+6).$$

We take the largest Q_2 such that

$$Q_2 \ll \min\{\max\{(F_1^{2k_2+2} N^{-k_2-2})^{1/(2K_2-1)}; (F_1^{6k_2+2} N^{-3k_2-4})^{1/(3K_2-3)}\}; Q_1; (N^2 Q_1^{-K_1})^{1/(6k_2-2)}\};$$

$$Q_2 \ll N^{2k_1+5} F^{-4} Q_1^{2-5K_1} \quad \text{for} \quad k_2 = 1;$$
$$Q_2 \ll N^{5k_1} F^{-10} Q_1^{5-11K_1} \quad \text{for} \quad k_2 = 2;$$
$$Q_2 \ll \max\{(F_1^{38} N^{-25})^{1/30}; (F_1^{118} N^{-77})^{1/114}\} \quad \text{for} \quad k_2 = 3.$$

If $\min\{d_{k_2+5}; d_{k_2+6}\} \ll Q_2^{1-3k_2}$, or $\min\{d_2, \ldots, d_{k_2+4}\} \ll Q_2^{-1/2(3k_2+1)}$, then it is easy to see that $Q_1^{1-K_1}\sum_h |D_4| \ll Q_2^{-1/2}$, and we can estimate the sum in (8) below trivially to get the needed estimate. Now we assume that

$$a \equiv \min\{d_2^{2/(3k_2+1)}, \ldots, d_{k_2+4}^{2/(3k_2+1)}, d_{k_2+5}^{1/(3k_2-1)}, d_{k_2+6}^{1/(3k_2-1)}\} \gg Q_2^{-1},$$

and use Lemma 1 to get

$$(8)\qquad (S/N)^{K_1} \ll Q_1^{-K_1/2} + F_1 Q_1^{1-K_1}/\sqrt{c_{k_1}}\sum_h \Big|\sum_{u,v} e(g(u, v))\Big|,$$

where the domain of summation over (u, v) and $g(u, v)$ are as in Lemma 1,

$$Q_1^{1-K_1}\sum_h \Big|\sum_{u,v} 1\Big| \ll Q_1^{1-K_1} F_1^2 N^{-2} c_{k_1}\sum_h |D_4| \ll F_1^2 N^{-1} a c_{k_1} \equiv N_1.$$

So, if $a^2 c_{k_1} \ll Q_2^{-1}$, then

$$(S/N)^{K_1} \ll Q_1^{K_1/2} + F_1 N^{-1} Q_2^{-1/2} \ll Q_2^{-K_1/2}.$$

Now we assume that $a^2 c_{k_1} \gg Q_2^{-1}$. If $k_2 \geq 2$ and $(X/Y)^{k_2-1} \gg Q_2^{K_2/2-1} + NQ_2^{3K_2/2-2} N_0^{-k_2-1/3}$ (where $N_0 = F_1^2/N$), then we apply Lemma 2 (k_2-1) times with $n = 1$ and then continue as below to prove the needed estimate. Also, if $X/Y \gg Q_2 + NQ_2^{K_2} N_0^{-k_2-1/3}$, then we apply Lemma 2 once with $n = 1$ and then continue as below. Now we assume that $X/Y \ll \min\{Q_2; NQ_2^{K_2} N_0^{-k_2-1/3}\}$, and consider the following distinct cases:

(a) $$d_{k_2+5}^{3(K_2+k_2-1)/(3k_2-1)} \ll N^{6K_2-k_2-6} Q_1^{2K_1-2K_1K_2} F_1^{6+2k_2-6K_2}.$$

We apply Lemma 2 with $n = 1$ k_2 times:

$$|\Sigma_2|^{K_2} \equiv Q_1^{1-K_1}\Big(\sum_h \Big|\sum_{u,v} e(g(u, v))\Big|\Big)^{K_2}$$
$$\ll N_1^{K_2} Q_3^{-K_2/2} + N_1^{K_2-1} Q_3^{1-K_2}\sum_{h^{(1)}} Q_1^{1-K_1}\sum_h \Big|\sum_{u,v} e(g_1(u, v))\Big|,$$

where

$$g_1(u, v) = \int_0^1 \ldots \int_0^1 \frac{\partial^{k_2}}{\partial t_1 \ldots \partial t_{k_2}}\big(g(u, v + t_1 h_1 + \ldots + t_{k_2} h_{1,k_2})\big)\,dt,$$

$$h^{(1)} = (h_{11}, \ldots, h_{1,k_2});\qquad |h_{1j}| \leq Q_3^{2^{j-1}} \qquad (j = 1, \ldots, k_2);$$

$$Q_3 = \min\{(F_1^2 Y^{-2})^{1/K_2};\ \max\{(F_1^{2k_2+2} N^{-2} Y^{-2k_2}/d_{k_2+5})^{1/(3K_2-2)};$$
$$[F_1^{3k_2+1} N^{-2} Y^{-3k_2} (d_{k_2+4} d_{k_2+5})^{-1} a^2 c_{k_1}^2]^{1/3(K_2-1)}\}\}.$$

Now we use Lemma A:

$$(\Sigma_2)^{K_2} \lll N_1^{K_2} Q_3^{-K_2/2} + N_1^{K_2} Q_3^{K_2-1} Y^{k_2} \cdot NF_1^{-k_2-1}/\sqrt{d_{k_2+5}} +$$
$$+ N_1^{K_2-1} (F_1^{k_2+3} N^{-2} Y^{-k_1} Q_3^{1-K_2} d_{k_2+4}^{-1})^{1/2} + N_1^{K_2-1} Y (F_1^2 d_{k_2+5})^{-1/2}$$
$$\ll (F_1^2 \sqrt{c_{k_1}/Q_2}\, N^{-1})^{K_2},$$

so that $(S/N)^{K_1} \ll Q_1^{K_1/2}$.

(b) $a^{6(K_2+3k_2-1)(K_2+k_2-1)/K_2} \ll F_1^{2k_2+6-6K_2} Q_1^{2k_2-2k_2k_1} N^{6K_2-6-k_2}$.

Similarly to (a) we obtain:

$$(S/N)^{K_1} \ll Q_1^{-K_1/2}.$$

(c) the conditions of (a), (b) are not satisfied. We apply Lemma 2 with $n = 2$:

$$|S_1|^2 \equiv Q_1^{1-K_1} \Big(\sum_h \Big|\sum_{u,v} e\big(g(u,v)\big)\Big|\Big)^2$$
$$\ll N_1^2 Q_2^{-1} + N_1 Q_2^{-1} \sum_{h(1)} \Big|\sum_{u,v} e\big(g_1(u,v)\big)\Big|,$$

where

$$g_1(u,v) = \int_0^1 \frac{\partial}{\partial t} g(u+h_{11}t,\ v+h_{21}t)\,dt.$$

We divide the domain D_4 into subdomains where

$$a_4 \delta_2^{3k_2+1} \delta_{1,1} \leqslant |\psi_{0,1}(\theta)| \leqslant 2a_4 \delta_2^{3k_2+1} \delta_{1,1},$$
$$b_4 \delta_2^{3k_2+1} \delta_{1,1} \leqslant |\psi_{2,1}(\theta)| \leqslant 2b_4 \delta_2^{3k_2+1} \delta_{1,1},$$
$$c_4 \delta_2^{3k_2+1} \delta_{1,1} \leqslant |\psi_{1,1}(\theta)| \leqslant 2c_4 \delta_2^{3k_2+1} \delta_{1,1},$$
$$c_{4,j} \delta_2^{6k_2+2} \delta_{1,1}^2 \leqslant |\psi_1^j(\theta)| \leqslant 2c_{4,j} \delta_2^{6k_2+2} \delta_{1,1}^2 \qquad (j = 1, 2).$$

Here we denote $\delta_{1,m} = \prod_{j=1}^{m} (y/x + |h_{2j,1}/h_{2j-1,1}|)$. We assume that the domain D_5, defined by the above inequalities, corresponds to the largest subsum. One can see that

(9) $\max\{a_4, c_4\} \geqslant d_{k_2+5} c_{k_1}^2$,

$$\max\{a_4, b_4\} \geqslant d_{k_2+6} c_{k_1}^2 \quad \text{and} \quad c_{4,1} + a_4 b_4 + a_4^2 \geqslant d_{k_2+5}^2 c_{k_1}^4.$$

We consider the following distinct cases.

1. $c_{4,1} \ll d_{k_2+5}/Q_2$ or $c_{4,2} \ll d_{k_2+6}/Q_2$.

In either case we have

$$Q_2^{-1} \sum_{h(1)} |D_5| \ll Q_2^{-1/2} |D_4|.$$

If $c_{4,1} \geqslant b \equiv a^{2K_2-2} c_{k_1}^{K_2-2} \min\{Q_2^{1+K_2/2} NF_1^{-2};\ Q_2^{1-K_2/2} (NF_1^{-2})^{1/3}\}$ then, using (9), one can show that either

$$a_4^2 \geqslant Q_2^{4-3K_2} c_{4,1}^2 a^{2K_2-8} c_{k_1}^{6K_2-8} \min\{1;\ Q_2^{2-2K_2} (F_1^2/N)^{4/3}\}$$

or $c_{4,1} \geqslant bX/Y$ and $b_4 \geqslant d_{k_2+5} c_{k_1}^2$. Applying Lemma 2 (k_2-1) times with $n = 1$ and appropriate Q to minimize the obtained expression and, denoting

$$g_2(u,v) = \int_0^1 \ldots \int_0^1 \frac{\partial^{k_2-1}}{\partial t_1 \ldots \partial t_{k_2-1}} \big(g_1(u,\ v+h_{31}t_1 + \ldots + h_{k_2+1,1} t_{k_2-1})\big)\,dt,$$

we use Lemma A to get

$$|S_1|^{K_2} \ll (N_0 \sqrt{c_{k_1}/Q_2})^{K_2} + N_1^{K_2-1} Q_2^{(2-K_2)/4} Q^{1-K_2/2},$$
$$\sum_{h(1)} \Big|\sum_{u,v} e\big(g_2(u,v)\big)\Big| \lll (N_0 \sqrt{c_{k_1}/Q_2})^{K_2} + N_1^{K_2-1} Q_2^{-K_2/4} +$$
$$+ ac_{k_1} [(c_{4,1} Q_2 NQ^{K_2-2} F_1^{2-2K_2} Y^{2k_2-2})^{1/2} +$$
$$+ \min\{(F_1^{2k_2+6} Q_2 \cdot Q^{2-K_2} Y^{2-2k_2} N^{-5} a_4^{-2})^{1/4} +$$
$$+ F_1 X^{-1} \sqrt{Q_2/c_{4,1}};$$
$$F_1^{2k_2+6} Q_2 Q^{2-2K_2} Y^{2-2k_2} N^{-5} b_4^{-2})^{1/4} +$$
$$+ F_1 Y^{-1} \sqrt{Q_2/c_{4,1}}\}]$$
$$\ll (N_0 \sqrt{c_{k_1}/Q_2})^{K_2} + N_0^{K_2-1/3} c_{k_1}^{K_2/2} \equiv R_0.$$

If $c_{4,1} \ll b$, then we use Lemma B with $k = 2$ and, arguing as above, we obtain the same result.

2. $\min\{c_{4,1};\ c_{4,2}\} \geqslant d_{k_2+5}/Q_2$ and

$$\min\{a_4/d_{k_2+4};\ b_4/d_{k_2+2};\ c_4/d_{k_2+3};\ (c_{4,2}/d_{k_2+6})^{1/2}\} \ll \delta^{(3K_2-2)/(6K_2-8)}$$

or

$$c_{4,1} \ll \delta d_{k_2+5},$$

where

$$\delta^{3K_2/2-1} = a^{4-3K_2} \cdot \max\{N_0^{k_2-8/3K_2} F_1^{-2} Q_2^{-1};\ N_0^{k_2+1} N^{-1} Q_2^{3-3K_2}\}.$$

As in 1 we can show that

$$Q_2^{-1}\sum_{h(1)}|D_5| \ll (1/\sqrt{Q_2}+\min\{a_4/d_{k_2+4};\ b_4/d_{k_2+2};\ c_4/d_{k_2+3};\ \sqrt{c_{4,1}/d_{k_2+5}};\ \sqrt{c_{4,2}/d_{k_2+5}}\})\cdot|D_4|$$

and

$$|S_1|^{K_2} \ll N_1^{K_2}(Q_2^{-K_2/2}+\delta^{K_2/4}Q^{-K_2/4})+ +N_1^{K_2-1/3}+N_0^{K_2}c_{k_1}^{K_2/2}\delta^{K_2/4}(Q_2N^{k_2+2}F_1^{-2k_2-2}\delta d_{k_2+5}\cdot Q^{K_2-2})^{1/2} \ll R_0.$$

3. The inequalities of 2 are not satisfied. If $k_2 = 1$, then, as in 2, we obtain

$$|S_1|^{K_2} \ll R_0+F_1^4c_{k_1}N^{-2}(Q_2N^3F_1^{-4})^{1/2} \ll R_0.$$

If $k_2 \geqslant 2$, then we apply Lemma 2 with $n = 2$ and $Q = Q_2^2$ once more:

$$|S_1|^4 \ll N_1^4Q_2^{-2}+N_1^3Q_2^{-3}\sqrt{\delta_3}\sum_{h(1)}\Big|\sum_{u,v}e(g_3(u, v))\Big|, \tag{10}$$

where

$$g_3(u, v) = \int_0^1 \frac{\partial}{\partial t}(g_1(u+h_{31}t, v+h_{41}t))\,dt,$$

$$\delta_3 = \min\{1;\ c_{4,1}/d_{k_2+5};\ c_{4,2}/d_{k_2+6};\ a_4^2/d_{k_2+4}^2;\ b_4^2/d_{k_2+2}^2;\ c_4/d_{k_2+3}^2\}.$$

We divide the domain D_5 into subdomains and take one of them, D_6, corresponding to the largest subsum, where, say,

$$a_5\delta_2^{3k_2+1}\delta_{1,2} \leqslant |\psi_{0,2}(\theta)| \leqslant 2a_5\delta_2^{3k_2+1}\delta_{1,2},$$
$$b_5\delta_2^{3k_2+1}\delta_{1,2} \leqslant |\psi_{2,2}(\theta)| \leqslant 2b_5\delta_2^{3k_2+1}\delta_{1,2},$$
$$c_5\delta_2^{3k_2+1}\delta_{1,2} \leqslant |\psi_{1,2}(\theta)| \leqslant 2c_5\delta_2^{3k_2+1}\delta_{1,2},$$
$$c_{5,j}\delta_2^{6k_2+2}\delta_{1,2}^2 \leqslant |\psi_2^j(\theta)| \leqslant 2c_{5,j}\delta_2^{6k_2+1}\delta_{1,2}^2 \qquad (j = 1, 2).$$

Here

$$\max\{a_5; b_5\} \geqslant c_{4,2}, \qquad \max\{a_5; c_5\} \geqslant c_{4,1},$$

and, denoting

$$\delta_4 = \min\{\sqrt{c_{5,1}/c_{4,1}},\ \sqrt{c_{5,2}/c_{4,2}};\ a_5/a_4;\ b_5/b_4;\ c_5/c_4;\ 1\},$$

we get

$$Q_2^{-3}\sum_{h(1)}|D_6| \ll \delta_4|D_5|.$$

So, if $\delta_3\delta_4a^4c_{k_1}^2 \ll Q_2^{-2}$, then, estimating the sum in (10) trivially, we get

$$|S_1|^{K_2} \ll R_0.$$

Now we assume that $\delta_3\delta_4a^4c_{k_1}^2 \geqslant Q_2^{-2}$, and consider the following distinct cases:

(a) $\delta_4 \ll Q_2^{-1}$. If

$$c_{5,1} \geqslant a^{2K_2-2}c_{k_1}^{K_2-2}\delta_3^{(1/2)(K_2-2)}\delta_4^{(1/2)(K_2-4)}\min\{Q_2^{K_2}NF_1^{-2};\ (NF_1^{-2})^{1/3}\}\cdot X/Y \equiv d,$$

then we denote $g_4(u, v) = g_3(u, v)$ if $k_2 = 2$ and $g_4(u, v) = g_3(u, v+h_{5,1})-g_3(u, v)$ if $k_2 = 3$ and apply Lemma 2 (k_2-2) times with $n = 1$ and appropriate Q first and Lemma A after that:

$$|S_1|^{K_1} \ll R_0+(F_1^2ac_{k_1}/N)^{K_1}(\delta_3\delta_4)^{K_1/4}Q^{-1}+ +(F_1^2ac_{k_1}/N)^{K_1-1}\delta_3^{(K_1-2)/4}\delta_4^{(K_1-4)/4}Q_2^{-3}Q^{1-K_1/4}\sum_{h(1)}\Big|\sum_{u,v}e(g_4(u, v))\Big|$$

$$\lll R_0+(F_1^2ac_{k_1}/N)^{K_2}(\delta_3\delta_4)^{K_2/4}(Q^{-1}+\sqrt{Q_2^3Q^{K_2/2-2}Y^{2k_2-4}N^4F_1^{-2k_2-2}})+$$
$$+(F_1^2ac_{k_1}/N)^{K_2-1}\delta_3^{(K_2-2)/4}\delta_4^{(K_2-4)/4}\cdot\min\{F_1/(X\sqrt{c_{5,1}})+$$
$$+\sqrt[4]{F_1^{2k_2+6}N^{-6}Q_2^{-3}Q^{2-K_2/2}Y^{4-2k_2}a_5^{-2}};\ F_1/(Y\sqrt{c_{5,1}})+$$
$$+\sqrt[4]{F_1^{2k_2+8}N^{-6}Q_2^{-3}Q^{2-K_2/2}Y^{4-2k_2}b_5^{-2}}\}$$

$$\ll R_0+[(ac_{k_1})^{K_2-2/3}\cdot\delta_3^{K_2/4-1/3}\delta_4^{K_2/4-2/3}a_4^{-1/3}]\cdot(F_1^2/N)^{K_2-1/3} \ll R_0.$$

If $c_{5,1} \ll d$, then we use the inequality

$$c_{4,1} \ll \max\{a_5; c_5\} \ll \max\{a_5; \sqrt{a_5b_5+c_{5,1}}\}$$

and, applying Lemma B, obtain

$$|S_1|^{K_2} \lll R_0.$$

(b) $Q_2^{-1} \ll \delta_4 \ll (Q_2^{-2}+N_1^{-4/3K_1})(a^4c_{k_1}^2\delta_3)^{-1}(F_1^{2k_2-2}Q_2^{-3}N^{-4}Y^{4-2k_2})^{8/K_2^2} \equiv f$, or $k_2 = 2$ and $\delta_4 \geqslant Q_2^{-1}$.

Similarly to (a) we get

$$|S_1|^{K_2} \ll R_0.$$

(c) $k_2 = 3$, $\delta_4 \geqslant f$. We apply Lemma 2 with appropriate Q and $n = 2$:

$$|S_1|^{K_2} \ll (N_1\sqrt{c_{k_1}/Q_2})^{K_2}+(F_1^2ac_{k_1}/N)^{K_2-1}\delta_3^{(K_2-2)/4}\times \delta_4^{(K_2-4)/4}Q_2^{-3}Q^{-1}\sum_{h(1)}\Big|\sum_{u,v}e(g_5(u, v))\Big|+N_1^{K_2}\cdot(\delta_3\delta_4)^{K_2/4}Q^{-1}, \tag{11}$$

where

$$g_5(u, v) = \int_0^1 \frac{\partial}{\partial t}\big(g_3(u+h_{5,1}t,\ v+h_{6,1}t)\big)\,dt.$$

We again divide the domain D_6 into subdomains and take one of them, D_7, corresponding to the largest subsum, where, say,

$$a_6\delta_2^{3k_2+1}\delta_{1,3} \leqslant |\varphi_{0,3}(\theta)| \leqslant 2a_6\delta_2^{3k_2+1}\delta_{1,3},$$
$$b_6\delta_2^{3k_2+1}\delta_{1,3} \leqslant |\varphi_{2,3}(\theta)| \leqslant 2b_6\delta_2^{3k_2+1}\delta_{1,3},$$
$$c_6\delta_2^{3k_2+1}\delta_{1,3} \leqslant |\varphi_{1,3}(\theta)| \leqslant 2c_6\delta_2^{3k_2+1}\delta_{1,3},$$
$$c_{6,j}\delta_2^{6k_2+2}\delta_{1,3}^2 \leqslant |\psi_3^j(\theta)| \leqslant 2c_{6,j}\delta_2^{6k_2+2}\delta_{1,3}^2 \quad (j = 1, 2).$$

Here $\max\{a_6; b_6\} \geqslant c_{5,2}$, $\max\{a_6; c_6\} \geqslant c_{5,1}$ and $\max\{a_5; \sqrt{a_5 b_5 + c_{6,1}}\} \geqslant c_{5,1}$. We consider all possible distinct cases.

(d) $c_{6,1} \geqslant NF_1^{-2}(ac_{k_1})^{14}\delta_3^2\delta_4^2[Q_2^{-8}+(NF_1^{-2})^{2/3}]^{-1} \equiv g$. Applying Lemma A to the sum in (11) and choosing Q to minimize the obtained expression, we get

$$|S_1|^8 \lll R_0 + N_1^7 N_0 \delta_3^{3/2}\delta_4 \cdot \min\{\sqrt[6]{a^2\delta_3\delta_4^2 a_6^{-2}(NF_1^{-2})^2 c_{6,1}} + YF_1^{-1}/\sqrt{c_{6,1}};$$
$$\sqrt[6]{\delta_3\delta_4^2 a^2 b_6^{-2}(NF_1^{-2})^2 c_{6,1}} + XF_1^{-2}/\sqrt{c_{6,1}}\}.$$

If $c_{6,1} \geqslant gX/Y$, then, because $\max\{a_6; b_6\} \geqslant c_{5,1} \geqslant \delta_3\delta_4 a^{16}$, the above expression is $\ll R_0$; if $c_{6,1} \ll gX/Y$ but $a_6 \geqslant a^{22}\delta_3^5\delta_4^4 c_{k_1}^9\sqrt{c_{6,1}}$, then we get the same result; otherwise we use inequality $\max\{a_6; \sqrt{a_6+c_6}\} \geqslant c_{5,1}$ to prove that

$$|S_1|^8 \lll R_0.$$

(e) $c_{6,1} \leqslant g$. Applying Lemma B with $k_2 = 2$, we choose Q to minimize the obtained expression and obtain:

$$|S_1|^8 \lll R_0 + N_1^8\delta_2^2\delta_3^2\sqrt{c_{6,1}/c_{5,1}}\cdot(X^4Q_2^3N^3F_1^{-8})^{1/5}$$
$$\leqslant R_0 + N_0^8 c_{k_1}^4(N_0Q_2^{-8}+N_0^{1/3})\cdot(X^4Q_2^3N^3F_1^{-8})^{1/5} \leqslant R_0.$$

This completes the proof of Theorem 1.

Theorem 2 can be proved similarly to Theorem 2 in [3]. The only change is in showing that $|D_2| \leqslant \sqrt{a_{31}}\,N$. This can be done by proving that $H(g(x)) \approx P(\theta_1, \theta_2, \theta_3)$, where $\theta_j = h_j/x_j$ and P is a homogeneous polynomial of degree 3 such that $P(\theta) = P_{\theta_1} = P_{\theta_2} = P_{\theta_3} = P_{\theta_1^2} = P_{\theta_2^2} = P_{\theta_3^2} = 0$ has no non-zero solutions.

To prove Theorem 3, we apply Lemma 2 with $m = 3$ and $q = q_1q_2q_3$, to be defined later:

$$(12)\qquad (S/N)^2 \ll q^{-1} + (qN)^{-1}\sum_{|h_1|=1}^{q_1}\sum_{|h_2|=1}^{q_2}\sum_{|h_3|=1}^{q_3}\Big|\sum_{x\in\mathcal{D}_1} e(g(x))\Big| + \Big|\sum_{h,x}\Big|,$$

where

$$g(x) = \int_0^1 \frac{\partial}{\partial t}\big(f(x_1+h_1t,\ x_2+h_2t,\ x_3+h_3t,\ x_4, \ldots, x_n)\big)\,dt$$

and $|\sum_{h,x}|$ is small compared with the estimate we obtain for the first sum. Let $h = (h_1, h_2, h_3)$ be fixed, $\varrho = |h_1/X_1|+|h_2/X_2|+|h_3/X_3|$, $\theta_j = h_j/x_j\varrho$, $F_1 = F\varrho$. Then

$$H(g(x)) \cong [(P_1(\theta))^{n-3}\cdot P_2(\theta) + O(\Delta)]\cdot F_1^n N^{-2},$$

where $P_1(\theta) = \alpha_1\theta_1+\alpha_2\theta_2+\alpha_3\theta_3$, $P_2(\theta) = P_1^3 - 2(\theta_1+\theta_2+\theta_3)P_1^2 + (\alpha_1\theta_1^2 + \alpha_2\theta_2^2 + \alpha_3\theta_3^2 + 2\theta_1\theta_2 + 2\theta_1\theta_3 + 2\theta_2\theta_3)P_1 + (\alpha_1+\alpha_2+\alpha_3-2)\theta_1\theta_2\theta_3$. Also, denoting by $g_{i_1,\ldots,i_j}(x)$ the function obtained from $g(x)$ by fixing all of x_i except of $x_{i_1}, \ldots, x_{i_j}$, we obtain

$$H(g_{1,2,3}(x)) \cong (P_2(\theta)+O(\Delta))F_1^3(X_1X_2X_3)^{-2};$$
$$H(g_i(x)) \cong (P_1(\theta)-2\theta_i+O(\Delta))F_1X_i^{-2} \quad (i = 1, 2, 3);$$
$$H(g_{i,j}(x)) \cong [(\alpha_i-1)(\alpha_j-1)(P_1-2\theta_i)(P_1-2\theta_j) - \alpha_i\alpha_j(P_1-\theta_i-\theta_j)^2 + O(\Delta)]F_1^2(X_iX_j)^{-2}$$
$$\equiv (P_{ij}(\theta)+O(\Delta))F_1^2(X_iX_j)^{-2} \quad (0 < i < j = 1, 2, 3).$$

We divide D_1 into $\ll N^\varepsilon$ subdomains: One of them has $\ll NF_1^{-1/2}$ points. Each of the remaining subdomains is of the type

$$a_1 \leqslant |P_1(\theta)| \leqslant a_1(1+\varepsilon_0),$$
$$a_2 \leqslant |P_2(\theta)| \leqslant a_2(1+\varepsilon_0),$$
$$a_{i+2} \leqslant |P_1(\theta)-2\theta_i| \leqslant a_{i+2}(1+\varepsilon_0) \quad (i = 1, 2, 3),$$
$$a_{i,j} \leqslant |P_{i,j}(\theta)| \leqslant a_{i,j}(1+\varepsilon_0) \quad (0 < i < j = 1, 2, 3).$$

We suppose that the domain D_2, defined by the above inequalities, corresponds to the largest subsum, and consider all possible cases.

1. $a_1 \geqslant \varepsilon_0$, $a_2 \geqslant \varepsilon_0$. Then $H(g(x)) \geqslant F_1^n N^{-2}$, and, applying Lemma 1, we obtain

$$S_1 = \Big|\sum_{x\in D_2} e(g(x))\Big| \lll F_1^{n/2} + NF_1^{-1/2}.$$

2. $a_1 \leqslant \varepsilon_0$. In this case

$$|D_2| \leqslant (\Delta + a_2) N, \quad \max\{a_2, a_{i,j}\} \geqslant \varepsilon_0, \quad \max_{i,j}\big(a_{i,j}(a_{i+2} + a_{j+2})\big) \geqslant \varepsilon_0.$$

If $a_2 \geqslant F_1^{\varepsilon - 1/3}$, then we use Lemma 1 with $k = 1$ three times (changing the order of summation, if necessary):

$$S_1 \lll X_i F_1^{-1/2} \sum_m \Big|\sum_x e\big(f_1(m, x)\big)\Big| + N F_1^{-1/2}$$
$$\lll N F_1^{-1/2} + X_i X_j (F_1)^{-1} \sum_{m_1, m_2} \Big|\sum_x e\big(f_2(m, x)\big)\Big|$$
$$\lll N F_1^{-1/2} + X_1 X_4 \ldots X_n \cdot F_1/a_2 + (\Delta + \min\{a_1, a_2\}) \cdot X_4 \ldots X_n F_1^{3/2}.$$

If a_2 is small, then we can use Lemma 1 with $k = 2$ and get

$$S_1 \lll X_1 X_4 \ldots X_n F_1 (\Delta + a_2) + N F_1^{-1/2}.$$

Comparing two last estimates, we get

$$S \lll X_1 X_4 \ldots X_n F_1 \Delta + N F_1^{-1/2} + \Delta X_4 \ldots X_n F_1^{3/2} + X_1 X_4 \ldots X_n F_1^{2/3}.$$

3. $a_1 \geqslant \varepsilon_0$, $a_2 \leqslant \varepsilon_0$. In this case $|D_2| \leqslant (\Delta + \sqrt{a_2}) N$ (see comments to Theorem 2),

$$|D_2| \leqslant (\Delta + \max_{i,j} a_{i,j}) N, \quad \max_{3 \leqslant i \leqslant 5} a_i \geqslant \varepsilon_0.$$

If, say, $a_{2,3} = \max a_{i,j} \geqslant F_1^{\varepsilon - 1/3}$, then $a_2 \geqslant \varepsilon_0$ (or $a_3 \geqslant \varepsilon_0$) and, applying twice Lemma 1 with $k = 1$, we get

$$S_1 \lll X_2 F_1^{-1/2} \sum_m \Big| \sum_{x_1, x_3, \ldots, x_n} e\big(f_1(m, x)\big)\Big| + N/\sqrt{F_1}$$
$$\lll (\sqrt{\Delta} + \sqrt{a_2}) F X_1 X_4 \ldots X_n + N/\sqrt{F_1} + X_1 X_2 X_4 \ldots X_n/\sqrt{a_{2,3}};$$

also, using Lemma 1 with $m = 1$, we get

$$S_1 \lll (\Delta + a_{2,3}) X_1 X_2 X_4 \ldots X_n \sqrt{F_1} + N/\sqrt{F_1}.$$

If $a_2 \geqslant F_1^{\varepsilon_0 - 1/3}$, then we apply Lemma 1 twice with $k = 1$ and once with $k = n-2$ and obtain:

$$S_1 \lll X_2 F_1^{-1/2} \sum_m \Big| \sum_{x_1, x_3, \ldots, x_n} e\big(f_1(m, x)\big)\Big| + N/\sqrt{F_1}$$
$$\lll X_2 X_3 F_1^{-1/2} \sum_{m_1, m_2} \Big| \sum_{x_1, x_4, \ldots, x_n} e\big(f_1(m, x)\big)\Big| +$$
$$+ N/\sqrt{F_1} + X_1 X_2 X_4 \ldots X_n/\sqrt{a_{2,3}}$$
$$\lll F_1^{n/2} + \sqrt{F_1}\, X_1 X_4 \ldots X_n/\sqrt{a_2} + X_1 X_2 X_4 \ldots X_n/\sqrt{a_{2,3}} + N/\sqrt{F_1}.$$

Comparing all three estimates, we obtain

$$S_1 \lll F_1^{n/2} + N/\sqrt{F_1} + N F_1^{1/6} X_3^{-1} + F_1^{3/4} N (X_2 X_3)^{-1} +$$
$$+ F_1^{5/6} N (X_2 X_3)^{-1} + N\Delta \sqrt{F_1}/X_3 + N F_1 \sqrt{\Delta}\,(X_2 X_3)^{-1} \equiv R_0$$

so, in all three cases we get

$$S_1 \lll R_0 + \Delta N F_1^{3/2}/X_1 X_2 X_3 \equiv R_1.$$

Substituting this into (12), we choose q to minimize the resulting expression:

$$(S/N)^2 \lll q^{-1} + (qN)^{-1} \sum_h R_1 \lll q^{-1} + N^{-1}\Big[\big(F \sqrt[3]{q (X_1 X_2 X_3)^{-1}}\big)^{n/2} +$$
$$+ N \sqrt{X_1 X_2 X_3 q^{-1} F^{-3}} + N X_3^{-1} \sqrt[18]{F^3 q (X_1 X_2 X_3)^{-1}} +$$
$$+ N (X_2 X_3)^{-1} \sqrt[4]{F^3 q (X_1 X_2 X_3)^{-1}} + N \Delta X_3^{-1} \sqrt[6]{F^3 q (X_1 X_2 X_3)^{-1}} +$$
$$+ N (X_2 X_3)^{-1} \sqrt[18]{F^{15} q^5 (X_1 X_2 X_3)^{-5}} +$$
$$+ N \sqrt{\Delta}\, (X_2 X_3)^{-1} \sqrt[3]{F^3 q (X_1 X_2 X_2)^{-1}} +$$
$$+ N \Delta \sqrt{F^3 q (X_1 X_2 X_3)^{-3}}\Big]$$
$$\leqslant \big(F^{3n} N^{-6} (X_1 X_2 X_3)^{-n}\big)^{1/(n+6)} +$$
$$+ X_3^{-1} \sqrt[19]{F^3 X_1^{-1} X_2^{-1}} + X_2^{-1} X_3^{-1} \sqrt[23]{F^{15}/X_1^5} +$$
$$+ X_3^{-1} \sqrt[7]{F^3 \Delta^6 X_1^{-1} X_2^{-1}} + X_2^{-1} X_3^{-1} \sqrt[8]{F^6 \Delta^3 X_1^{-2}} +$$
$$+ N^{-1/(n+1)} + X_3^{-3/4} + \sqrt{\Delta/X_3}.$$

This proves Theorem 3.

Now we assume that (1) and (4) are true and prove Theorems 4 and 5. First we notice that using m.e.p. one can never obtain a better estimate than $S \leqslant \sqrt{N}$. This can be proved by induction on the total number of times we apply (1) and (4). If we apply (1) once, we get $S \leqslant F^{m/2} X_{m+1} \ldots X_n$, where $F^{m/2} > X_1 \ldots X_m$. (Note that the condition $F \geqslant X_1$ is not essential; we can remove it and substitute (1) with

$$S \leqslant N_m F^{-m/2} S(F; F/X_1 + 1, \ldots, F/X_m + 1, X_{m+1}, \ldots, X_n).$$

One can however do it better: If $F \leqslant \varepsilon_0 X_1$, then

$$S = \Big| \sum_{x_2, \ldots, x_n} \sum_{x_1 = X}^{X'} e\big(f(x)\big)\Big|$$
$$\leqslant \Big| \sum_{x_2, \ldots, x_n} \int_X^{X'} e\big(f(x)\big)\,dx_1\Big| + \sum_{m=1}^{\infty} \frac{1}{m} \Big| \sum_{x_2, \ldots, x_n} \int_X^{X'} e\big(f(x) - m x_1\big)\,dx_1\Big|$$
$$\leqslant \frac{X_1}{F} S(F; X_2, \ldots, X_n) + \ldots)$$

If we apply (4) once, we get $S^2 \ll N^2/q+(N/q)S(F\varrho;\, X,\, q)$, where $N^2/q \gg N$. Suppose, our statement is always true if we apply (1) and (4) k times. If we apply (1) and (4) $(k+1)$ times, and if we start with (4), we can never get a better estimate than $S \ll \sqrt{N}$ (the same reasoning as above). If we start with applying (1), then we get

$$S \ll N_m F^{-m/2} S(F;\, F/X_1,\, \ldots,\, F/X_m,\, X_{m+1},\, \ldots,\, X_n),$$

and we apply (1) and (4) k times to the last sum. So, we can never obtain a better estimate than $S \ll N_m \cdot F^{-m/2}(F/X_1 \ldots F/X_m \cdot X_{m+1} \ldots X_n)^{1/2} = \sqrt{N}$. To prove Theorem 4, we suppose that $(\lambda_0,\, \ldots,\, \lambda_n) \in E_n$. Using (1), we obtain:

$$\begin{aligned} S &\ll N_m F^{-m/2} \cdot S(F;\, F/X_1,\, \ldots,\, F/X_m,\, X_{m+1},\, \ldots,\, X_n) \\ &\ll N_m F^{-m/2} \cdot S(F;\, F/X_1,\, \ldots,\, F/X_m,\, X_{m+1},\, \ldots,\, X_n) \\ &= F^{k-m/2+\lambda_1+\ldots+\lambda_m} X_1^{1-\lambda_1} \ldots X_m^{1-\lambda_m} X_{m+1}^{\lambda_1} \ldots X_n^{\lambda_n}, \end{aligned}$$

i.e., $(\tilde{\lambda}_0,\, \ldots,\, \tilde{\lambda}_n) \in E_n$. Theorem 5 can be proved similarly by using (4):

$$\begin{aligned} S^2 &\ll N^2/q+Nq^{-1} S(F \cdot \sqrt[m]{q/N_m};\, X,\, q) \\ &\ll N^2/q+Nq^{-1}(F \cdot \sqrt[m]{q/N_m})^{\lambda_0} X_1^{\lambda_1} \ldots X_n^{\lambda_n} q_1^{\lambda_1^*} \ldots q_m^{\lambda_1^*} \\ &= N^2/q+F^{\lambda_0} N X_1^{\lambda_1+\lambda_1^*} \ldots X_m^{\lambda_m+\lambda_m^*} X_{m+1}^{\lambda_{m+1}} \ldots X_n^{\lambda_n} q^{-1} \cdot (q/N_m)^{(\lambda_0+|\lambda^*|)/m}. \end{aligned}$$

Choosing q which minimizes the obtained expression we prove Theorem 5.

6. Final remarks. While Theorems 1–3 can be applied to a wide class of function $\in \mathscr{F}$, they may not give a "good" result in some cases. The general strategy in using m.e.p. is the following. Knowing the estimate of an exponential sum one wants to obtain, he needs to apply the algorithm (based on (1) and (4)) described in the introduction to determine whether it is possible and (if it is possible) a sequence of operations of A and B one needs to apply in order to obtain the result. The last step is to prove the result using the ideas contained in the proof of Theorem 1. For a specific function it can be done much simpler than in the general case.

References

[1] Kenji Asada and Daisuke Fujiwara, *On some oscillatory integral transformations in* $L^2(R^n)$, Japan J. Math. 4 (1978), pp. 299–360.

[2] J. G. Van der Corput, *Verscharfung der Abschatzung beim Teilerproblem*, Math. Ann. 87 (1922), pp. 39–65.

[3] G. Kolesnik, *An improvement of the method of exponent pairs*, Colloquia Math. Soc. János Bolyai, to be published.

[4] — *On the order of* $\zeta(\frac{1}{2}+it)$ *and* $\Delta(R)$, Pacific J. Math. 96 (1982), pp. 107–122.

[5] Eric Phillips, *The zeta function of Riemann: Further developments of van der Corput's method*, Quart. J. Math. 4 (1933), pp. 209–225.

[6] R. A. Rankin, *Van der Corput's method and the theory of exponent pairs*, Quart. J. Math., Second Series, 6 (1955), pp. 147–153.

[7] B. R. Srinivasan, *The lattice point problem in many dimensional hyperboloids, III*, Math. Ann. 160 (1965), pp. 280–311.

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE
CALIFORNIA STATE UNIVERSITY
Los Angeles, California 90032

Received on 4.8.1983 (1369)