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On the gaps of the Lagrange spectrum

by

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The *Lagrange spectrum* L is the set of the inverses of the values of Perron's modular function. This function is defined by

$$M(\alpha) = \liminf n \min\{|n\alpha - q| : q \text{ is an integer}\}$$

where α is irrational and n varies over all positive integers (see [10]). A *gap of the Lagrange spectrum* is an open interval (a, b) such that a and b are in L and the intersection of L with (a, b) is empty.

In [2], T. W. Cusick showed that L is closed. Here, the following results are proved: Let (a, b) be a gap of the Lagrange spectrum. Then there is an irrational number α having a continued fraction expansion determined by four finite sequences of integers such that $M(\alpha) = 1/a$. Moreover, a reduced real binary indefinite quadratic form $f(x, y) = Ax^2 + Bxy + Cy^2$ with roots that are quadratic surds is constructed, such that $\sqrt{B^2-4AC}/a$ is the smallest value numerically represented by f. a is the sum of the roots of f. The lengths of the periods and preperiods of the continued fraction expansion of these roots are bounded by

$$2[a]^{[(\ln((b-a)/2))/\ln(2/(1+\sqrt{5}))]+3}.$$

There is no statement made on irrationals or forms related to the right endpoint b. Here we are not concerned with the Markov (Markoff) spectrum, which is the set of the inverses of the arithmetic minima of real indefinite binary quadratic forms. Similar results for the Markov spectrum are given in [4].

In their often quoted essay on quadratic irrationals in the lower Lagrange spectrum (see [3]), Nancy Davis and John Kinney state for both spectra that the endpoints of a gap not exceeding $\sqrt{12}$ are the sum of quadratic irrationals. Here, new tools are used that permit an extension beyond $\sqrt{12}$ for left endpoints of gaps.

Let $A = \ldots, a_{-1}, a_0, a_1, \ldots$ be a doubly infinite sequence of positive integers. A is called *eventually periodic on both sides* if there are integers i, k

such that $a_i, a_{i+1}, \ldots$ and $a_k, a_{k-1}, \ldots$ are periodic. For any doubly infinite sequence A, we define the sequence $\bar{A} = \ldots, \bar{a}_{-1}, \bar{a}_0, \bar{a}_1, \ldots$ by $\bar{a}_k = a_{-k}$ for all k. Let $B_k = \ldots, b_{-1}^{(k)}, b_0^{(k)}, b_1^{(k)}, \ldots$ where k and $b_i^{(k)}$ are positive integers. We say that the sequences $B_1, B_2, \ldots$ converge to A and write $\lim B_k = A$ if for all positive integers n there is an m such that $b_i^{(k)} = a_i$ for all $k \geqslant m$ and all integers i with $|i| \leqslant n$. Let $[a_0, a_1, a_2, \ldots]$ denote the regular simple continued fraction allocated to the sequence of integers $a_0, a_1, \ldots$ Let $M_k(A) = [a_k, a_{k+1}, \ldots] + [0, a_{k-1}, a_{k-2}, \ldots]$ for each integer k, $M(A) = \sup M_k(A)$ and $L(A) = \limsup M_k(A)$.

The set of all $L(A)$ is the Lagrange spectrum (see [10]). The set of all $M(A)$ is the Markov spectrum (see [12]).

The subsequent lemma is classical (see [11]).

LEMMA 1. *Let n, $a_0, a_1, \ldots, b_1, b_2, \ldots$ be positive integers and $\varepsilon_n = ((1+\sqrt{5})/2)^{-2n+1}$. Then*

$$|[a_0, a_1, \ldots] - [a_0, a_1, \ldots, a_n, b_1, b_2, \ldots]| < \varepsilon_n.$$

The following lemmas are essential for our proof. Lemma 2 is related to the compactness theorem for the Markov spectrum (see [13], Theorem 19, [3], Lemma 3, [12]). Lemma 2 implies that the Lagrange spectrum is a subset of the Markov spectrum. This fact is well known (see [2]).

LEMMA 2. *Let a be a finite element of the Lagrange spectrum. Then there are doubly infinite sequences of integers A, B with the following properties:*

$$L(B) = M(A) = M_0(A) = a, \tag{1}$$

there is an increasing sequence of integers $n_1, n_2, n_3, \ldots$ such that

$$\lim \ldots, b_{-1}^{(n_k)}, b_0^{(n_k)}, b_1^{(n_k)}, \ldots = A,$$

where

$$b_i^{(n_k)} = b_{n_k+i} \quad \textit{for all } i, k, \tag{2}$$

$$\lim M_{n_k}(B) = a. \tag{3}$$

Proof. Since a is in the Lagrange spectrum, there is a sequence $B = \ldots, b_{-1}, b_0, b_1, \ldots$ such that $\limsup M_k(B) = a$. Hence, there is a sequence $m_1, m_2, \ldots$ such that $\lim M_{m_k}(B) = a$. Without loss of generality the sequence $m_1, m_2, \ldots$ is increasing for otherwise we can consider $\bar{B}$. Since a is finite, B is bounded. Therefore, it can be shown by induction, that for all integers i, there is a subsequence $N_i = n_1^{(i)}, n_2^{(i)}, \ldots$ of $m_1, m_2, \ldots$ with the property that $b_{n_1+i}^{(i)} = b_{n_k+i}^{(i)}$ for all k, and that N_i is a subsequence of the sequences $N_{|i|}$ and $N_{1-|i|}$. Thus (2) holds with $n_k = n_k^{(k)}$ and $A = \ldots, a_{-1}, a_0, a_1, \ldots$, where $a_k = b_{n_1+k}^{(k)}$. Since $n_1, n_2, \ldots$ is a subsequence of $m_1, m_2, \ldots$, property (3) holds. Lemma 1 and (2) imply that $\lim M_{n_k}(B) = M_0(A)$. This and (3) show that $M_0(A) = a$. Lemma 1 and (2) imply that for all integers k there is a strictly increasing sequence of integers $s_1(k), s_2(k), \ldots$ and a sequence $\delta_1, \delta_2, \ldots$ with $\lim \delta_i = 0$ such that

$$M_k(A) \leqslant M_{s_i(k)}(B) + \delta_i \quad \text{for all } i.$$

Therefore,

$$M_k(A) \leqslant \limsup M_{s_i(k)} \leqslant a \quad \text{for all } k.$$

Thus $M(A) \leqslant a$.

LEMMA 3. *Let $a_0, a_1, a_2, \ldots$ be a sequence of integers bounded by an integer s. Let n be an integer and $u = 2s^{2n}$. Let $\alpha_k = [a_k, a_{k+1}, \ldots]$ for all k. If the numbers $\alpha_0, \alpha_1, \ldots, \alpha_u$ are pairwise distinct, then there are positive integers i, j, r with the subsequent properties:*

$$r > 2n, \tag{4}$$

$$[a_0, a_1, \ldots] < [a_0, a_1, \ldots, a_i, a_j, a_{j+1}, \ldots, a_{j+r}, \alpha] \quad \textit{for all positive numbers } \alpha \geqslant 1, \tag{5}$$

$$a_{j+k} = a_{i+k+1} \quad \textit{for} \quad k = 0, 1, \ldots, 2n-1. \tag{6}$$

Proof. Let $A_k = (a_k, a_{k+1}, \ldots, a_{k+2n-1})$ for all k. By the pigeon hole principle, there are pairwise distinct nonnegative integers $k_1, k_2, k_3 \leqslant u$ such that $A_{k_1} = A_{k_2} = A_{k_3}$. Without loss of generality we may assume that $\alpha_{k_1} < \alpha_{k_2} < \alpha_{k_3}$. This implies that

$$[a_0, a_1, \ldots] < [a_0, a_1, \ldots, a_i, a_j, a_{j+1}, \ldots], \tag{7}$$

where $i = k_2 - 1$, and $j = k_3$ if i is odd and $j = k_1$ if i is even. Let $\alpha^* = [a_0, a_1, \ldots, a_i, a_j, a_{j+1}, \ldots]$ and $\delta = \alpha^* - \alpha_0$. By (7) and Lemma 1, there is an integer $r > 2n$ such that

$$\alpha^* - [a_0, \ldots, a_i, a_j, a_{j+1}, \ldots, a_{j+r}, \alpha] < \delta \quad \text{for all } \alpha \geqslant 1.$$

Thus, (5) holds.

THEOREM 1. *Let (a, b) be a gap of the Lagrange spectrum. Then there is a sequence A, eventually periodic on both sides, such that $M(A) = M_0(A) = a$. The lengths of the periods and preperiods of A are bounded by*

$$2[a]^{[(\ln((b-a)/2))/\ln(2/(1+\sqrt{5}))]+3}.$$

Remark. Replacing the term $(b-a)/2$ by $b-a$ in the above formula produces a bound that is valid for the Markov spectrum (see [4]).

Proof. Let A and B be sequences existing according to Lemma 2. Let n be a positive integer such that $\varepsilon_n < (b-a)/2 \leqslant \varepsilon_{n-1}$. We choose n_1 of Lemma 2 such that

(8) $$M_k(B) < b-2\varepsilon_n \quad \text{for all } k > n_1.$$

We now show, that two of the numbers $\alpha_0, \alpha_1, \ldots, \alpha_u$ as defined in Lemma 3, where $u = 2[a]^{2n}$, are equal. To this end, we assume the contrary and try to find a contradiction. Since $a_k < M_k(A) \leqslant a$ for all k, Lemma 3 is applicable with $s = [a]$. By (2) there is an g such that

(9) $$a_j, a_{j+1}, \ldots, a_{j+r} = b_{n_g+j}, b_{n_g+j+1}, \ldots, b_{n_g+j+r}.$$

According to Lemma 3, there is an $m > 0$ such that

(10) $$\varepsilon_m < \min\{[a_0, a_1, \ldots, a_i, a_j, a_{j+1}, \ldots, a_{j+r}, c_1, c_2, \ldots]: c_k < a \text{ for all } k\} - [a_0, a_1, \ldots].$$

By (2) there is an h such that $n_h - m \geqslant n_g + j$ and

(11) $$b_{n_h-m}, b_{n_h-m+1}, \ldots, b_{n_h-1}, b_{n_h}, \ldots, b_{n_h+i+2n} = a_{-m}, a_{-m+1}, \ldots, a_{i+2n}.$$

Let $p = n_h - n_g - j + i + 1$ and define $A^* = \ldots, a^*_{-1}, a^*_0, a^*_1, \ldots$ by

$$a^*_0, a^*_1, \ldots, a^*_{p-1} = a_0, a_1, \ldots, a_i, a_j, a_{j+1}, \ldots, a_{j+r}, b_{n_g+j+r+1}, b_{n_g+j+r+2}, \ldots, b_{n_h-1}$$

and

$$a^*_k = a^*_{p+k} \quad \text{for all } k.$$

(10), (11) and Lemma 1 imply

(12) $$a = M_0(A) < M_0(A^*).$$

By (6), (9) and (11), $a^*_{i+k} = b_{n_g+k+j-1}$ for $k = 1, 2, \ldots, p+2n$. Hence, by Lemma 1, for all k there is an $s_k > n_1$ such that

(13) $$M_k(A^*) \leqslant M_{s_k}(B) + 2\varepsilon_n < b.$$

The last inequality is due to (8). Since A^* is purely periodic, (12) and (13) imply $a < L(A^*) < b$. This contradiction shows that our assumption does not hold. Hence, $\alpha_{k_1} = \alpha_{k_2}$, where $0 \leqslant k_1 < k_2 \leqslant u$. This implies that $a_0, a_1, \ldots$ is eventually periodic with bounds on the periods and preperiods as indicated (note that $a \geqslant 1$ (see [10])). The sequences $\bar{A}$, $\bar{B}$ satisfy (1), (2), (3) with the decreasing sequence $n_1, n_2, \ldots$ Applying the above argument to $\bar{A}$ and $\bar{B}$ implies that $a_{-1}, a_{-2}, \ldots$ is eventually periodic as well.

A second type of special sequence with Lagrange value a can be found which is proposed to be called 'almost periodic':

Definition. Let $n_1, n_2, \ldots$ and $m_1, m_2, \ldots$ be increasing sequences of integers. Let B, D, P, Q be finite sequences of integers. Let $P^{(k)} = P$ and $Q^{(k)} = Q$ for all k and $P_k = P^{(1)}, P^{(2)}, \ldots, P^{(n_{|k|})}$ and $Q_k = Q^{(1)}, Q^{(2)}, \ldots, Q^{(m_{|k|})}$.

A doubly infinite sequence of integers is called *almost periodic* if it has the form

$$\ldots, B, P_{-1}, D, Q_{-1}, B, P_0, D, Q_0, B, P_1, D, Q_1, \ldots$$

It is denoted by (B, P_k, D, Q_k).

Theorem 2. *Let (a, b) be a gap of the Lagrange spectrum. Then there is an almost periodic sequence of positive integers C such that $L(C) = M(C) = a$.*

Proof. Choose A, B according to Lemma 2. We may assume that there is an $\varepsilon > 0$ such that

(14) $$M_k(B) < b - \varepsilon \quad \text{for all } k > n_1.$$

By Theorem 1 there are positive integers r, s, p, q such that

$$A = \overline{a_{-r-p}, a_{-r-p+1}, \ldots, a_{-r-1}}, a_{-r}, a_{-r+1}, \ldots, a_s, \overline{a_{s+1}, \ldots, a_{s+q}}.$$

The bars label the periodically repeated sections of A. Let n be a positive integer such that

(15) $$2\varepsilon_n < \varepsilon.$$

By (2) there are positive integers $g, h, d_1, d_2, \ldots, d_i$ such that

(16) $$b_{n_g}, b_{n_g+1}, \ldots, b_{n_h-1} = a_0, a_1, \ldots, a_{s+nq}, d_1, d_2, \ldots, d_i, a_{-r-np}, a_{-r-np+1}, \ldots, a_{-1}.$$

For each k, let

$$C_k = a_0, a_1, \ldots, a_{s+(|k|+n)q}, d_1, d_2, \ldots, d_i, a_{-r-(|k|+n)p}, a_{-r-(|k|+n)p-1}, \ldots, a_{-1}$$

and let l_k be the lengths of C_k. Define $C = \ldots, c_{-1}, c_0, c_1, \ldots$ by $C = \ldots, C_{-1}, C_0, C_1, \ldots$ and $C_0 = c_0, c_1, \ldots, c_{l_0-1}$. Let $m_k = \sum_{i=0}^{k-1} l_i$ if $k \geqslant 0$ and $m_k = -\sum_{i=k}^{-1} l_i$ if $k < 0$. Then $\lim M_{m_k}(C) = M_0(A) = a$ and

(17) $$a \leqslant L(C).$$

Now we want to show that $M(C) \leqslant a$. To this end we assume that $M_m(C) > a$ for some m. Choose t, v such that $m_t < m < m_v$ and $2\max\{\varepsilon_{m-m_t}, \varepsilon_{m_v-m}\} < (M_m(C)-a)/2$. Then the sequence $E = \ldots, e_{-1}, e_0, e_1, \ldots$ with $e_1, e_2, \ldots, e_{m_v-m_t} = c_{m_t}, c_{m_t+1}, \ldots, c_{m_v-1}$ and $e_{m_v-m_t+k} = e_k$ for all k satisfies

(18) $$a < M_{m-m_t+1}(E).$$

For each k there is an $s_k > n_1$ such that $M_k(E) < M_{s_k}(B) + 2\varepsilon_n$. Hence, (14) and (15) give $M_k(E) < b$. By (18), $a < L(E) < b$. This contradiction shows that our assumption does not hold. Thus, (17) implies $M(C) = L(C) = a$.

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On sign-changes in the remainder-term of the prime-number formula, II

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1. In the first paper of this series [3] we have proved that the differences

$$\Delta_3(x) = \psi(x) - x = \sum_{n \leqslant x} \Lambda(n) - x \tag{1.1}$$

and

$$\Delta_2(x) = \Pi(x) - \mathrm{li}\, x = \sum_{m \geqslant 1} \frac{1}{m} \pi(x^{1/m}) - \int_0^x \frac{du}{\log u} \tag{1.2}$$

which are the remainder-terms in the prime-number formula change sign at least

$$\frac{\gamma_0}{4\pi} \log T \tag{1.3}$$

times, in the interval $[2, T]$, $T \geqslant T_0$, where $\gamma_0 = 14.13\ldots$ denotes here the imaginary part of the "lowest" zero of the Riemann zeta function. T_0 stands for a positive, effectively computable numerical constant.

We have two other remainders in the prime-number theorem; the most intensively studied in the literature

$$\Delta_1(x) = \pi(x) - \mathrm{li}\, x = \sum_{p \leqslant x} 1 - \mathrm{li}\, x \tag{1.4}$$

and also

$$\Delta_4(x) = \vartheta(x) - x = \sum_{p \leqslant x} \log p - x. \tag{1.5}$$

As in [3] let us denote by $V_j(T)$, $1 \leqslant j \leqslant 4$, the number of sign-changes of $\Delta_j(x)$ in $[2, T]$, $T \geqslant 2$.

Let

$$\Theta = \sup_{\zeta(\varrho)=0} \mathrm{Re}\, \varrho \tag{1.6}$$

where $\zeta(s)$ is the Riemann zeta-function.