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On sign-changes in the remainder-term of the prime-number formula, II

by

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1. In the first paper of this series [3] we have proved that the differences

(1.1) $$\Delta_3(x) = \psi(x) - x = \sum_{n \leqslant x} \Lambda(n) - x$$

and

(1.2) $$\Delta_2(x) = \Pi(x) - \mathrm{li}\, x = \sum_{m \geqslant 1} \frac{1}{m} \pi(x^{1/m}) - \int_0^x \frac{du}{\log u}$$

which are the remainder-terms in the prime-number formula change sign at least

(1.3) $$\frac{\gamma_0}{4\pi} \log T$$

times, in the interval $[2, T]$, $T \geqslant T_0$, where $\gamma_0 = 14.13\ldots$ denotes here the imaginary part of the "lowest" zero of the Riemann zeta function. T_0 stands for a positive, effectively computable numerical constant.

We have two other remainders in the prime-number theorem; the most intensively studied in the literature

(1.4) $$\Delta_1(x) = \pi(x) - \mathrm{li}\, x = \sum_{p \leqslant x} 1 - \mathrm{li}\, x$$

and also

(1.5) $$\Delta_4(x) = \vartheta(x) - x = \sum_{p \leqslant x} \log p - x.$$

As in [3] let us denote by $V_j(T)$, $1 \leqslant j \leqslant 4$, the number of sign-changes of $\Delta_j(x)$ in $[2, T]$, $T \geqslant 2$.

Let

(1.6) $$\theta = \sup_{\zeta(\varrho) = 0} \mathrm{Re}\, \varrho$$

where $\zeta(s)$ is the Riemann zeta-function.

Ingham, [2], proved in 1936 that if there is a zero $\varrho = \theta + i\gamma$ of $\zeta(s)$ on the line $\sigma = \theta$ then there exist two positive constants c_1 and T_1 such that $\Delta_1(x)$ changes sign in every interval of the form $(T, c_1 T)$, $T \geqslant T_1$. This implies that

(1.7) $$V_1(T) \geqslant c_2 \log T$$

for certain positive c_2 and sufficiently large T.

The only fault of this beautiful and deep theorem is the very strong condition which in particular implies that $\theta < 1$ (the quasi-Riemann hypothesis). In this paper we shall prove (1.7) unconditionally.

Our theorem is the following one.

THEOREM 1. *There exists a positive but ineffective constant c_3 such that for sufficiently large T we have*

(1.8) $$V_1(T) \geqslant c_3 \log T.$$

The same estimate holds also for $V_4(T)$.

The problem of the size of $V_1(T)$ was first considered by J. E. Littlewood [9] who proved that $V_1(T) \to \infty$ as T tends to infinity, disproving in this way an old conjecture of B. Riemann. More precise informations about $V_1(T)$ can be obtained by the application of Turán's power sum method (compare S. Knapowski [4], [5], S. Knapowski–P. Turán [6], [7] and J. Pintz [10], [11]). The strongest estimate proved by this method was achieved by J. Pintz [11]:

(1.9) $$V_1(T) \geqslant 10^{-11} \frac{\log T}{(\log \log T)^3} \quad \text{for} \quad T > T_1$$

where T_1 is an ineffective constant.

Dr J. Pintz has kindly informed the author that actually he is able to prove (1.9) with an effective T_1.

Our next aim is to improve a theorem of Pólya.

Pólya, [12], proved in 1930 that

(1.10) $$\overline{\lim_{T \to \infty}} \frac{V_3(T)}{\log T} \geqslant \frac{\gamma_1}{\pi},$$

where γ_1 is defined as follows. If $\zeta(s)$ has any zeros $\varrho = \theta + i\gamma$ on the line $\sigma = \theta$, then γ_1 denotes the least positive γ corresponding to these zeros; otherwise $\gamma_1 = +\infty$.

We shall prove the following stronger result:

THEOREM 2.

(1.11) $$\underline{\lim_{T \to \infty}} \frac{V_3(T)}{\log T} \geqslant \frac{\gamma_1}{\pi},$$

where γ_1 is defined as in (1.10).

From the proof of this theorem it follows that the estimate (1.11) is true also for $V_4(T)$ provided $\theta > 1/2$. The case of $V_j(T)$, $j = 1, 2$ is somewhat more complicated because of the logarithmic singularities of the involved Dirichlet series. These cases can be treated in a similar way as in the first paper of this series and we can prove that (1.11) is true for $V_2(T)$ and also for $V_1(T)$ if $\theta > 1/2$. It is clear that Theorem 1 is a consequence of such extended version of Theorem 2 and Ingham's result (1.7).

2. Proof of Theorem 2 in the case $\gamma_1 = +\infty$. According to the definition of γ_1 we know that there are no zeros on the line $\sigma = \theta$. We can choose a sequence

(2.1) $$\varrho_m = \beta_m + i\gamma_m, \quad m = 1, 2, 3, \ldots$$

of zeros of $\zeta(s)$ such that

(2.2) $$\gamma_m \to \infty \quad \text{as} \quad m \to \infty,$$

(2.3) $$\beta_m \to \theta \quad \text{as} \quad m \to \infty,$$

(2.4) $$\beta_m > \theta^* := \tfrac{1}{2}(\theta + \tfrac{1}{2}) \quad \text{for} \quad m \geqslant 1,$$

and the region

(2.5) $$s = \sigma + it, \quad \sigma > \beta_m, \ |t| < \gamma_m$$

is zero-free.

As in [3] we define the operators δ_n, $n \geqslant 1$ by

(2.6) $$\delta_n(f; x) := \int_0^x \int_0^{\xi_{n-1}} \ldots \int_0^{\xi_1} f(\xi) \frac{d\xi}{\xi} \ldots \frac{d\xi_{n-2}}{\xi_{n-2}} \frac{d\xi_{n-1}}{\xi_{n-1}}$$

where $x > 0$ and f is any complex-valued function for which the integrals on the right-hand side of (2.6) do exist.

It is well known that there is a broken line L in the vertical strip $s = \sigma + it$, $1/20 \leqslant \sigma \leqslant 1/10$, $-\infty < t < +\infty$ symmetrical to real axis, consisting of horizontal and vertical segments alternately and increasing monotically from $-\infty$ to $+\infty$ on which the following inequality

(2.7) $$\frac{\zeta'}{\zeta}(s) \ll \log^2(|t|+2)$$

holds. We can also assume that

(2.8) $$[\tfrac{1}{10} - 10i, \tfrac{1}{10} + 10i] \subset L.$$

We have:

$$(2.9)\qquad \delta_{n-1}(\Delta_3;x)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\left\{-\frac{\zeta'}{\zeta}(s)\right\}\frac{x^s}{s^n}\,ds-x$$

$$=-\sum_{\varrho>L}\frac{x^\varrho}{\varrho^n}+O\left(\left|\int_L\frac{\zeta'}{\zeta}(s)\frac{x^s}{s^n}\,ds\right|\right).$$

The summation in (2.9) runs over all zeros $\varrho=\beta+i\gamma$ of $\zeta(s)$ lying to the right of the line L.

Further, (2.7)–(2.9) easily implies that for $x>1$ and $m\geqslant 1$ we have

$$(2.10)\qquad \delta_{n-1}(\Delta_3;x)=-\sum_{\varrho>L}\frac{x^\varrho}{\varrho^n}+O(10^n x^{0.1})$$

$$=-\sum_{\substack{\beta\geqslant\theta^*\\|\gamma|\leqslant|\varrho_m|^2}}\frac{x^\varrho}{\varrho^n}+O_m(10^n x^{0.1}+x^\theta|\varrho_m|^{-2n}+x^{\theta^*}).$$

For a fixed m denote by

$$(2.11)\qquad \theta^*<\beta_m^{(1)}<\beta_m^{(2)}<\ldots<\beta_m^{(k_m)}<\theta$$

the real parts of the zeros $\varrho=\beta+i\gamma$ lying in the region

$$(2.12)\qquad \theta^*\leqslant\sigma<\theta,\qquad |t|\leqslant|\varrho_m|^2.$$

Then

$$(2.13)\qquad \delta_{n-1}(\Delta_3;x)=-\sum_{1\leqslant\nu\leqslant k_m}\Phi_\nu(x)+O_m(10^n x^{0.1}+x^\theta|\varrho_m|^{-2n}+x^{\theta^*})$$

where

$$(2.14)\qquad \Phi_\nu(x)=\sum_{\substack{\beta=\beta_m^{(\nu)}\\|\gamma|<|\varrho_m|^2}}\frac{x^\varrho}{\varrho^n}.$$

Denote further by

$$(2.15)\qquad \varrho_m^{(\nu)}=\beta_m^{(\nu)}+i\gamma_m^{(\nu)}=|\varrho_m^{(\nu)}|e^{i\varphi_m^{(\nu)}},\qquad 1\leqslant\nu\leqslant k_m,\qquad \gamma_m^{(\nu)}>0,$$

the "lowest" zero on the line $\sigma=\beta_m^{(\nu)}$. Then

$$(2.16)\qquad \Phi_\nu(x)=\frac{2\varkappa_\nu x^{\beta_m^{(\nu)}}}{|\varrho_m^{(\nu)}|^n}\{\cos(\gamma_m^{(\nu)}\log x-n\varphi_m^{(\nu)})+r_\nu(x)\}$$

where

$$(2.17)\qquad \varkappa_\nu=\operatorname{ord}\varrho_m^{(\nu)}$$

and

$$(2.18)\qquad |r_\nu(x)|\leqslant\sum_{\substack{\varrho=\beta_m^{(\nu)}+i\gamma\\\gamma_m^{(\nu)}<|\gamma|\leqslant|\varrho_m|^2}}\left|\frac{\varrho_m^{(\nu)}}{\varrho}\right|^n.$$

For $n>n_0$, we have

$$(2.19)\qquad |r_\nu(x)|<0.1$$

and thus

$$(2.20)\qquad \delta_{n-1}(\Delta_3;x)=-2\sum_{1\leqslant\nu\leqslant k_m}\frac{\varkappa_\nu x^{\beta_m^{(\nu)}}}{|\varrho_m^{(\nu)}|^n}\{\cos(\gamma_m^{(\nu)}\log x-n\varphi_m^{(\nu)})+\bar{O}(0.1)\}+$$

$$+O_m(10^n x^{0.1}+x^\theta|\varrho_m|^{-2n}+x^{\theta^*}).$$

We have used in (2.20) the symbol $\bar{O}$. The notation $f(x)=\bar{O}(g(x))$ for $x\in\Omega\subset\boldsymbol{R}$ means that $|f(x)|\leqslant g(x)$ for $x\in\Omega$.

3. Now we restrict the range for x as follows:

$$(3.1)\qquad x=e^\xi,$$

$$(3.2)\qquad 0.9\log T<\xi<\log T.$$

Let us put

$$(3.3)\qquad n:=\left[2(\theta-\beta_m)\frac{\log T}{\log|\varrho_m|}\right]$$

and define the functions:

$$(3.4)\qquad u_m^{(\nu)}(\xi)=\beta_m^{(\nu)}\xi-n\log|\varrho_m^{(\nu)}|.$$

Let $\varepsilon>0$ and

$$(3.5)\qquad A(T,\varepsilon):=\{\xi|\ 0.9\log T<\xi<\log T,$$

$$|u_m^{(\nu)}(\xi)-u_m^{(\nu')}(\xi)|\geqslant\varepsilon \text{ for every } \nu\neq\nu',\ 1\leqslant\nu,\nu'\leqslant k_m\}.$$

If $\xi\notin A(T,\varepsilon)$ then

$$(3.6)\qquad \left|\xi-n\frac{\log|\varrho_m^{(\nu)}/\varrho_m^{(\nu')}|}{\beta_m^{(\nu)}-\beta_m^{(\nu')}}\right|<\frac{\varepsilon}{|\beta_m^{(\nu)}-\beta_m^{(\nu')}|}$$

and we can estimate the measure of $A(T,\varepsilon)$ as follows:

$$(3.7)\qquad |A(T,\varepsilon)|\geqslant 0.1\log T-2\varepsilon b$$

where

$$(3.8)\qquad b=\sum_{\substack{\nu\neq\nu'\\1\leqslant\nu,\nu'\leqslant k_m}}\sum\frac{1}{|\beta_m^{(\nu)}-\beta_m^{(\nu')}|}.$$

Thus

$$|A(T, 0.05b^{-1}\log T)| \geqslant 0.05 \log T. \tag{3.9}$$

The set $A(T, 0.05b^{-1}\log T)$ is contained in the sum of the finite number of disjoint intervals:

$$I_\nu := \{\xi \in (0.9\log T, \log T) |\ u_m^{(\nu)}(\xi) \geqslant \max_{1\leqslant \mu \leqslant k_m} u_m^{(\mu)}(\xi) + 0.05\, b^{-1}\log T\}. \tag{3.10}$$

Some of these intervals can be empty but at least one is not.

For $\xi \in I_\nu$ we have

$$\delta_{n-1}(\Delta_\varrho; x) = -2\varkappa_\nu e^{u_m^{(\nu)}(\xi)}\{\cos(\gamma_m^{(\nu)}\xi - n\varphi_m^{(\nu)}) + \tag{3.11}$$

$$+\bar{O}(0.1) + O_m\big(\sum_{\mu\neq\nu} e^{u_m^{(\mu)}(\xi)-u_m^{(\nu)}(\xi)} + 10^n e^{0.1\xi - u_m^{(\nu)}(\xi)} +$$

$$+ e^{\theta\xi - u_m^{(\nu)}(\xi)}|\varrho_m|^{-2n} + e^{\theta^*\xi - u_m^{(\nu)}(\xi)}\big)\}$$

and

$$\sum_{\mu\neq\nu} e^{u_m^{(\mu)}(\xi) - u_m^{(\nu)}(\xi)} \leqslant k_m T^{-0.05b^{-1}}, \tag{3.12}$$

$$\begin{aligned} 10^n e^{0.1\xi - u_m^{(\nu)}(\xi)} &\leqslant 10^n \exp(0.1\xi - \beta_m\xi + n\log|\varrho_m|) \\ &\leqslant \exp\Big(0.1\xi - \beta_m\xi + 2(\theta-\beta_m)\log T + 2\log 10(\theta-\beta_m)\frac{\log T}{\log|\varrho_m|}\Big) \\ &\leqslant T^{-0.9(\beta_m - 0.1) + 2(\theta-\beta_m)(1+\log 10/\log|\varrho_m|)} \\ &\leqslant T^{-0.9(\beta_m-0.1)+4(\theta-\beta_m)}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} e^{\theta\xi - u_m^{(\nu)}(\xi)}|\varrho_m|^{-2n} &\leqslant \exp(\theta\xi - \beta_m\xi + n\log|\varrho_m| - 2n\log|\varrho_m|) \\ &\leqslant |\varrho_m|\exp\big((\theta-\beta_m)\log T - 2(\theta-\beta_m)\log T\big) \\ &\leqslant |\varrho_m|\, T^{-(\theta-\beta_m)} \end{aligned} \tag{3.14}$$

and finally

$$\begin{aligned} e^{\theta^*\xi - u_m^{(\nu)}(\xi)} &\leqslant \exp(\theta^*\xi - \beta_m\xi + n\log|\varrho_m|) \\ &\leqslant \exp\big(\theta^*\xi - \beta_m\xi + 2(\theta-\beta_m)\log T\big) \\ &\leqslant T^{-0.9(\beta_m-\theta^*)+2(\theta-\beta_m)}. \end{aligned} \tag{3.15}$$

It is evident that if $m \geqslant m_0$ all these functions tend to zero as $T\to\infty$. Hence for $\xi\in I_\nu$ and sufficiently large T we can write

$$\delta_{n-1}(\Delta_3, x) = -2\varkappa_\nu e^{u_m^{(\nu)}(\xi)}\{\cos(\gamma_m^{(\nu)}\xi - n\varphi_m^{(\nu)}) + \bar{O}(0.5)\}. \tag{3.16}$$

One can see that $\delta_{n-1}(\Delta_3, x)$ has at least

$$\frac{\gamma_m^{(\nu)}|I_\nu|}{\pi} - 2 \tag{3.17}$$

sign-changes in the interval I_ν. Thus $\delta_{n-1}(\Delta_3, x)$ has at least

$$\begin{aligned} &\frac{1}{\pi}\sum_{\substack{1\leqslant\nu\leqslant k_m\\ I_\nu\neq\emptyset}} \gamma_m^{(\nu)}|I_\nu| - 2k_m \\ &\geqslant \frac{\gamma_m^{(0)}}{\pi}|A(T, 0.05b^{-1}\log T)| - 2k_m \geqslant \frac{0.05\gamma_m^{(0)}}{\pi}\Big(1 + O\Big(\frac{k_m}{\log T}\Big)\Big)\log T \end{aligned} \tag{3.18}$$

sign-changes in the interval $0.9\log T \leqslant \xi \leqslant \log T$, where

$$\gamma_m^{(0)} = \min\{\gamma_m^{(\nu)}|\ I_\nu \neq \emptyset\}. \tag{3.19}$$

As it was explained in [3], $V_3(T)$ is greater than or equal to the number of sign-changes of $\delta_{n-1}(\Delta_3, x)$ in the interval $(2, T]$. Therefore (3.18) implies that

$$\varliminf_{T\to\infty}\frac{V_3(T)}{\log T} \geqslant \frac{0.05\gamma_m^{(0)}}{\pi}. \tag{3.20}$$

4. To finish the proof in this case it suffices to prove that $\gamma_m^{(0)}$ tends to infinity together with m.

If $I_\nu \neq \emptyset$ for certain ν, $1\leqslant\nu\leqslant k_m$ then there exists ξ, $0.9\log T \leqslant \xi \leqslant \log T$ such that

$$u_m^{(\nu)}(\xi) = \beta_m^{(\nu)}\xi - n\log|\varrho_m^{(\nu)}| \geqslant \beta_m\xi - n\log|\varrho_m|. \tag{4.1}$$

Therefore

$$\begin{aligned} \beta_m^{(\nu)} &\geqslant \beta_m - n\frac{\log|\varrho_m|}{\xi} + n\frac{\log|\varrho_m^{(\nu)}|}{\xi} \\ &\geqslant \beta_m - n\frac{\log|\varrho_m|}{\xi} \geqslant \beta_m - 2(\theta-\beta_m)\frac{\log T}{\xi} \\ &\geqslant \beta_m - \frac{2}{0.9}(\theta-\beta_m) > \beta_m - 4(\theta-\beta_m). \end{aligned} \tag{4.2}$$

Thus $\beta_m^{(\nu)} \to \theta$ as $m\to\infty$ and

$$\gamma_m^{(0)} = \min\{\gamma_m^{(\nu)}|\ I_\nu\neq\emptyset\}\to\infty \quad \text{as} \quad m\to\infty. \tag{4.3}$$

This finishes the proof of Theorem 2 in this case.

5. Proof of Theorem 2 in the case $\gamma_1 < \infty$. Let

$$\varrho_0 = \theta + i\gamma_0 = |\varrho_0| e^{i\varphi_0}, \quad \gamma_0 > 0, \quad \varkappa_0 := \operatorname{ord}\varrho_0 \tag{5.1}$$

denote the "lowest" zeta-zero on the line $\sigma = \theta$. Then for any real $\Gamma > \gamma_0$

(5.2) $$\delta_{n-1}(\Delta_3, x) = -\sum_{\substack{\beta=\theta \\ |\gamma|<\Gamma}} \frac{x^{\varrho}}{\varrho^n} + O(x^{\theta}\Gamma^{2-n} + 10^n x^{0.1} + x^{\eta})$$

where

(5.3) $$\eta = \eta(\Gamma) := \sup_{\substack{\varrho=\beta+i\gamma \\ \beta<\theta, |\gamma|<\Gamma}} \operatorname{Re} \varrho.$$

If Γ is sufficiently near γ_0, then

(5.4) $$\delta_{n-1}(\Delta_3, x) = -\frac{2\varkappa_0 x^{\theta}}{|\varrho_0|^n}\Big\{\cos(\gamma_0 \log x - n\varphi_0) + O\Big(\Gamma^2\Big(\frac{|\varrho_0|}{\Gamma}\Big)^n + (10|\varrho_0|)^n x^{0.1-\theta} + |\varrho_0|^n x^{\eta-\theta}\Big)\Big\}.$$

Denote by c_0 the constant implied by O-notation in (5.4).
If

(5.5) $$n \geqslant n_0(\Gamma)$$

then

(5.6) $$\Gamma^2\Big(\frac{|\varrho_0|}{\Gamma}\Big)^n < \frac{1}{10c_0}$$

and for $x \geqslant T_0 = T_0(n)$

(5.7) $$(10|\varrho_0|)^n x^{0.1-\theta} < 1/10c_0,$$

(5.8) $$|\varrho_0|^n x^{\eta-\theta} < 1/10c_0.$$

Hence for $x \geqslant T_0$

(5.9) $$\delta_{n-1}(\Delta_3, x) = -\frac{2\varkappa_0 x^{\theta}}{|\varrho_0|^n}\{\cos(\gamma_0 \log x - n\varphi_0) + \bar{O}(\tfrac{1}{2})\}$$

and it is evident that $\delta_{n-1}(\Delta_3, x)$ changes sign at least

(5.10) $$\frac{\gamma_0 \log T - \gamma_0 \log T_0}{\pi} - 2 \geqslant \Big(\frac{\gamma_0}{\pi} - \varepsilon\Big)\log T$$

times in the interval $(T_0, T]$. Thus

(5.11) $$V_3(T) \geqslant \Big(\frac{\gamma_0}{\pi} - \varepsilon\Big)\log T$$

for every $\varepsilon > 0$ which is equivalent to

(5.12) $$\varliminf_{T\to\infty} \frac{V_3(T)}{\log T} \geqslant \frac{\gamma_0}{\pi}.$$

6. Proof of Theorem 1. As we have already seen it suffices to prove the estimate (1.11) for $V_1(T)$ in the case $\theta > 1/2$. Since the arguments are similar to those from the first paper of this series we shall be very brief. Let $l(x)$ be the function defined by the formula (8.2) in [3]. Then $l(x) = \operatorname{li} x$ for $x \geqslant 2$ and $l(x) > 0$ for $0 < x \leqslant 2$.

As in [3] we can prove a formula analogous to (2.10)

(6.1) $$\delta_{n-1}(\pi(\xi) - l(\xi), x) = \sum_{\substack{\varrho=\beta+i\gamma \\ \beta\geqslant\theta^*, |\gamma|<|\varrho_m|^2}} \mathscr{I}_n(\varrho, x) + O(10^n x^{0.1} + x^{\theta}|\varrho_m|^{-2n} + x^{\theta^*} + 2^n x^{0.5})$$

where $\mathscr{I}_n(\varrho, x)$ denote certain contour integrals with

(6.2) $$\Big|\mathscr{I}_n(\varrho, x) - \frac{x^{\varrho}}{\varrho^n \log x}\Big| \leqslant \frac{x^{\beta}}{|\varrho|^{n+1}\log x} + O\Big(\frac{x^{\beta}}{|\varrho|^n \log^2 x}\Big)$$

for $x \geqslant x_0$ and $1 \leqslant n \leqslant \frac{1}{2}\log x$.

Thus the formula (2.20) becomes

(6.3) $$\delta_{n-1}(\pi(\xi) - l(\xi), x) = -2\sum_{1\leqslant\nu\leqslant k_m} \frac{\varkappa_\nu x^{\beta_m^{(\nu)}}}{|\varrho_m^{(\nu)}|^n \log x}\{\cos(\gamma_m^{(\nu)}\log x - n\varphi_m^{(\nu)}) + \bar{O}(0.1)\} + O(10^n x^{0.1} + x^{\theta}|\varrho_m|^{-2n} + x^{\theta^*} + 2^n x^{0.5})$$

and the same arguments as before, lead to the required result.

7. Remarks. The method developed in the papers I and II is of a more general character and can be successfully applied to a large class of similar problems. The generality of this method rests on the very useful properties of the operator δ. Recall that for $x > 0$ and any complex-valued function f we define

$$\delta(f)(x) = \delta(f, x) := \int_0^x f(\xi)\frac{d\xi}{\xi}$$

if the integral does exist. This is some kind of a mean-value operator with respect to the Haar measure on the group R^+. All proofs in our papers rest on the following two properties of the operator δ:

(1) If $V(f, T)$ denotes the number of sign-changes of a real-valued function f in the interval $(0, T]$ then

$$V(f, T) \geqslant V(\delta(f), T).$$

(2) Every quasicharacter q^s, $\operatorname{Re} s > 0$ of the group R^+ is an eigenfunction of δ with the eigenvalue $1/s$.

Owing to (1) we do not lose control of the number of sign-changes after applying δ. On the other hand if we write the Mellin inversion formula for f in the form:

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\, x^s\, ds, \qquad c > 0,$$

then the computation of $\delta(f)$ becomes very easy since owing to (2) we have

$$\delta(f)(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(s)\frac{x^s}{s}\, ds.$$

Let us notice that the factor $1/s$ improves the convergence of the above integral. This makes the whole analysis simpler especially if we repeat this procedure a number of times.

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Theta series of quaternary quadratic forms over Z and $Z[(1+\sqrt{p})/2]$

by

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In an earlier work [3] we mentioned that the arithmetic method introduced to prove linear independence of theta series should be applicable to other genera as well as to number fields provided the basic structural features of the quadratic forms in these genera could be overcome. In this paper, we give evidence to this remark. All quadratic forms shall be even positive definite and all genera will be uniquely determined by their discriminants. For convenience we denote by $G := G(n, D)$ the genus of n-ary quadratic forms over Z of discriminant D, and if G is replaced by $\mathfrak{G}$ the corresponding genus over the ring of integers in $Q(\sqrt{p})$, where p throughout shall be an arbitrarily fixed prime congruent to 1 (mod 4). Specifically, we investigate the linear independence of theta series of degrees one and two arising from the forms in $G(4, p^2)$ and $\mathfrak{G}(4, 1)$. We consider each genus separately even though both are closely linked to the genus $G(4, p)$ studied in [3].

A key ingredient of our arithmetic approach is to analyze for each form f its theta series $\theta_f^{(d)}$ (of degree d) modulo q-powers where q is a prime factor of the order of the unit group $O(f)$ of f. For this, we need a rather detailed, albeit technical, knowledge of the arithmetic structures of f and $O(f)$ which we shall determine. However, several new phenomena arise; e.g. (1) the symmetries of f — in the $G(4, p^2)$ case — are not controlled by the minimal vectors, (2) the unit groups $O(f)$ — in the $\mathfrak{G}(4, 1)$ case — are not generated by $\pm$ symmetries of f, (3) the "glueing" construction process of a form $\tilde{f} \in \mathfrak{G}(4,1)$ from an $f \in G(4, p)$ may introduce new minimal vectors. The latter, in the language of quaternion algebras, means that if $\mathfrak{A}$ is the rational quaternion algebra with discriminant p^2 and $\tilde{\mathfrak{A}} = \mathfrak{A} \otimes Q(\sqrt{p})$ then the

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