On the method of exponent pairs

by

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1. Introduction. The method of exponent pairs (henceforth abbreviated to m.e.p.) was introduced by van der Corput [2] for estimating certain one-dimensional exponential sums. We write, as is customary, $e(x)$ for $\exp(2\pi i x)$. The sums in question have the form

$$\sum_{X \leq x \leq X + Y} e(f(x)),$$

where $Y \leq X$ and $f(x)$ behaves rather like a power of $x$ (not a positive integer power). We shall introduce notation below to make explicit the similarity required between $f(x)$ and the power of $x$ (for one- and several-dimensional functions $f(x)$).

As is well known, such exponential sums are of great importance in analytic number theory. Let us suppose that $f(x)$ is similar to $FX^{-k}X^l$, then, in the refined form given by E. Phillips, the m.e.p. leads to an estimate of the form $| \sum_{X \leq x \leq X + Y} e(f(x))| \ll X^k X^l$, where $(k, l)$ is an exponent pair (e.p.). (We have modified Phillips' definition slightly for the purpose of this paper; in his definition $f' \approx F$.) It is trivial to see that $(0, 1)$ is an e.p. All other e.p. are produced by two processes $A$ and $B$. Process $A$ is essentially an application of Cauchy–Schwarz inequality, and process $B$ is essentially the combination of the Poisson summation formula with the method of stationary phase. Rankin [6] proved that $\inf \inf_{k/l \in \mathbb{Z}} (2k+l) = 0.829021356...$ (in fact, $(2k+l)$ corresponds to $k+l$ in his definition, where $E$ denotes the set of all e.p. obtained by the one dimensional m.e.p. Consequently the one dimensional m.e.p., when applied to the problem of bounding $\zeta(\frac{1}{2}+it)$, cannot give a better result than $\zeta(\frac{1}{2}+it) \ll t^\theta$ for $\theta = 0.16451067...$ As Rankin observed, this is inferior to what may be proved by using two dimensional exponential sums. Srinivasan [7] developed a m.e.p. for the estimation of many dimensional exponential sums $\sum_{x \in D} e(f(x))$ where $D \in \mathcal{D}, f \in \mathcal{F}$ (see Section 2), but he fails to take full advantage of the Cauchy–Schwarz inequality, and so it leads
to weaker results than one would like. The m.e.p. gives the same estimate for all \( f \in \mathcal{F}, D \in \mathcal{Y} \): 
\[
S \equiv \sum_{x \in D} e(\varphi(x)) \ll F^{\frac{n}{2}} X_1^{\frac{1}{2}} \ldots X_n^{\frac{1}{2}}. 
\]
The vector \( \lambda = (\lambda_0, \ldots, \lambda_n) \) is called an exponent \((n+1)\)-tuple, or we write \( \lambda \in E_n \) (note that the above definition differs from the corresponding definitions in [5]–[7]). As in the one-dimensional case, m.e.p. consists of repeated applications of processes \( A \in \mathcal{A} \) and \( B \in \mathcal{A} \), defined respectively by Lemmas 2 and 1 (in [5]–[7] Lemma 2 is used with \( k = 1 \) only). After applying processes \( A \) and \( B \) several times and estimating the last sum trivially, we obtain an exponent pair (which depends on the number of times we apply each of processes \( A \) and \( B \)).

The transformation \( B \) is applied whenever \( F^{\frac{n}{2} - 1/3} \ll A_k^{\frac{1}{2}} \equiv H_k(f(x)) \ll 1 \) and the remainder \( R = N^{1+\epsilon}/A_k \) in Lemma 1 is smaller than the principal one. If we could ignore the error term, then, denoting by \( S(F; X) \) the best estimate of the exponential sum \( \sum_{x \in D} e(f(x)) \) which can be obtained by m.e.p. for \( f \in \mathcal{F}(F, X, D) \), we could in such a case write Lemma 1 in the following form:

\[
(1) \quad S(F; X) \ll X^2 F^{-\frac{n}{2}} S(F; F/X_1, x_2, \ldots, F/X_n, X_{n+1}, \ldots, X_n) 
\]
for any \( k \in [1, n] \).

If we estimate the last sum trivially, then we get

\[
(2) \quad S(F; X) \ll N/F^{\frac{n}{2}}(X_1 \ldots X_n)^{-1} \ll N|H_k(f(x))|^{1/2}
\]
which gives a non-trivial estimate for \( S(F; X) \) if \( |H_k(f(x))| \) is small. If \( |H_k(f(x))| \) is not small, one applies a transformation \( A \) sufficiently many times to make it small. Lemma 2 reduces the estimation of \( S \) to the estimation of another exponential sum, \( \sum_{x \in D} e(g(x, h)) \), where \( g(x, h) = h_1f_1(x) + \cdots + h_nf_n(x) + g_1(x, h) \), where \( g_1(x, h) \) is usually small so that we have

\[
|H_k(g(x, h))| \ll |H_k(f(x))|^{\epsilon}, \quad \text{where} \quad \epsilon = h_1/X_1 + \cdots + h_n/X_n.
\]
If we take \( m = 1 \) (or \( A \in \mathcal{A} \)), then \( g(x, h) \in \mathcal{F}(P_1/X_1, X, D + h_1/X_1) \); also, after applying Lemma 1 to the last sum we get a new sum with a function, say, \( \varphi(x, h) \) such that for \( h_1 \in [H_1, 2H_1] \), \( \varphi \in \mathcal{F}(P_1/X_1, X, H_1, D + H_1/X_1) \). So, we can write transformation \( A \) for \( A \in \mathcal{A} \) as follows:

\[
(3) \quad |S(F; X)|^2 \ll N^{\epsilon} q + N^{-\epsilon} \max_{1 \leq h \leq q} S(F/\varphi, X; H) = N^{\epsilon} q + N^{-\epsilon} S(Fq/X_1, X; q)
\]
for any \( q \leq X \). If \( m > 1 \), then \( g(x, h) \notin \mathcal{F}. \) Also, while applying Lemma 2 with \( m > 1 \) allows us to reduce the values of \(|H_k(g(x, h))| \) much faster than if we take \( m = 1 \) and in a generic case \( |H_k(g(x, h))| \equiv |H_k(f(x))|^\epsilon \), it can be too small at some points of \( \mathcal{A} \) and we cannot use Lemma 1. Suppose, however, that \( \mathcal{F}_0 \subset \mathcal{F}, \triangledown \mathcal{F}_0 = \emptyset, \mathcal{A}_0 \subset \mathcal{A} \) are such that for all \( f \in \mathcal{F}_0, A \in \mathcal{A}_0, B \in \mathcal{A}_0 \), the inequality (1) holds and suppose that we can ignore the points at which \( H_k(g(x, h)) \) is very small so that the processes \( A \) can be written as

\[
(4) \quad S(F; X)^3 \ll N^2 Q_{-1}^{-1} + N^2 Q_{-1}^{-1} S(Fq, X, Q),
\]
where \( q = q_1/X_1 = \ldots = q_m/X_m = \sqrt{Q_{-1}/N_0}, \quad Q = q_1 \ldots q_m, \quad N_0 = X_1 \ldots X_m, \quad 1 \leq m \leq m(n, \mathcal{A}_0) \leq n. \) One can see that the conjecture that both (1) and (4) are true is equivalent to one conjecture:

\[
S(F; X_1, \ldots, X_n) \ll S(F; X_1) \ldots S(F; X_n),
\]

In particular, (1) and (4) hold if we take \( \mathcal{A}_0 = \mathcal{A}_1 \), \( \mathcal{A}_0 = \emptyset, \mathcal{F}_0 = \mathcal{F} \). Using (1) and (3), one can develop an improvement of the m.e.p. of [7] for \( A \in \mathcal{A} \).

One can, however, obtain a better estimate by using our Theorems 1–3.

**Theorem 1.** Let \( k_1, k_2 \) be positive integers with \( 1 \leq k_1, k_2 \leq 3 \), and let \( \alpha, \beta \) be real numbers such that \( \alpha, \beta \neq \text{integers}, \alpha + \beta \neq k_1, k_2 \), \( (x - k_1)^2 - \beta(x - k_2) \neq 0 \). Let \( A \) be small and let \( f(x, y) \) be a real \( C^\infty \) function in

\[
D = \{(x, y) \mid X \leq x \leq 2X, Y \leq y \leq 2Y} 
\]
such that

\[
f(x, y) \sim_\mathcal{F} Fx^{-\alpha} y^{-\beta} x^\alpha y^\beta, \quad X \geq Y, XY = N.
\]

Then

\[
S \equiv \sum_{(x, y) \in D} e(f(x, y)) \ll N/\sqrt{Q_1},
\]

where \( Q_1 \) is the largest number satisfying the following inequalities:

\[
(5) \quad Q_1 \ll \min \{(N^{2k_1}k_1^{-2k_1} - k_1^{-2k_1} + 2, N^{2k_2}k_2^{-2k_2} - k_2^{-2k_2} + 2), (N^{2k_1}k_2^{-2k_2} - 2k_2^{-2k_2} + 2, N^{2k_1}k_2^{-2k_2} - 2k_2^{-2k_2} + 2)\},
\]

where \( K_1 = k_1^{-2k_1}, K_2 = k_2^{-2k_2} \); also for \( k_1 = 3 \),

\[
4\sqrt{NA^2} \leq Q_1 \ll \min \{(N^2 F^{-3})^{11/15}, (N^2 F^{-3})^{1/13}, (N^4 (F^{-1})^{1/15})\};
\]

for \( k_1 = 2, \)

\[
\sqrt{NDA^2} \leq Q_1 \ll \min \{(F N^{-1})^{1/5}, (N^2 F^{-1})^{9/29}\};
\]

for \( k_1 = 1, Q_1 \ll \min \{A^{-1}, N^{2/7}\} \).
Note that the restrictions $1 \leq k_1, k_2 \leq 3$ and $(x-k_1-1)^2 + \beta(x-k_1) \neq 0$ can be removed and introduced to simplify the proof. Also, it seems that using the idea of the proof of the theorem one can develop the method of exponent pairs for double and possibly multiple sums. Note that $O(N^k)$ can be replaced with $O(\log^k N)$. Using Theorem 1, one can similarly to [4] show that $T(\zeta(\frac{1}{2} + it)) \ll \tau^{139/858}$, $A(R) \leq R^{139/420}$, and many other constants can be improved.

**Theorem 2.** Let $\alpha, \beta, \gamma \geq 1$, be real numbers, $\alpha + \beta + \gamma \neq 1$, $\alpha + \beta + \gamma \neq 2$, $\alpha \beta \gamma(\beta - 1)(\gamma - 1) \neq 0$, $\Delta \leq \varepsilon_0$, and let $f(x, y, z)$ be a real $C^\infty$ function such that $f(x, y, z) \geq F_{X^{-\alpha} Y^{-\beta} Z^{-\gamma}} x^{\alpha} y^{\beta} z^{\gamma}$ throughout

$$D = \{ (x, y, z) \mid X \leq x \leq 2Y, Y \leq y \leq 2Z, Z \leq z \leq 2Z \}.$$ 

Then

$$S \leq \sum_{x,y,z \in D} e(f(x, y, z)) \ll N \left[ F N^{-1} + N^{-1/2} + \frac{1}{N} + \frac{1}{N^2} + \frac{1}{N^2} \right] \log^2 N.$$ 

**Theorem 3.** Let $\alpha_1, \ldots, \alpha_n$ be real numbers such that $\alpha_1 + \alpha_2 + \ldots + \alpha_n \neq 0$, $\alpha_1 + \alpha_2 + \ldots + \alpha_n \neq 2$, $\alpha_1 + \alpha_2 + \ldots + \alpha_n \neq 1$ for any $\{x_1, \ldots, x_n\} \subset \{1, \ldots, n\}$. Let $n \geq 4$, $A \leq \varepsilon_0$, $F \geq X_1 \geq X_2 \geq \ldots \geq X_4$, $X_1, X_2, \ldots, X_4 \geq N$, and let $f(x)$ be a real $C^\infty$ function such that for $x \in D = \{ x \in X_i \leq x_i \leq 2X_i, i = 1, \ldots, n \}$

$$f(x) \geq F_{X_1^{-\alpha_1} \ldots X_n^{-\alpha_n} x_1^{\alpha_1} \ldots x_n^{\alpha_n}}.$$ 

Then

$$S \leq \sum_{x \in D} e(f(x)) \ll N^{1+\delta} \left[ (F^{1+\delta} X_1^{-\alpha_1} X_2^{-\alpha_2} \ldots X_n^{-\alpha_n} N^{-\delta})^{1/(\alpha+n)} + + X_3^{-1} \left( \sqrt{F^{3/2} X_3^{-1} X_1^{-1} X_2^{-1}} + X_2^{-1} X_3^{-1} \sqrt{F^{1/2} X_1^{1/2}} + + X_3^{-1} \sqrt{F^{3/2} X_1^{1/2} X_2^{1/2} + X_2^{-1} X_3^{1/2} \sqrt{F^{3/2} X_1^{1/2} X_2^{1/2} + X_2^{-1} X_3^{1/2} \sqrt{F^{3/2} X_1^{1/2} X_2^{1/2} + N^{-1/2(n+1)}}}} + + X_3^{-3/2} + \sqrt{\log^2 N} \right) \right] \log^2 N.$$

Theorems 1–3 are applied in the case when we reduce the value of $H(f(x))$ so that (2) gives a non-trivial estimate of $S(F; X)$. If the determinant is still “large”, then one can apply (3) several times and the corresponding Theorem 1, 2 or 3 after that. We can improve the above theorems under the restriction that we apply the transformations $A$ and $B \leq K$ times. Then similarly to [4] one can show that there exists a polynomial $P$, depending on $K$ only, such that for all $f \in \mathcal{F}$ with $P(\alpha) \neq 0$ one can use (1) and (4) $\leq K$ times.

Unfortunately, it is difficult to find $P$ explicitly. We think however (see Section 4) that (1) and (4) hold for all $f \in \mathcal{F}$ and all $m$. Assuming this, one can develop a general m.e.p.

**Theorem 4.** Let $m \in [1, n]$, $(\lambda_0, \ldots, \lambda_m) \in E_m$. Then $(\lambda_0, \lambda_1, \ldots, \lambda_m) \in E_m$, where $\lambda_0 = \lambda_0 + \lambda_1 + \ldots + \lambda_m - m/2$, $j = 1 - \lambda_j$ $(j = 1, \ldots, m)$, $j = \lambda_j$ $(j = m + 1, \ldots, n)$.

**Theorem 5.** Let $\lambda_j \geq 0$ $(j = 0, \ldots, n)$, $m \in [1, n]$, $(\lambda_0, \ldots, \lambda_m) \geq 0$ $(j = 1, \ldots, m)$, $\lambda_0 + |\lambda_j| \geq \lambda_0 + \lambda_1 + \ldots + \lambda_m \geq m$, $(\lambda_0, \lambda_1, \ldots, \lambda_m)$, $\lambda_0 + \lambda_1 + \ldots + \lambda_m \in E_{m+n}$. Then $\lambda_0, \lambda_1, \ldots, \lambda_m \in E_m$, where $\lambda_0 = \lambda_0 m/2(\lambda_0 + |\lambda_j|)$, $\lambda_j = (\lambda_j + \lambda_j - 1)m/2(\lambda_0 + |\lambda_j|)$ + + 1/2, for $j = 1, \ldots, m$, $\lambda_j = (\lambda_j - 1)m/2(\lambda_0 + |\lambda_j|) + 1$ for $j = m + 1, \ldots, n$.

While these theorems are conditional upon the truth of (1) and (4) for all $f \in \mathcal{F}$, they allow to estimate the upper bound of the estimates one can hope to obtain by using the general m.e.p.

Using the conditional Theorems 5 and 6, one can write a program so that a computer can find various possible exponent pairs (the algorithm is due to Enrico Bombieri). Indeed, we start with e.p. $(0, 0, 0, 0)$ and apply Theorem 4, then we apply Theorem 5 several times, etc.

Ending the process at any time, we obtain an exponent pair. Another algorithm allows to estimate the limits of the m.e.p. Suppose, we know the sizes of $F, X_1, X_2, \ldots, X_n$ depending on some parameter $T$. We apply (1) with some $k$ such that $X_1^2 \geq \ldots \geq X_n^2 \geq k^3$. If such $k$ do not exist, then we take $Q = N^2 T^{-\alpha}$ and if $Q \geq N$, then it is impossible to prove that $S(F; X) \ll T^\alpha$ by m.e.p. Otherwise we apply (4), etc. Using the algorithm, we discovered that one cannot show that $\zeta(\frac{1}{2} + it) \ll \tau^{1.608}$ by using the m.e.p. only. Under the condition that Theorems 5 and 6 are true, we found that

$$(6966 \mid 99727, 99727) \in E_2,$$

$$(47728 \mid 274910, 274910) \in E_2,$$

and $\zeta(\frac{1}{2} + it) \ll \tau^{1.608}$, where 47728/274910 = 0.16183920. For comparison, the best published unconditional result is $\zeta(\frac{1}{2} + it) \ll \tau^{1.608}$, [4], where 35/216 = 0.162037; Theorem 1 of this paper implies a little better constant: $\zeta(\frac{1}{2} + it) \ll \tau^{1.608}$, where 139/858 = 0.162004. Similar calculations can be done for other problems.

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2. **Notation.** Throughout the paper we suppose that all considered domains $D, D_1, \ldots$ belong to the set $\mathcal{O} = \mathcal{O}(x)$ of all subdomains of $\{ x \mid x_i \leq x_i \leq 2X_i, x_i \leq x_i \leq 2X_i, i = 1, \ldots, n \}$ such that the boundary of each of them

$$(6966 \mid 99727, 99727) \in E_2,$$

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and $\zeta(\frac{1}{2} + it) \ll \tau^{1.608}$, where 47728/274910 = 0.16183920. For comparison, the best published unconditional result is $\zeta(\frac{1}{2} + it) \ll \tau^{1.608}$, [4], where 35/216 = 0.162037; Theorem 1 of this paper implies a little better constant: $\zeta(\frac{1}{2} + it) \ll \tau^{1.608}$, where 139/858 = 0.162004. Similar calculations can be done for other problems.

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consists of \( C \) algebraic curves of degrees \( C \), \( x = (x_1, \ldots, x_n) \), \( i = (i_1, \ldots, i_k) \), etc. \( X_1, \ldots, X_n \) are sufficiently large numbers such that \( X_1 \geq \cdots \geq X_n \), \( X_1 \geq X_2 \), \( \cdots \), \( X_1 \geq X_n \geq N \); \( X_1 \cdots X_n = N \).

\( f(x) \leq g(x) \) means that \( f(x) = O(g(x)) \).

\( f(x) = g(x) \) means that \( f(x) = O(g(x)) \).

\( f(x) \gg g(x) \) means that \( f(x) \geq N^g(x) \).

\( f(x) \ll g(x) \) means that \( f(x) \leq N^g(x) \).

\( f(x) \sim g(x) \) means that \( f(x) = g(x) = O(g(x)) \) for all \( x \) and \( i \) for which it makes sense.

\( \mathcal{F} \equiv \mathcal{F}(F; X; d) \) is the set of all sufficiently many times differentiable real functions \( f(x) \) in \( D \) such that \( f(x) \sim F x^{1-i_1} \cdots x^{1-i_k} \) in \( X \), where \( x_{i_1} \cdots x_{i_k} \neq 0 \) for any \( \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\} \), \( X \Rightarrow \Upsilon \).

\( R(P_1(x_1, \ldots, x_k), \ldots, P_k(x_1, x_k)) \) — the resultant of polynomials \( P_1, \ldots, P_k \).

\( H_i(f(x)) \) is the determinant of the \( i \)-th principal minor of the Hessian.

\( H(f(x)) = H_i(f(x)) \).

\( C, C_2, \ldots \) — appropriate constants.

\( D(m; f) = \{(y_{i_1}, \ldots, y_{i_k}, x_{i_1} + \cdots + x_{i_k}, \ldots, x_{i_k}) \mid y_{i_j} = f_s(x), j = 1, \ldots, m; x \in \mathcal{F}\} \).

\( D_B = \{x \mid x \in D, j \leq x_j \leq j + 1, j = 1, \ldots, n\} \).

\( e(f) = \exp(2nf) \).

3. **Transformation B.** Transformation B is a combination of Poisson summation formula (or some variation) with the method of stationary phase:

**Lemma 1.** Let \( f(x) \) be a real \( C^\infty \) function such that for all \( x \in \mathcal{F} \) we have:

(i) For all integral \( a_i \in [-c, c] \) the functions \( \sum_{i \in a_i} \prod_{j \in a_i} f_i(x) \) do not change sign, where \( i = (i_1, \ldots, i_k), i_j = (i_1, \ldots, i_n), i_j \leq c \).

(ii) \( f^i(x) \leq A_{x_i}^{-1} x_{i_k}^{1-i} \) for some \( F \).

(iii) \( \int |f^i(x)| \leq (A_k)^{-1} \) for some \( k = 1, \ldots, n \) and some \( A_k < (X_1 \cdots X_n)^{2x} \times x^{k+i}\).

Then there exist real numbers \( B_j \) such that for \( D_1 = D_2(B_1, B_2, \ldots, B_{n-1}, B_{n+1}, D_3 = D_4(B_1, B_2, \ldots, B_{n-1}, B_{n+1}, B_{n+2}, a) \) we have

\( S = \sum_{x \in D} e(f(x)) \ll \sum_{x \in D} e(f_1(m, x_{k+1}, \ldots, x_n)) + O(N^2) + O(N^{1+c} \sqrt{F^{-1} A_k N_{k+1}^{1/2}}) \).

\( \ll |A_k^{1/2} + N^{1+c} \sqrt{F^{-1} A_k N_{k+1}^{1/2} + N^2 + N N_{k+2}^{1/2} A_{k+2}^{1/2} + X_{k+1} + \cdots + X_{k+2} (F/X + 1) \sqrt{A_k}, \quad \text{where} \quad m = (m_1, \ldots, m_i), \quad f_i(m, x_{k+1}, \ldots, x_n) = f_i(x_1, \ldots, x_n) = -f_i(x_1, \ldots, x_n), \quad f_i, f_2, \ldots, f_k \text{ are the solutions of the system} \quad f_j(x_1, \ldots, x_n) = m_j \quad \text{if} \quad j = 1, 2, \ldots, k. \)

**Proof.** We suppose that \( k = n \), the general case can be proved similarly.

Using the Poisson summation formula, we write

\[ S = \sum_{x \in D} \int e(f(x) - m \cdot x) dx + O(N^2), \]

where the last sum is over an \( n \)-dimensional rectangle containing \( D(n; f) \).

Denoting

\[ \varphi(x) \equiv \varphi_n(x) = F^{-1} (f(x) - m \cdot x) \]

and

\[ g(x) = \int_{\mathbb{R}^n} x_{i_k} (x_1 - y_1 \delta_1, \ldots, x_n - y_n \delta_n) U(y) dy, \]

where \( \delta_j = X_j / F^{1/4} A_k N_{k+1}^{1/2} \), \( U(y) = U_1(y_1) \cdots U_1(y_n) \) is the characteristic function of \( D \),

\[ U_1(y) = \begin{cases} c^{-1} e^{(2 \pi)^{-1} y} & \text{if } |y| < 1, \\ 0 & \text{if } |y| \geq 1, \end{cases} \]

\[ c = 1 \int e^{(2 \pi)^{-1} y} dy, \]

we obtain:

\[ S = \sum_{x \in D} \int g(x) e(F \varphi(x)) dx + O(N^2) + \sum_{x \in D} e(F \varphi(x)) dx + O(N^{1+c} \sqrt{F^{-1} A_k N_{k+1}^{1/2}}) + O(N^2). \]

Denoting

\[ g_1(x) = g(X_1 x_1, \ldots, X_n x_n), \quad \varphi(x) = \varphi(x_1, x_2, \ldots, x_n), \]

we obtain:

\[ S = N \sum_{x \in D} \int g_1(x) e(F \varphi(x)) dx + O(N^{1+c} \sqrt{F^{-1} A_k N_{k+1}^{1/2}}). \]

Now we use (3.17) of [1] to find an asymptotic expansion for the last integral:

\[ I(F) \equiv \int_{\mathbb{R}^n} g_1(x) e(F \varphi(x)) dx = F^{-n/2} |H(\varphi(x))|^{-1/2} (n/8) \ln H(\varphi(x)) + F + \varphi(x)^2). \]

\[ \{g_1(x) + \sum_{k=1}^n \delta_k (x)^{-1/2} \left( \left(-i 2F H^{-1}(x^2) D_x, D_x \right)^k x \right) \times (g_1(x) e(F h(x))) \]
for \( P_\delta(F; x^0) \) is too long; since it will be shown to be small later, we do not include it here); \( x^0 = (x_1^0, \ldots, x_n^0) \) is a stationary point of \( \varphi_1(x) \);

\[
\langle H^{-1}(x^0) D_x, D_x \rangle = \sum_{ij} h_{ij} B_{ij,j}(x);
\]

\( h_{ij} \) is the \((i, j)\) entry of the inverse matrix for the Hessian \( H(x) \) of \( \varphi_1(x) \) at \( x = x^0 \); Ind \( H(x) \) is the number of negative eigenvalues of \( H(x) \);

\[
h(x) = \varphi_1(x) - \varphi_1(x^0) - \frac{1}{2} \sum_{ij} (\varphi_{1,ij}(x^0)(x_i - x_i^0)(x_j - x_j^0)).
\]

For \( |i| \leq 4n/\delta = 2K \) we have

\[
\left| \frac{\partial^{\lceil |i|/2 \rceil}}{\partial x_1 \cdots \partial x_n} \varphi_1(x^0) \right| \leq F^{\lceil |i|/2 \rceil + \delta^{-1} A_1^2 \cdots A_n^2 \delta^{-1} A_n^{2n}};
\]

also, if for all \( x \in D \)

\[
|H(f(x))| \leq \delta F^s N^{-2} \quad \text{with} \quad \delta = F^{-s} N^{2s} A^{-1} > F^{d^{-1/2}},
\]

then

\[
h_{ij} \leq 1/\delta \leq F^{d^{-1/2} - \epsilon_0}
\]

so that each term in (7) is piecewise monotone as a function of each variable and bounded by an absolute constant; as in the proof of (3.40) of [1], one can show that

\[
P_k(F; x^0) \leq (\delta F^{-1/2})^{-k-1} \leq F^{-\epsilon_0 k^{1/2}} \leq F^{-n}.
\]

Using this together with (6) and (7) and applying Abel's summation formula, we obtain

\[
S \leq N \delta^{-1/2} F^{-n/2} \left| \sum_{m \geq d_2} e\left( f_1(m) \right) \right| + N^{1/4} \sqrt{A_n N^{-2}},
\]

where \( f_1(m) = F \varphi_1(x^0) = f(x(m)) - m \cdot x(m) \) and \( x(m) \) is the solution of the system \( \nabla f(x) = m \), where \( \nabla \) is the gradient vector.

This lemma is in applications better than Lemma 1 of [3], because the remainder contains only one \( A_k \), while the remainder in Lemma 1 of [3] contains \( A_1, \ldots, A_k \). However, the error term is too large in some cases and needs to be improved. If

\[
f(x) \geq A x_1 \cdots x_n,
\]

where

\[
A = F X_1^{-s_1} \cdots X_n^{-s_n},
\]

then

\[
f_1(m) \geq B m_1^{\beta_1} \cdots m_n^{\beta_n},
\]

where

\[
B = (1 - \xi_1 - \cdots - \xi_n) (x_1 \cdots x_n)^{1/(s_1 + \cdots + s_n - 1)},
\]

\[
\beta_j = \xi_j (x_1 + \cdots + x_n - 1) \quad (j = 1, \ldots, n),
\]

\[
m_j \geq F/X_j = M_j \quad (j = 1, \ldots, n), \quad f_1(m) \geq F.
\]

So, if \( f \in \mathcal{F}(F; X, d) \), then \( f_1 \in \mathcal{F}(F; M, d) \), and Lemma 1 implies that

\[
S(F; X) \leq N \delta^{-1/2} S(F; F/X_1, \ldots, F/X_n, X_{1,1}, \ldots, X_{n,1}) + N F^{-1/2}.
\]

While this inequality differs from (1) by a summand \( NF^{-1/2} \), in the applications this term is usually smaller than the principal term.

It may also be possible to improve it. Indeed, the error term comes from the estimate for \( R = \sum_x g_1(x) \), where \( g_1(x) = x_1 \cdots x_n - g(x), g_1(x) \neq 0 \) in a domain with the volume \( \leq \sqrt{A_n} F^{-1} \), \( 0 \leq g_1(x) \leq 1 \). In the case where \( D \) is such that it can be divided into \( \leq C \) subdomains in each of which

\[
\int_{-1}^{1} \cdots \int_{-1}^{1} \left| \frac{\partial^p}{\partial t_1 \cdots \partial t_n} \left( g_1(x_1 + t_1, \ldots, x_n + t_n) \right) \right| dt
\]

does not change sign (this is true if \( D \) is a parallelepiped; using some devices, one can often reduce the problem of estimating \( S \) to such case), then we can apply Abel's summation formula, and obtain

\[
R \leq \sum_{x \in D_0} e(f(x)), \quad |D_0| \leq \sqrt{A_n} F^{n-1},
\]

and it is possible to estimate it non-trivially. Because the last sum is of the same type as \( S \) but over a much smaller domain, it is natural to expect \( R \leq \epsilon_2 S \). In such case we get (1). If \( A_k \geq N_k F^{-1/2} \) (this does not happen if \( f \in \mathcal{F} \)), then one needs to estimate the sum differently. One expects that it is possible to find a subsum \( \sum_{x_{k_1}, \ldots, x_{k_m}} e(f(x)) \) such that \( H_{x_{k_1}, \ldots, x_{k_m}}(f(x)) \) is not small. Applying Lemma 1 to the above subsum, we can find a nontrivial estimate of \( S \).

4. Transformation \( A \).

**Lemma 2.** Let \( f(x) \) be a real function;

\[
f_1(x, h) = \int_0^1 \frac{d}{dt} f(x + th) \, dt.
\]

Let \( q_1, \ldots, q_m \) be positive numbers such that

\[
q_1/X_1 = \cdots = q_m/X_m = \sqrt{Q/N_m}, \quad \text{where} \quad Q = q_1 \cdots q_m, N_m = X_1 \cdots X_m;
\]

\[
D_1 = \{ [x, h], \quad x \in D, (x+h) \in D, h \neq 0, |h| \leq q_j \};
\]

\[
|D| \geq \sum_{x \in b} 1; \quad q_j/X_j \leq |D|/N.
\]
Then

$$|S|^2 \equiv \left| \sum_{x \in \mathcal{D}} e(f(x)) \right|^2 \leq |D|^2 Q^{-1} + |D| Q^{-1} \left| \sum_{(x, h) \in \mathcal{D}} e(f_1(h, h)) \right|.$$  

In applications the terms in the above sum with some of $h_j = 0$ are easier to estimate so that we will ignore them in the future.

While we do not know how to prove our conjecture (4) at this moment, we can try to "justify" it. If we apply Lemma 2 $k$ times, we obtain

$$|S|^{2k} \leq |D|^{2k} Q^{-2k-1} + |D|^{2k-1} Q^{-1-2k} \left| \sum_{(x, h) \in \mathcal{D}} e(f_2(x, h)) \right|,$$

where

$$h = (h^{(1)}, \ldots, h^{(k)}), \quad h^{(0)} = (h_{1j}, \ldots, h_{kj}),$$

$$\delta_{ij} = \langle q_{ij} \rangle, \quad q_{ij} \cdots q_{ijkl} = Q_j = Q_j^{2k-1} \quad (j = 1, \ldots, k),$$

$$D_2 = \{(x, h) | (x + h^{(1)} + \cdots + h^{(k)}) \in \mathcal{D} \text{ for } j = 0, \ldots, k \},$$

$$f_2(x, h) = \left\{ \sum_{0 \leq k \leq 1} \frac{\partial^k}{\partial \lambda_1 \cdots \partial \lambda_k} \right\} (f(x + \lambda_1 h^{(1)} + \cdots + \lambda_k h^{(k)})) \, dt \ll F_2 \cdots F_k \equiv F_2,$$

where

$$q_j = \left\{ \frac{|h_{ij}|}{X_1} + \cdots + \frac{|h_{m,j}|}{X_j} \right\}.$$  

We divide $D_2$ into $O(N^2)$ subdomains of the type:

$$D_3 = \{(x, h) | H_{ij} \leq \langle h_{ij} \rangle \leq (1 + e_0) H_{ij}, \quad X_i \leq \langle x \rangle \leq (1 + e_0) X_i \},$$

$$H_{ij}(f_2(x, h)) \equiv A_{ij}^{-1} = \delta F_2^{2k-1} (x, h) \in \mathcal{D}_2.$$  

If $\delta \gg F_2^{2k-1/3}$, then we use Lemma 1; otherwise, in a generic case, $|D_3| \ll \delta N \prod_{i \neq j} H_{ij}$, and there is a $j$ such that $\partial^2 \partial x_j f_2(x, h) \equiv F_2 X_j^{-2}$, and we can use Lemma 1 with $n = 1$. In both cases we obtain

$$|S|^{2k} \leq N^{2k} Q^{-2k-1} + N^{2k} Q^{-2k} \left| \sum_{(x, h) \in \mathcal{D}} e(f_3(m, h)) \right| +$$

$$+ \left( N^{1+e} F_2^{2k-1/2} + N^{1+e} F_2^{2k-1/3} X_k^{2/3} \prod_{i \neq j} H_{ij} \right) \sum_{i \neq j} H_{ij}.$$

where $D_4 = D_3(n, f_3), |D_4| \ll \delta N \prod_{i \neq j} H_{ij}$. So,

$$|S|^{2k} \ll N^{2k} Q^{-2k-1} + N^{2k} Q^{-2k} \left| \sum_{h_1 \in \mathcal{D}_1} \Delta^{1-2k} F_2^{-2k} \sum_{m, h} e(f_3(m, h)) \right| +$$

$$+ \left( N^{1+e} F_2^{2k-1/2} + N^{1+e} F_2^{2k-1/3} X_k^{2/3} \right) \sum_{i \neq j} H_{ij}.$$  

The remainder in the above inequality is usually smaller than the first term, and it can also be improved. If we can ignore the remainder, then

$$|S|^{2k} \ll N^{2k} Q^{-2k-1} + N^{2k} Q^{-2k} \max_{H_{ij} \neq \mathcal{D}_1} \delta^{1-2k} F_2^{-2k} \sum_{m, h} e(f_3(m, h)).$$

We can apply this to the last sum, etc. We can see that the maximum is attained when $\delta = 1$ and that we get similar inequalities as if (taking $k = 1$) we had $f_2(x, h) \in \mathcal{F}(F_2; X, H)$ with $H = (H_1, \ldots, H_m)$, $H_1 \leq q_j$, $q = H_1 X_1 + \cdots + H_m X_m$. This gives some justification to our conjecture that

$$|S|^{2k} \ll N^{2k} Q^{-2k-1} + N^{2k} Q^{-2k} \max_{H_{ij} \neq \mathcal{D}_1} \delta^{1-2k} F_2^{-2k} \sum_{m, h} e(f_3(m, h)).$$

Because the entries of any exponent $n$-tuple are non-negative, the maximum is attained at $H_j = q_j$, and the last inequality is the same as (4).

5. Estimates. In this section we will prove Theorems 1 and 3 and, assuming (1) and (4), Theorems 4 and 5. To prove Theorem 1, we need the following results:

**Lemma A.** Let $f(x, y)$ be a real $C^n$ function such that for any $c$, $f_{x,c} \in C^n$ and has $O(N^n)$ solutions; $|f_{x,1}| |f_{y,1}| F_X^{-1} Y^{-1}$ for some $F$ and all $(x, y) \in \mathcal{D} = \{(x, y) | x \leq x \leq 2X, \quad 0 \leq y \leq 2Y \}$; $|f_{x,1}| \equiv M_X^{-1} \gg F^{1+e} X^{-2}$; $|f_{x,2} f_{x,2} - f_{x,3} f_{x,3}| \equiv M_X^{-1} \gg M_Y^{-1}$; $N = XY$. Then

$$\left| \sum_{(x, y) \in \mathcal{D}} e(f(x, y)) \right| \ll |K|^{-\frac{1}{2}} \left| (M_X Y + M_X + F \sqrt{M_X}) X \right|.$$

**Lemma B.** Let $f(x, y)$ be a real $C^n$ function such that $|f_{x,1}| \equiv a M_X^{-1} X^{-2}$, $|f_{x,2} f_{x,2} - f_{x,3} f_{x,3}| \equiv b M_X^{-1} Y^{-2}$ for some $M$ and all $(x, y) \in \mathcal{D} = \{(x, y) | x \leq x \leq 2X, \quad 0 \leq y \leq 2Y \}$; $K = 2k, k \gg 2$. Then

$$\left| \sum_{(x, y) \in \mathcal{D}} e(f(x, y)) \right| \ll \min \left\{ \left| \mathcal{O}(a M_X^{-1} X^{-3}) \right|^{1/k^2-2} + \left| G_1^{-1-2k^2} X^{2k^2} \right| \right.$$  

$$\left. + \left| G_1^{-1} \Delta \left| (a M_X^{-1} X^{-3} \Delta)^{1/k^2-2} + |a|^{-1} \Delta \left( \Delta X^{2k^2} \right) \right| \right\}$$

$$+ \left| \mathcal{O}(b M_X^{-1} Y^{-3}) \right|^{1/k^2-2} + \left| G_1^{-1} \Delta \left| (b M_X^{-1} Y^{-3} \Delta)^{1/k^2-2} + |a|^{-1} \Delta \left( \Delta X^{2k^2} \right) \right| \right\}.$$  

**Lemma A** is obtained by applying Lemma 1 to the sum over $x$ and using van der Corput's estimate to the sum over $y$ after that. **Lemma B** can be proved by induction on $k$ and applying **Lemma 2** with $n = 1$ (for $k = 2$, **Lemma B** follows from van der Corput's estimate).

We denote $K = 2k^2, K_{ij} = 2^j, F_1 = F \sqrt{Q^{-1} X^{-N}}$. If $N \leq 31$ and $k_1 = 1$, then we use **Lemma B** with $k = 2$ and $k = 3$ and get:

$$S \ll N \cdot \min \left\{ \left| F^7 X^{-8} + F^{-1/2} \right|, \left| F^7 X^{-12} + X^{-17} + F^{-1/4} \right| \right\}$$

which proves **Theorem 1** in the considered case. If $k_1 \geq 2$ and $X^{2k_1-2}$
\[ L_{n+1} \leq N^{k_1+1} - F - Q_1^{2-S_{k_1/2}} \]
then we apply Lemma 2 \((k_1-1)\) times with \(n = 1\) and \(Q_1\) as in (5) and, using Theorem 1 of [3] (which, after some obvious changes connected with application of Lemma 1 of this paper) can be stated as follows:

\[ S \ll (\sqrt{F^2 N^3 + N^{5/6} + \sqrt{\delta^2 F^2 N^3 X^{-4}}}) \]

we obtain:

\[ (S/N)^{k_1} \ll Q_1^{-k_1/2} + N^{-1} Q_1^{-k_1/2} \sum_{h \neq 0} e(f_0(x, y)) \]

\[ \ll Q_1^{-k_1/2} + N^{-1} \left( \sqrt{F^2 Q_1^{-2} X^{-2k_1-2} N^3} + N^{5/6} + \sqrt{\delta^2 F^2 Q_1^{-2} X^{-2k_1-2} N^3} \right) \ll Q_1^{-k_1/4}. \]

In the above we have written \(f_0(x, y)\) for the integral

\[ \int_0^1 \int_0^1 \delta_{1-t} \cdot \left( f(x + h_1 t_1 + \ldots + h_{k_1+1} t_{k_1+1} - y) \right) \, dt_1 \ldots dt_{k_1+1}. \]

The obtained estimates prove Theorem 1 in the considered cases. Now we assume \(N^{3/2} \leq Y \leq \sqrt{N}\) and \(X^{2k_1-2} \geq N^{2k_1-1} F - Q_1^{-2+5k_1/2}\) for \(k_1 \geq 2\), and apply Lemma 2 with \(n = 2\) and \(Q_1\) as in (5).

We obtain:

\[ (S/N)^2 \ll Q_1^{-1} + N^{-1} Q_1^{-1} \sum_{h \neq 0} e(f_1(x, y)) \]

where

\[ f_1(x, y) = \int_0^1 \int_0^1 \delta_{1-t} \cdot \left( f(x + h_1 t_1 + \ldots + h_{k_1} t_{k_1} - y) \right) \, dt_1 \ldots dt_{k_1}. \]

For a fixed \((h_1, h_2)\) with \(h_1, h_2 \neq 0\) (otherwise the sum is easier to estimate and the estimate is smaller) we divide the domain \(D\) into subdomains with

\[ \delta_0 \sigma_{11} \ll |C_{30} \theta + C_{21} \sigma_{11}| \ll 2\delta_0 \sigma_{11}, \]

\[ \delta_0 \sigma_{12} \ll |C_{12} \theta + C_{03} \sigma_{11}| \ll 2\delta_0 \sigma_{11}, \]

\[ \delta_0 \sigma_{11} \ll |C_{30} \theta + C_{21} \sigma_{11}|C_{21} \theta + C_{03} \sigma_{11}| - (C_{21} \theta + C_{12} \sigma_{11})^2 \ll 2\delta^2 \sigma_{11}, \]

where \(\theta = y/x\), \(\sigma_{11} = h_2/h_1\), \(\delta_0 = \min(x, |\sigma_{11}|, C_{21} \delta_0 \sigma_{11}), \)

\[ \frac{\partial^2}{\partial x^2} f(x + h_1 t_1 + \ldots + h_{k_1} t_{k_1} - y) \]

and one domain which contains all remaining \((O(\sqrt{N})\)\) points. We assume that a subdomain \(D_1\), corresponding to the largest subsum, is defined by the above inequalities, and consider the following distinct cases (in each case we assume that the conditions of all previous cases are not satisfied):

I. \(c_1 \ll Q_1^{-1}\). Then we have

\[ |D_1| \ll \sqrt{N} + N \sigma_1 \ll N/Q_1, \quad S_1 \ll N/\sqrt{Q_1}. \]

II. \(c_1 \ll Q_1^{-1/2} \left( N^{1/2} F^{-2} (X/Y)^{-k_1/4} \right) \ll (3k_1 - 3)\).

We take \(Q_2 = \left( N^{1/2} F^{-2} (X/Y)^{-k_1/4} \right) \ll (3k_1 - 3)\) and apply Lemma 2 \((k_1-1)\) times with \(n = 1\):

\[ (S/N)^{k_1} \ll Q_1^{-k_1/2} + (N^{-1} Q_1^{-1} \sum_{h \neq 0} e(f_1(x, y)))^{k_1/2} Q_1^{-k_1/4} + \]

\[ + (N^{-1} Q_1^{-1} \sum_{h \neq 0} e(f_1(x, y))^{k_1/2} N^{-1} Q_1^{-1} \sum_{h \neq 0} e(f_1(x, y)))^{k_1/2} \ll Q_1^{-k_1/2}. \]

where

\[ k = (h_1, h_2, \ldots, h_{k_1+1}), \quad |h_1| \ll Q_2 = \left( N^{1/2} F^{-2} (X/Y)^{-k_1/4} \right) \ll (3k_1 - 3) \]

Here \(|D_1| \ll N \sigma_1 \ll N/\sqrt{Q_1}\); also since \(C_{30} \sigma_{11} \neq C_{21} \sigma_{11}\) and \(|\sigma_{11}| \ll \min(x, C_{21} \delta_0 \sigma_{11}) \ll c_1/\delta_0 \sigma_{11}\). To estimate the last sum, we use Lemma A and get:

\[ (S/N)^{k_1} \ll Q_1^{-k_1/2} + c_1^{k_1/2} \sqrt{F^2 Q_1^{-1}} c_2^{k_1/2} N^{-k_1/2} (X/Y)^{-k_1/4} + \]

\[ + c_1^{k_1/2} \left( F^2 Q_1^{-1} N^{-k_1/2} (X/Y)^{-k_1/4} \right) \ll Q_1^{-k_1/2}. \]

III. \(c_1 \gg Q_1^{-(1+2k_1/2) (X/Y)^{-k_1/4}} c_1^{k_1/2} N^{-k_1/2} (X/Y)^{-k_1/4} \ll Q_1^{-k_1/2}. \)

If \(k_1 = 1\), then we continue as in (7) below. Otherwise we apply Lemma 2 with \(n = 2\) and \(Q = Q_1^{1/2}\):

\[ (S/N)^2 \ll Q_1^{-1} + N^{-1} Q_1^{-1} \sum_{h \neq 0} e(f_2(x, y)) \]

where

\[ f_2(x, y) = \int_0^1 \int_0^1 \frac{\partial^2}{\partial x^2} f(x + h_1 t_1 + \ldots + h_{k_1} t_{k_1} - y) \, dt_1 \ldots dt_{k_1}. \]

Denoting

\[ \sigma_{12} = h_2/h_1 + h_3/h_1, \quad \sigma_{21} = h_2/h_1 + h_3/h_1, \quad \delta_1 = (y/x + h_2/h_1)(y/x + h_2/h_1), \]

we divide \(D_1\) into subdomains with

\[ \delta_1 \sigma_{12} \ll |C_{40} \sigma_{12} + C_{13} \sigma_{12} \theta + C_{22} \sigma_{22} | \ll |P_1(\theta)| \ll 2\delta^2 \sigma_{12}, \]

\[ \delta_1 \sigma_{21} \ll |C_{22} \sigma_{21} + C_{13} \sigma_{12} \theta + C_{04} \sigma_{22} | \ll |P_2(\theta)| \ll 2\delta^2 \sigma_{21}, \]

\[ \delta_2^2 \sigma_{22} \ll |P_1(\theta) P_2(\theta) - (C_{32} \sigma_{22} + C_{22} \sigma_{22} \theta + C_{13} \sigma_{22} ^2) | \ll |P_0(\theta)| \ll 2\delta^2 \sigma_{22} \]

and one domain which contains \(O(\sqrt{N})\) integral points. We suppose that the domain \(D_2\), defined by the above inequalities, corresponds to the largest subsum, and consider the following distinct cases:
Now we apply Lemma 1 and get
$$
(W_1^{k}\mid_{q_{1}}^{1/4} \leq \eta_{1}^{k_{1}-2} + \eta_{1}^{k_{1}-2} \mid (F(2, q_{1}^{k_{1}}) \mid_{q_{1}}^{1/2} - c_{0} - \epsilon_{1}^{k_{1}-2} \mid (N^{2} X^{2} - 2k_{1}^{1/4} - 1/X \sqrt{c_{0}} + c_{1}^{k_{1}-2} \mid Y_{1}^{1/4}(X \sqrt{c_{0}} - 1) \leq \eta_{1}^{k_{1}}/2.}
$$

C. \( c_{2} \geq (N^{4} F - 2) \mid_{q_{1}}^{1/2} \mid X^{2k_{1} - 4, 18}(15k_{1} - 22) \)

If \( k_{1} = 2 \), we continue as in 7) below. If \( k_{1} = 3 \), we apply Lemma 2 once more with \( n = 2 \) and \( k = q_{1}^{1/4} \) and get
$$
(S/N)^{k} \leq \eta_{1}^{k} + \eta_{1}^{k} \mid \sum_{h_{1} \neq h_{2}} \eta_{1}^{k} \mid f_{3}(x, y) dt.
$$

We denote
$$
\phi_{ij}(\theta) = C_{i_{j}, 2} + C_{i_{j}+1, 2} \theta \mid_{q_{1}}^{1/2} \sigma_{1} + C_{i+1, 2} \theta \mid_{q_{1}}^{1/2} \sigma_{2} + C_{i_{j}, j+3} \sigma_{3},
$$

where \( \sigma_{1}, \sigma_{2}, \sigma_{3} \) are the elementary symmetric functions of \( h_{1}/h_{3}, h_{1}/h_{3}, h_{1}/h_{3} \); 

$$
\phi_{ij}(\theta) = \eta_{1}^{k} \mid_{q_{1}}^{1/4} \mid \eta_{1}^{k} \mid (x + h_{1} t_{1} + h_{3} t_{2} + h_{5} t_{3} + y + h_{1} t_{1} + h_{4} t_{2} + h_{6} t_{3}) dt.
$$

For a fixed \( h = (h_{1}, \ldots, h_{6}) \) we divide the domain \( D_{3} \) into one subdomain containing \( O(\sqrt{N}) \) integral points and domains defined by inequalities of the type

$$
\delta_{1} a_{3} \leq \phi_{2} a_{3} \leq 2\delta_{2} a_{3}, \quad \delta_{2} b_{3} \leq \phi_{2} b_{3} \leq 2\delta_{2} b_{3}, \quad \delta_{3} c_{3} \leq \phi_{2} c_{3} \leq 2\delta_{3} c_{3}.
$$

We suppose that a \( D_{3} \), corresponding to the largest subsum, is defined by the above inequalities, and consider the following cases:

1. \( \sqrt{N^{4} F - 2} \mid_{q_{1}}^{1/2} \mid X^{2k_{1} - 4, 18}(15k_{1} - 22) \)

Using Lemma 1, we obtain
$$
(S/N)^{k} \leq \eta_{1}^{k} + \eta_{1}^{k} \mid \sum_{h_{1} \neq h_{2}} \eta_{1}^{k} \mid f_{3}(x, y) dt.
$$
Applying Lemma 1, we obtain

\[ N^{-1} Q_1^{-7} \sum_{k} \| \sum_{x,y} e(f_3(x,y)) \| \ll F^{\sqrt{Q_1} c_3 N^{-5}} + (F c_3 \sqrt{Q_1} N^{-3})^{-1/2} \ll Q_1^{-4}. \]

3) \( \max |a_3, b_3, c_3| \ll \sqrt{N^3 F^{-2} Q_1^3}, \ c_3 \gg \sqrt{Q_1^4/N}. \)

\[ \max |a_3, b_3| \gg (c_3^2 + c_3 Q_1^2) \sqrt{N^3 F^{-2} Q_1^3}. \]

We denote \( \gamma_1 = \max |a_3, b_3, c_3 \sqrt{c_3}|, \ \gamma_2 = \min |a_3, b_3, c_3| \). If \( \gamma_1 \) is small, then, since \( \varphi_3 \) we have

\[ \max |\varphi_2 \theta(\theta)|, |\varphi_2 \varphi_2(\theta)|, |\varphi_2 \varphi_2(\theta)| \ll \gamma_1 \delta_2, \]

we denote \( u = \sigma_3/\theta, v = \sigma_3/\theta^2, w = \sigma_3/\theta^3 \) and solving the above system for \( u, v, w, \) we obtain that \( \sigma_1 = d_2 \theta + o(\gamma_1, \theta), \ \sigma_2 = d_2 \theta^2 + o(\gamma_1, \theta); \ c_3 = d_3 \theta + O(\gamma_1, \theta); \)

\[ h_d/h_3 = d_2 h_2/h_1 + o(\gamma_1, \theta), \quad h_e/h_3 = d_2 h_2/h_1 + o(\gamma_1, \theta), \]

\[ N^{-1} Q_1^{-7} \sum_{h} |D_3| \ll (\gamma_2 (1 + Q_1^{-2})^2 (1 + Q_1^{-1})), \quad Q_1^{-7} \sum_{h} \ll (1 + Q_1^{-2}) (1 + Q_1^{-1}). \]

Applying Lemma A, we obtain

\[ N^{-1} Q_1^{-7} \sum_{k} \sum_{x,y} e(f_3(x,y)) \ll N^{-1} Q_1^{-7} \sum_{h,x,y} \sqrt{F^2 Q_1^3 N^{-5}} c_3 + \]

\[ + Q_1^{-7} \sum_{h} \| [F^2 Q_1^3 N^{-5}]^{1/4} \max |a_3^{1/2}, b_3^{1/2}, c_3^{1/2}] + Y^{-1} c_3^{1/2} \ll Q_1^{-4}. \]

4) \( \sqrt{Q_1^4/N} \ll \gamma_1 \ll \sqrt{N^3 F^{2} Q_1^3}, \)

\[ \max |a_3, b_3| \ll (c_3^2 + c_3 Q_1^{-2}) \sqrt{N^3 F^{2} Q_1^3}. \]

Acting similarly to 3), one can show that if \( \max |a_3, b_3, c_3| \ll 1, \) then

\[ \frac{\partial^3}{\partial x^3} (f_3(x,y)) \ll F X^{-3} \]

and, using Lemma B with \( k = 3, \)

\[ (S/N)^8 \ll Q_1^{-4} + N^{-1} Q_1^{-7} \sum_{h} \sum_{x,y} (F^4 X^{-3})^{1/6} + X^{-1/4} Q_1^{-7} (\sum_{h,x,y} 1/N)^{1/4} + \]

\[ + (F_1)^{1/4} Q_1^{-7} (\sum_{h,x,y} 1/N)^{1/2} \]

\[ \ll Q_1^{-4} + (F^2 Q_1^3 N^{-5})^{1/6} + X^{-1/4} Q_1^{-7} (\sum_{h,x,y} 1/N)^{1/4} + \]

\[ + (F_1)^{1/4} Q_1^{-7} (\sum_{h,x,y} 1/N)^{1/2} \ll Q_1^{-4}. \]

5) \( c_3 \ll \sqrt{Q_1^4/N}, \max |a_3, b_3| \ll \sqrt{N^3 F^{-2} Q_1^3}. \)

If \( \max |a_3, b_3| \ll (c_3^2 + c_3 Q_1^{-2}) \sqrt{N^3 F^{-2} Q_1^3}, \) then \( N^{-1} Q_1^{-7} \sum_{h} |D_3| \ll Q_1^{-4}, \)

and \((S/N)^8 \ll Q_1^{-4}. \) Otherwise we use Lemma B with \( k = 2 \) to get (if, say, \( b_3 \geq a_3 \), which is the worst case):

\[ N^{-1} Q_1^{-7} \sum_{h} \sum_{x,y} e(f_3(x,y)) \ll \ll Q_1^{-4} + N^{-1} Q_1^{-7} \sum_{h} \sum_{x,y} (F_1 X^{-3})^{1/6} + X^{-1/4} Q_1^{-7} \ll Q_1^{-4}. \]

6) \( a_3 \gg \sqrt{N^3 F^{2} Q_1^3}, \)

\[ a_3 \ll c_3 \ll \sqrt{Q_1^4/N}, \]

\[ c_3 \ll \sqrt{Q_1^4/N}. \]

Dividing \( D_3 \) into subdomains where \( \| \varphi_2 (\theta) \| \gg a \cdot (y/x)^2, \) we obtain

\[ N^{-1} Q_1^{-7} \sum_{h,x,y} e(f_3(x,y)) \ll \ll F^{\sqrt{Q_1} c_3 N^{-5}} N^{-1} Q_1^{-7} \sum_{h,x,y} |D_3| + Q_1^{-4} \]

\[ \ll Q_1^{-4} + \sqrt{Q_1} N^{-5} \sum_{h} \sum_{x,y} (F_1^{1/2} Q_1^{1/2} N^{-5})^{1/2} \ll Q_1^{-4}. \]

If \( a_3 \ll \sqrt{N^3 F^{2} Q_1^3}, \) then we divide \( D_3 \) into subdomains with \( \| \varphi_2 (\theta) \| \gg a \cdot (y/x)^2, \) and if \( a \gg \sqrt{F^2 Q_1^{11} N^{-5} Y^{-4}} \), then \( |D_3| \ll N c_3/a \) and, as above, we get

\[ (S/N)^8 \ll \ll Q_1^{-4} + \sqrt{F^2 Q_1^{11} N^{-5} Y^{-4}} + N^{-1} |D_3| \ll Q_1^{-4}. \]

If \( a \ll \sqrt{F^2 Q_1^{11} N^{-5} Y^{-4}} \), then, as in 3), one can show that

\[ Q_1^{-7} \sum_{h} \ll \| (a_3 + c_3 + a + Q_1^{-2}) \| (a_3 + c_3 + Q_1^{-1} + a) \| D_3 \| \ll \sqrt{a_3} |D_3|. \]

If \( a_3 \gg (c_3 + a^2 + Q_1^{-4}) \sqrt{N^3 F^{2} Q_1^3}, \) then as above we obtain

\[ (S/N)^8 \ll Q_1^{-4} + Q_1^{-7} (F^2 Q_1^{13} N^{-5} a_3)^{-1/4} \ll Q_1^{-4}; \]

otherwise use Lemma B to get

\[ (S/N)^8 \ll Q_1^{-4} + \sqrt{F^2 Q_1^{13} N^{-5} Y^{-4}} + N^{-1} Q_1^{-7} \sum_{h} \sum_{x,y} 1 \ll Q_1^{-4} \]

\[ \ll Q_1^{-4} + \sqrt{F^2 Q_1^{13} N^{-5} Y^{-4}} + (F^2 Q_1^{13} N^{-5} Y^{-4}) \times (\sqrt{F^2 Q_1^{13} N^{-5} Y^{-4}} + Q_1^{-1}) \ll Q_1^{-4}. \]

\]
7) $c_3 \gg N^5 Q_1^{-15} F^{-2}$. We define

$$
\Phi_{0,0}(f) = -f_2; \quad \Phi_{0,1}(f) = f_2; \quad \Phi_{0,2}(f) = -f_2; \quad M(f) = f_2 f_3 - (f_0)^2;
$$

$$
\Phi_{i,i+1}(f) = [H(f)]^{2i+2j-1} \left[ -\frac{\partial}{\partial x}(\Phi_{i,i}(f)(H(f)^{3-2i})) f_2 y + \right.
$$

$$
\left. + \frac{\partial}{\partial y}(\Phi_{i,i}(f)(H(f)^{3-2i})) f_2 x \right];
$$

and

$$
\Phi_{i+1,1}(f) = [H(f)]^{2i+2j-1} \left[ \frac{\partial}{\partial x}(\Phi_{i,i}(f)(M(f)^{3-2i})) f_2 y - \right.
$$

$$
\left. - \frac{\partial}{\partial y}(\Phi_{i,i}(f)(M(f)^{3-2i})) f_2 x \right];
$$

for $i, j \geq 2$; $\varphi_2(\theta), \ldots, \varphi_{k_2+4}(\theta)$ are the polynomials obtained from $\Phi_{k_2+2,0}(f(x, y)), \ldots, \Phi_{0,0,k_2+4}(f(x, y))$ respectively by replacing $f_{k_2}$ with $\varphi_i(\theta)$ (they all are polynomials of degree $k_1(3k_2+1)$ in $\theta$ and of degree $(3k_2+1)$ in $\sigma_1, \sigma_2, \sigma_3$).

$$
\psi_{i,m}(\theta) = \sum_{j=0}^{m} \varphi_{i+j+2}(\theta) \partial^{m-j} \sigma_{i,m};
$$

$\sigma_{i,m}$ is the $i$th elementary symmetric function of $\sigma_1 = h_{2,2}/h_{1,1}, \ldots, \sigma_m = h_{2m}/h_{2m-1,1}$.

$$
\begin{align*}
\psi_{1,1} &= \psi_{2,2} = \psi_{4,4} = \psi_{6,6} = -\psi_{2,1} \psi_{1,1} - \psi_{3,2} \psi_{2,2} - \psi_{4,3} \psi_{3,3} - \psi_{5,4} \psi_{4,4} - \psi_{6,5} \psi_{5,5}; \\
\psi_{1,2} &= \psi_{2,3} = \psi_{3,4} = \psi_{4,5} = \psi_{5,6} = \psi_{6,7} = 0.
\end{align*}
$$

We divide the domain $D_3$ into $O(N^5)$ subdomains and take a subdomain $D_4$, corresponding to the largest subsum, where, say,

$$
d_1(\delta_2)^{3k_2+3} \leq |\varphi_j(\theta)| \leq 2d_1(\delta_2)^{3k_2+3} \quad (j = 2, \ldots, k_2 + 4),
$$

$$
d_2(\delta_2)^{3k_1 + 2k_2 - 3} \leq |\varphi_j(\theta)| \leq 2d_2(\delta_2)^{3k_1 + 2k_2 - 3} \quad (j = k_2 + 5, k_2 + 6).
$$

We take the largest $Q_2$ such that

$$
Q_2 \leq \min \begin{cases}
\max \{ F_{2k_2+2} N^{-k_2-2} 1/10k_2-1, \\
( F_{k_2+2} N^{-3k_2-2} 1/(3k_2-3), Q_1; (N^2 Q_1^{-1}) 1/(6k_2-2) \},
\end{cases}
$$

and

$$
Q_2 \leq N^{k_2} F^{-4} Q_{1-2k_1} \quad \text{for } k_2 = 1;
$$

$$
Q_2 \leq N^{k_2} F^{-10} Q_{1-11k_1} \quad \text{for } k_2 = 2;
$$

$$
Q_2 \leq \max \{ F_{18} N^{-2k_1} 1/30, (F_1 N^8 N^{-7}) 1/10 \} \quad \text{for } k_2 = 3.
$$

If $\min \{ d_{k_2+5,1}, d_{k_2+6,1} \} \leq Q_2^{2k_1-1}$, or $\min \{ d_{k_2+3,1}, d_{k_2+4,1} \} \leq Q_2^{-1/2(k_2+1)}$, then it is easy to see that $Q_1^{-1} \sum_k |D_k| \leq Q_2^{1/2}$, and we can estimate the sum in (8) below trivially to get the needed estimate. Now we assume that

$$
a_2 \equiv \min \{ d_{k_2+3,1}, d_{k_2+4,1}, d_{k_2+5,1}, d_{k_2+6,1} \} \ll Q_2^{-1},
$$

and use Lemma 1 to get

$$
(S/\sqrt{N})^{k_1} \ll Q_1^{-K_1} F_1 Q_2^{1-1/k_1} \sum_{k_1} \sum_{k=0} \sum_{\varphi(\theta, u, v)},
$$

where the domain of summation over $(u, v)$ and $\varphi(\theta, u, v)$ are as in Lemma 1,

$$
Q_1^{-1} \sum_{k=0} \sum_{\varphi(\theta, u, v)} \ll Q_1^{-k_1} F_1^{-N} Q_2^{-1} \sum_k |D_k| \ll F_1^{-N} Q_2^{-1} a c_{k_2+1} \equiv N_1.
$$

So, if $a^2 c_{k_2} \ll Q_2^{-1}$, then

$$
(S/\sqrt{N})^{k_1} \ll Q_1^{-k_1} F_1 Q_2^{-1/2} \ll Q_2^{-k_1}. \quad (8)
$$

Now we assume that $a^2 c_{k_2} \gg Q_2^{-1}$. If $k_2 \geq 2$ and $(X/Y)^{k_2-1} \gg Q_2^{2k_2-1} + N Q_2^{2k_2-2} N_0^{-k_2-1/3}$, then we apply Lemma 2 $(k_2-1)$ times with $n = 1$ and then continue as below to prove the needed estimates. Also, if $X/Y \gg Q_2 + N Q_2^{2k_2-2} N_0^{-k_2-1/3}$, then we apply Lemma 2 once with $n = 1$ and then continue as below. Now we assume that $X/Y \ll \min \{ Q_2; N Q_2^{2k_2-2} N_0^{-k_2-1/3} \}$, and consider the following distinct cases:

$$
a(\delta_2)^{3k_2-1} (X/Y)^{k_2-1} \ll N_{k_2-2} X_{k_2-2}^{1-2k_2} + Q_1^{-k_2} \sum_{k=0} \sum_{\varphi(\theta, u, v)} \sum_{k=0} \sum_{\varphi(\theta, u, v)},
$$

where

$$
g_{1}(u, v) = \int_{0}^{1} \int_{0}^{1} \frac{q^2}{\partial x \partial y} (q(u, v + t_{1} h_{1} + \ldots + t_{k_2} h_{k_2})) dt,
$$

$K(t) = (h_{1}, \ldots, h_{k_2}); \quad |h_{j}| \leq Q_2^{j-1} \quad (j = 1, \ldots, k_2);
We consider the following distinct cases.

1. \( c_{4,1} \ll d_{k_2 + 5}/Q_2 \) or \( c_{4,2} \ll d_{k_2 + 6}/Q_2 \).

In either case we have

\[
Q_2^{-1} \leq \sum_{u(1)} a_{g_2}(u, v) \ll (N_0^2 \sqrt{c_{4,1}/Q_2})^{K_2} + N_0^{K_2 - 1} \cdot Q_2^{2 - K_2/2} \cdot \frac{1}{\delta_{k_2 + 6}^2} \cdot \frac{\delta_{k_2 + 1}^2}{\delta_{k_2 + 1} \cdots \delta_{k_2 + 1}}(g_1(u, v + h_{21} t, v + h_{21} t)) dt,
\]

we use Lemma A to get

\[
|S|^{1/2} \ll (N_0 \sqrt{c_{4,1}/Q_2})^{K_2} + N_0^{K_2 - 1} \cdot Q_2^{2 - K_2/2} \cdot \frac{1}{\delta_{k_2 + 6}^2} \cdot \frac{\delta_{k_2 + 1}^2}{\delta_{k_2 + 1} \cdots \delta_{k_2 + 1}}(g_1(u, v + h_{21} t, v + h_{21} t)) dt,
\]

we divide the domain \( D_4 \) into subdomains where

\[
a_4 \cdot d_{k_2 + 2} \cdot \delta_{k_2 + 1} \leq |\varphi_{11}(\theta)| \leq 2a_4 \cdot d_{k_2 + 2} \cdot \delta_{k_2 + 1},
\]

\[
b_4 \cdot d_{k_2 + 2} \cdot \delta_{k_2 + 1} \leq |\psi_{11}(\theta)| \leq 2b_4 \cdot d_{k_2 + 2} \cdot \delta_{k_2 + 1},
\]

\[
c_4 \cdot d_{k_2 + 2} \cdot \delta_{k_2 + 1} \leq |\psi_{11}(\theta)| \leq 2c_4 \cdot d_{k_2 + 2} \cdot \delta_{k_2 + 1},
\]

\[
c_{4,2} \cdot d_{k_2 + 2} \cdot \delta_{k_2 + 1} \leq |\psi_{11}(\theta)| \leq 2c_{4,2} \cdot d_{k_2 + 2} \cdot \delta_{k_2 + 1}, \quad (j = 1, 2).
\]

Here we denote \( \delta_{1,m} = \prod_{j=1}^{m} (y/x + |h_{2j+1}/h_{2j+1}|) \). We assume that the domain \( D_5 \), defined by the above inequalities, corresponds to the largest sum. One can see that

(9) \[ \max \{a_4, b_4\} \gg d_{k_2 + 5} \cdot c_4^2, \quad \max \{a_{4,2}, b_4\} \gg d_{k_2 + 5} \cdot c_4^2, \quad \max |a_{4,2} + a_4 + b_4 + a_4^2| \gg d_{k_2 + 5} \cdot c_4^2 \]

If \( c_{4,1} \ll b \), then we use Lemma B with \( k = 2 \) and, arguing as above, we obtain the same result.

2. \( \min \{c_{4,1}, c_{4,2}\} \gg d_{k_2 + 5}/Q_2 \) and

\[
\min \{a_{4,2} d_{k_2 + 2}, b_d d_{k_2 + 2}, c_4 d_{k_2 + 6} \cdot |c_{4,2}/d_{k_2 + 6}|^{1/2} \} \ll \delta_{k_2 + 1} \cdot (6k_2 - 8)
\]

or

\[ c_{4,1} \ll d_{k_2 + 5}, \]

where

\[
\delta_{k_2 + 1} \gg d_{k_2 + 5} \cdot \max \{N_0^{k_2 - 8}/3k_3 F_1^{-1/2} Q_2^{-1}, N_0^{k_2 + 1} N^{-1} Q_2^{-3k_2} / 3k_3\}.
\]
As in 1 we can show that
\[ Q_2^{-1} \sum_{d \in D_5} |D_d| \leq \langle 1/\sqrt{Q_2} + \delta_{c_4} Q_2^{-2} \rangle |D_d| \]
and
\[ |S_1|^{K_2} \leq N_1^{K_2} (Q_2^{-k_2 + 2} + \delta_2 K_2^{2/4}) Q_2^{-k_2/4} + N_0^{K_2} Q_2^{-k_2/4} |\langle c_4, d_2, d_3, c_4^{-1}, d_2^{-1}, d_3^{-1} \rangle | D_d | \]
3. The inequalities of 2 are not satisfied. If \( k_2 = 1 \), then, as in 2, we obtain
\[ |S_1|^{K_2} \leq R_0 + F_1^* c_{k_1} N^{-2} (Q_2 N^2 F_1^{-4})^{1/2} \leq R_0. \]
If \( k_2 \geq 2 \), then we apply Lemma 2 with \( n = 2 \) and \( Q = Q_2^{-2} \) once more: 
\[ |S_1|^{K_2} \leq |S_1|^{K_2} + |S_2|^{K_2} \]
where
\[ \delta_3 = \min \left\{ 1; c_4, d_2, d_3, c_4, d_2, d_3, 6; a_2^{d_2 + 4} + b_2^{d_2 + 2} c_4 / d_2, \right\} \]
We divide the domain \( D_5 \) into subdomains and take one of them, \( D_6 \), corresponding to the largest sum, where, say,
\[ a_5 \delta_5^{d_5 + 1} \delta_5 \leq |\psi_{0,2}(\theta)| \leq 2 a_5 \delta_5^{d_5 + 1} \delta_5, \]
\[ b_5 \delta_5^{d_5 + 1} \delta_5 \leq |\psi_{2,1}(\theta)| \leq 2 b_5 \delta_5^{d_5 + 1} \delta_5, \]
\[ c_5 \delta_5^{d_5 + 1} \delta_5 \leq |\psi_{1,2}(\theta)| \leq 2 c_5 \delta_5^{d_5 + 1} \delta_5, \]
\[ c_5 \delta_5^{d_5 + 1} \delta_5 \leq |\psi_{2,1}(\theta)| \leq 2 c_5 \delta_5^{d_5 + 1} \delta_5, \]
Here
\[ \max \{ a_3; b_3 \} \gg c_4, \max \{ a_5; c_5 \} \gg c_4, \]
and, denoting
\[ \delta_4 = \min \left\{ \sqrt{c_4 / c_4}; \sqrt{c_4 / c_2}; a_4 / a_4; b_4 / b_4; c_4 / c_4; 1 \right\}, \]
we get
\[ Q_2^{-3} \sum_{d \in D_6} |D_d| \leq \delta_4 |D_d|. \]
So, if \( \delta_3 \delta_4 a_4 c_4 \ll Q_2^{-2} \), then, estimating the sum in (10) trivially, we get
\[ |S_1|^{K_2} \leq R_0. \]
Now we assume that \( \delta_3 \delta_4 a_4 c_4 \ll Q_2^{-2} \), and consider the following distinct cases:
(a) \( \delta_4 \ll Q_2^{-1} \). If
\[ Q_2^{-2} \gg a_2^{k_2 + 2} k_2^{-2} d_2^{-2} (d_2^{-2} + 2^{-1/2} - 2) \delta_2^{-2} \]
then we denote \( g_4(u, v) = g_3(u, v) \), if \( k_2 = 2 \) and \( g_4(u, v) = g_3(u, v + h_3, 1) \), if \( k_2 = 3 \) and apply Lemma 2 (\( k_2 = 2 \)) times with \( n = 1 \) and appropriate \( Q \) first and Lemma A after that:
\[ |S_1|^{K_2} \leq R_0 + (F_1^* a_2 c_{k_1} / N_1^{K_2} (d_2 \delta_2)^{K_2^{2/4}} Q_2^{-2} + 1) \]
\[ + (F_1^* a_2 c_{k_1} / N_1^{K_2} (d_2 \delta_2)^{K_2^{2/4}} Q_2^{-2} \frac{1}{2} \sum_{d \in D_6} |D_d| \]
\[ \ll R_0 + (F_1^* a_2 c_{k_1} / N_1^{K_2} (d_2 \delta_2)^{K_2^{2/4}} Q_2^{-2} + 1) \]
\[ + (F_1^* a_2 c_{k_1} / N_1^{K_2} (d_2 \delta_2)^{K_2^{2/4}} Q_2^{-2} \frac{1}{2} \sum_{d \in D_6} |D_d| \]
If \( c_{k_1} \ll \delta_4 \), then we use the inequality
\[ c_4, a_4 \ll a_3 / \sqrt{b_3 + c_5}, \]
and, applying Lemma B, obtain
\[ |S_1|^{K_2} \ll R_0. \]
(b) \( Q_2^{-2} \ll \delta_4 \ll (Q_2^{-2} + N_1^{-a_2 / k_2}) (d_2 a_4 c_4) \delta_2^{-2} (d_2 a_4 c_4) \delta_2^{-2} \]
\[ = f, \text{ or } k_2 = 2 \text{ and } \delta_4 \gg Q_2^{-1}. \]
Similarly to (a) we get
\[ |S_1|^{K_2} \ll R_0. \]
(c) \( k_2 = 3, \delta_4 \gg f \). We apply Lemma 2 with appropriate \( Q \) and \( n = 2 \):
\[ |S_1|^{K_2} \ll (N_1^{-c_4 / c_4} Q_2^{k_2} + (F_1^* a_2 c_{k_1} / N_1^{K_2} (d_2 \delta_2)^{K_2^{2/4}} \]
\[ \times d_2 ^{K_2 / 4} \sum_{d \in D_6} |D_d| \]
\[ \ll R_0. \]
where
\[ g_5(u,v) = \int_0^1 \frac{1}{t} g_3(u+h_{5,1}t, v+h_{6,1}t) dt. \]

We again divide the domain \( D_6 \) into subdomains and take one of them, \( D_7 \), corresponding to the largest subsum, where
\[
\begin{align*}
|a_6|^{\delta_{1,1}^{2}+1} &\leq |\varphi_{0,1}(0)| \leq 2|a_6|^{\delta_{1,1}^{2}+1},
|b_6|^{\delta_{1,1}^{2}+1} &\leq |\varphi_{3,1}(0)| \leq 2|h_6|^{\delta_{1,1}^{2}+1},
|c_6|^{\delta_{1,1}^{2}+1} &\leq |\varphi_{4,1}(0)| \leq 2|c_6|^{\delta_{1,1}^{2}+1},
|d_6|^{\delta_{1,1}^{2}+1} &\leq |\varphi_j(0)| \leq 2|d_6|^{\delta_{1,1}^{2}+1} (j = 1, 2).
\end{align*}
\]

Here \( \max \{a_6, b_6, c_6\} \geq c_{5,1} \), \( \max \{a_6, c_6\} \geq c_{5,1} \), and \( \max \{|a_5|, \sqrt{a_5}b_5+c_{6,1}\} \geq c_{5,1} \). We consider all possible distinct cases.

(d) \( c_{6,1} \geq NF_1^{-2}(a_6 b_6)^{1/2} \delta_{2,7} \delta_{2,7} (Q_2^{-8} + (NF_1^{-2})^{1/2})^{-1} \equiv g \). Applying Lemma A to the sum in (11) and choosing \( Q \) to minimize the obtained expression, we get
\[
|S_1| \ll N_0^{\delta_{2,7}^{2}+1} R_0 + NF_1^{-2} (X^4 Q_2^{-8} N F_1^{-1})^{1/5}.
\]

If \( c_{6,1} \geq g X / Y \), then because \( \max \{a_6, b_6\} \geq c_{5,1} \), the above expression is \( \ll R_0 \); if \( c_{6,1} \ll g X / Y \) but \( a_6 \gg a_6^{22} X^2 Q_2^{-8} \sqrt{c_{6,1}} \), then we get the same result; otherwise we use inequality \( \max \{a_6, \sqrt{a_6}+c_6\} \geq c_{5,1} \) to prove that
\[
|S_1| \ll \ll R_0.
\]

(e) \( c_{6,1} \ll g \). Applying Lemma B with \( k_2 = 2 \), we choose \( Q \) to minimize the obtained expression and obtain:
\[
|S_1| \ll R_0 + N_0^{\delta_{2,7}^{2}+1} \sqrt{c_{6,1}/c_{5,1}} (X^4 Q_2^{-8} N F_1^{-1})^{1/5}.
\]

This completes the proof of Theorem 1.

Theorem 2 can be proved similarly to Theorem 2 in [3]. The only change is in showing that \( |P_1| \ll \sqrt{a_3} N \). This can be done by proving that \( H(g(x)) \approx P(\theta_1, \theta_2, \theta_3) \), where \( \theta_j = h_j/x_j \) and \( P \) is a homogeneous polynomial of degree 3 such that \( P(0) = P_{a_3} = P_{b_3} = P_{c_3} = P_{d_3} = P_{e_2} = P_{f_2} = 0 \) has no non-zero solutions.

To prove Theorem 3, we apply Lemma 2 with \( m = 3 \) and \( q = q_1 q_2 q_3 \), to be defined later:
\[
(12) \quad (S_1 N) \ll g^{-1} + (q N)^{-1} \sum_{|a_1|=1}^{q_1} \sum_{|a_2|=1}^{q_2} \sum_{|a_3|=1}^{q_3} \sum_{x \in \mathbb{P}_1} e(g(x)) + \sum_{H \in \mathbb{P}_2} |H(\mathbb{X})|,
\]

where
\[
|g(x)| = \int_0^1 \frac{1}{t} f(x_1+h_1t, x_2+h_2t, x_3+h_3t, x_4, \ldots, x_6) dt,
\]

and \( \sum_{H \in \mathbb{P}_2} |H(\mathbb{X})| \) is small compared with the estimate we obtain for the first sum. Let \( h = (h_1, h_2, h_3) \) be fixed, \( q = |h_1/X_1| + |h_2/X_2| + |h_3/X_3| \), \( \theta_j = h_j/x_j \), \( F_1 = F_0 \).

Then
\[
H(g(x)) \equiv \left| \left( P_1(\theta_1) \right) - P_2(\theta_1) + O(\delta) \right| F_1^{-2},
\]

where
\[
P_1(\theta_1) = a_1 \theta_1 + a_2 \theta_2 + a_3 \theta_3, \quad P_2(\theta_1) = P_1^2 - 2(\theta_1 + \theta_2 + \theta_3) P_2 + (a_1 \theta_1^2 + a_2 \theta_2^2 + a_3 \theta_3^2) + 2a_1 \theta_1 \theta_2 + 2a_2 \theta_2 \theta_3 + 2a_3 \theta_3 \theta_1 P_1 + (a_1 + a_2 + a_3 - 2) \theta_1 \theta_2 \theta_3.
\]

Also, denoting by \( g_{i_1, \ldots, i_m}(x) \) the function obtained from \( g(x) \) by fixing all of \( x_i \) except of \( x_{i_1}, \ldots, x_{i_m} \), we obtain
\[
H(g_{i_1, \ldots, i_m}(x)) \equiv \left| \left( P_1(\theta_1) \right) - P_2(\theta_1) + O(\delta) \right| F_1^{-2} (X_i X_j)^{-2} (X_i X_j)^{-2}.
\]

We divide \( D_1 \) into \( \ll NF_1^{-1/2} \) subdomains: One of them has \( \ll NF_1^{-1/2} \) points. Each of the remaining subdomains is of the type
\[
\begin{align*}
a_1 &\ll |P_1(\theta_1)| \ll a_1 (1 + e_0),
a_2 &\ll |P_2(\theta_1)| \ll a_2 (1 + e_0),
a_4 &\ll |P_4(\theta_1)| \ll a_4 (1 + e_0),
\end{align*}
\]

We suppose that the domain \( D_2 \), defined by the above inequalities, corresponds to the largest subsum, and consider all possible cases.

1. \( a_1 \gg e_0, a_2 \gg e_0 \). Then \( H(g(x)) \gg F_1^{-2} \), and, applying Lemma 1, we obtain
\[
S_1 = \sum_{x \in \mathbb{P}_2} e(g(x)) \ll F_1^{-2} + NF_1^{-1/2}.
\]
2. \( a_1 \leq e_0 \). In this case
\[
|D_2| \leq (\Delta + a_2) N, \quad \max \{a_2, a_{i,j}\} \geq e_0, \quad \max \{a_2, (a_{i+2} + a_{j+2})\} \geq e_0.
\]
If \( a_2 \geq F_1^{-1/3} \), then we use Lemma 1 with \( k = 1 \) three times (changing the order of summation, if necessary):
\[
S_1 \ll X_i F_1^{-1/2} \sum_{m \neq x} e(f_1(m, x)) + NF_1^{-1/2}
\ll NF_1^{-1/2} + X_i X_j (F_1)^{-1} \sum_{m_1, m_2 \neq x} e(f_2(m, x))
\ll NF_1^{-1/2} + X_i X_4 \ldots X_n F_1 a_2 + (\Delta + \min \{a_1, a_2\}) X_4 \ldots X_n F_1^{2/3}.
\]
If \( a_2 \) is small, then we can use Lemma 1 with \( k = 2 \) and get
\[
S_1 \ll X_i X_4 \ldots X_n F_1 (\Delta + a_2) + NF_1^{-1/2}.
\]
Comparing two last estimates, we get
\[
S \ll X_i X_4 \ldots X_n F_1 \Delta + NF_1^{-1/2} + \Delta X_4 \ldots X_n F_1^{2/3} + X_i X_4 \ldots X_n F_1^{3/3}.
\]
3. \( a_1 \leq e_0, \quad a_2, a_3 \geq e_0 \). In this case \( |D_2| \leq (\Delta + \sqrt{a_2}) N \) (see comments to Theorem 2),
\[
|D_2| \leq (\Delta + \max \{a_2, a_3\}) N, \quad \max a_1 \geq e_0.
\]
If, say, \( a_{2,3} = \max a_{i,j} \geq F_1^{-1/3} \), then \( a_2 \geq e_0 \) (or \( a_3 \geq e_0 \)) and, applying twice Lemma 1 with \( k = 1 \), we get
\[
S_1 \ll X_2 F_1^{-1/2} \sum_{m \neq x} e(f_1(m, x)) + N/\sqrt{F_1}
\ll (\sqrt{\Delta} + \sqrt{e_0}) FX_4 \ldots X_n + N/\sqrt{F_1} + X_2 X_3 X_4 \ldots X_n / \sqrt{a_{2,3}};
\]
also, using Lemma 1 with \( m = 1 \), we get
\[
S_1 \ll (\Delta + a_{2,3}) X_2 X_4 \ldots X_n / \sqrt{F_1} + N/\sqrt{F_1}.
\]
If \( a_2 \geq F_1^{-1/3} \), then we apply Lemma 1 twice with \( k = 1 \) and once with \( k = n - 2 \) and obtain:
\[
S_1 \ll X_2 F_1^{-1/2} \sum_{m \neq x} e(f_1(m, x)) + N/\sqrt{F_1}
\ll X_2 X_3 F_1^{-1/2} \sum_{m_1, m_2 \neq x, m_3 \neq x} e(f_2(m, x)) +
+N/\sqrt{F_1} + X_2 X_3 X_4 \ldots X_n / \sqrt{a_{2,3}}
\ll F_1^{2/3} + \sqrt{F_1} X_2 \ldots X_n / a_1 + X_2 X_4 \ldots X_n / \sqrt{a_{2,3}} + N/\sqrt{F_1}.
\]
Comparing all three estimates, we obtain
\[
S_1 \ll F_1^{1/2} + N/\sqrt{F_1} + NF_1^{1/6} X_3^{-1} + F_1^{1/4} N (X_2 X_3)^{-1} +
+F_1^{1/6} N (X_2 X_3)^{-1} + N \Delta \sqrt{F_1} + N \Delta \sqrt{X_2 X_3} \ll R_0
\]
so, in all three cases we get
\[
S_1 \ll R_0 + \Delta NF_1^{1/2} / X_1 X_2 X_3 \equiv R_1.
\]
Substituting this into (12), we choose \( q \) to minimize the resulting expression:
\[
(S/N)^3 \ll q^{-1} + (qN)^{-1} \sum_R q^{-1} + N^{-1} [(F / q^{3/4} (X_1 X_2 X_3)^{-3})]^{1/2} +
+N \sqrt{X_1 X_2 X_3} q^{-1} F^{-3} + X_3^{-1/2} \sqrt{F_1} q (X_1 X_2 X_3)^{-1} +
+N (X_2 X_3)^{-1} \sqrt{F_1} q (X_1 X_2 X_3)^{-1} + N \Delta X_3^{-1} \sqrt{F_3} q (X_1 X_2 X_3)^{-1} +
+N (X_2 X_3)^{-1} \sqrt{F_1} q (X_1 X_2 X_3)^{-1} + N \Delta \sqrt{X_2 X_3} q (X_1 X_2 X_3)^{-1} +
+N \Delta \sqrt{F_3} q (X_1 X_2 X_3)^{-1}
\leq (F_3 N^{-6} (X_1 X_2 X_3)^{-1})^{1/6} +
+X_3^{-1/6} \sqrt{F_1 X_2 X_3}^{-1} + X_2^{-1} X_3^{-1} \sqrt{F_1^{1/5} / X_2} +
+X_2^{-1} \sqrt{F_3} A^{1/6} X_2^{-1} + X_2^{-1} X_3^{-1} \sqrt{F_3} A^{1/6} X_2^{-1} +
+N^{-1/6} + X_3^{3/4} + \sqrt{F_1} / X_3.
\]
This proves Theorem 3.

Now we assume that (1) and (4) are true and prove Theorems 4 and 5. First we notice that using m.e.p. one can never obtain a better estimate than \( S \ll \sqrt{N} \). This can be proved by induction on the total number of times we apply (1) and (4). If we apply (1) once, we get \( S \ll F m^{m-2} X_{m+1} \ldots X_n \), where \( F m^{m-2} > X_1 \ldots X_m \). (Note that the condition \( F \geq X_1 \) is not essential; we can remove it and substitute (1) with \( S \ll N_m F^{-m+1} S(F; F/X_1 + 1, \ldots, F/X_m + 1, X_{m+1}, \ldots, X_n) \).

One can however do it better: If \( F \leq e_0 X_1 \), then
\[
S = \left| \sum_{x_2 \ldots x_n} \sum_{x_1 \neq x} e(f(x)) \right| \ll \left| \sum_{x_2 \ldots x_n} \sum_{x_1 \neq x} e(f(x)) \right| + \sum_{m=1}^{n} \frac{1}{m} \left| \sum_{x_2 \ldots x_n} e(f(x) - mx_1) \right| \ll \frac{X_1}{F} S(F; X_2, \ldots, X_n + \ldots)
\]

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If we apply (4) once, we get
\[ S^2 \approx N^2/q + \left( N^2/q \right) S(F_q; X, q), \]
where \( N^2/q \gg N \).
Suppose, our statement is always true if we apply (1) and (4) \( k \) times. If we apply (1) and (4) \( (k+1) \) times, and if we start with (4), we can never get a better estimate than \( S \approx \sqrt{N} \) (the same reasoning as above). If we start with applying (1), then we get
\[ S \leq N^2 F^{-m} S(F; F/X_1, \ldots, F/X_m, X_{m+1}, \ldots, X_n), \]
and we apply (1) and (4) \( k \) times to the last sum. So, we can never obtain a better estimate than \( S \leq N^2 F^{-m} \left( F/X_1 \ldots F/X_m, X_{m+1} \ldots X_n \right)^{1/2} \approx \sqrt{N} \). To prove Theorem 4, we suppose that \((\lambda_0, \ldots, \lambda_m) \in \mathbb{E}_n\). Using (1), we obtain:
\[
S \leq N^2 F^{-m} S(F; F/X_1, \ldots, F/X_m, X_{m+1}, \ldots, X_n)
\]
\[
\leq N^2 F^{-m} S(F; F/X_1, \ldots, F/X_m, X_{m+1}, \ldots, X_n)
\]
\[
= F^{-m} X_1^{\lambda_1} \ldots X_m^{\lambda_m} X_{m+1}^{\lambda_{m+1}} \ldots X_n^{\lambda_n},
\]
i.e. \((\lambda_0, \ldots, \lambda_m) \in \mathbb{E}_n\). Theorem 5 can be proved similarly by using (4):
\[
S^2 \approx N^2/q + Nq^{-1} S(F; X, q)
\]
\[
\leq N^2/q + Nq^{-1} \left( F \cdot \sqrt{q/N_m} \right)^{1/2} X_1^{\lambda_1} \ldots X_m^{\lambda_m} q_1^{\lambda_1} \ldots q_m^{\lambda_m}
\]
\[
= N^2/q + Nq^{-1} N X_1^{\lambda_1} \ldots X_m^{\lambda_m} X_{m+1}^{\lambda_{m+1}} \ldots X_n^{\lambda_n} q^{-1} (q/N_m)^{(\lambda_0 + \lambda_n)/m}.
\]
Choosing \( q \) which minimizes the obtained expression we prove Theorem 5.

6. Final remarks. While Theorems 1–3 can be applied to a wide class of function \( \in \mathbb{F} \), they may not give a “good” result in some cases. The general strategy in using MEP is the following. Knowing the estimate of an exponential sum one wants to obtain, he needs to apply the algorithm (based on (1) and (4)) described in the introduction to determine whether it is possible and (if it is possible) a sequence of operations of \( A \) and \( B \) one needs to apply in order to obtain the result. The last step is to prove the result using the ideas contained in the proof of Theorem 1. For a specific function it can be done much simpler than in the general case.

References


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Received on 4.8.1983 (1369)