7. (Conway [1], p. 121)

$$T(x) = \begin{cases} 3x/2 & \text{if } 4|x, \\ (3x+1)/4 & \text{if } 4|x-1, \\ 3x/2 & \text{if } 4|x-2, \\ (3x-1)/4 & \text{if } 4|x-3. \end{cases}$$

(This transformation is 1-1.)

$$Q(4) = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

and is primitive. The stationary vector is $[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]^t$ and $S' = Z_4$ is the only ergodic set. Also

$$\prod_{\substack{i=0 \\ B(i,d)\in S'}}^{d-1} \left(\frac{m_i}{d}\right)^{\mu_{S'}(B(i,d))} = \left(\frac{3}{2}\right)^{\frac{1}{4}} \left(\frac{3}{4}\right)^{\frac{1}{4}} \left(\frac{3}{2}\right)^{\frac{1}{4}} \left(\frac{3}{4}\right)^{\frac{1}{4}} > 1.$$

Hence we expect most trajectories to be unbounded.

In conclusion, we wish to thank R. N. Buttsworth and G. Leigh for some useful comments.

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Algebraic numbers near the unit circle

by

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1. Introduction. Let α be an algebraic number of degree $n \geqslant 2$ over the rationals, with conjugates $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$. In this paper we investigate conditions on α which imply that α has one or more conjugates on the unit circle. Our conditions will be expressed in terms of the following two functions:

(1.1) $$\overline{|\alpha|} = \max_{1 \leqslant j \leqslant n} |\alpha_j|,$$

(1.2) $$\Lambda(\alpha) = \prod_{j=1}^{n} \max\{1, |\alpha_j|\}.$$

Also we require the notion of denominator of an algebraic number.

Definition 1. Let $P(x) = qx^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be the primitive minimal polynomial of α over the rationals, where $q > 0$. Then we say that q is the *denominator of* α (cf. Blanksby [3]).

It is clear that α has denominator 1 if and only if α is an algebraic integer.

It is easy to see that there exist positive functions $\varphi(n, q)$ and $\psi(n, q)$ such that for all algebraic numbers α of degree n and denominator q,

1) either $\overline{|\alpha|} \leqslant 1$ or

(1.3) $$\overline{|\alpha|} \geqslant 1 + 1/\varphi(n, q),$$

2) either $\Lambda(\alpha) = 1$ or

(1.4) $$\Lambda(\alpha) \geqslant 1 + 1/\psi(n, q).$$

There arises the problem of finding the best possible functions φ and ψ satisfying (1.3) and (1.4) respectively. There has been much recent work on this problem in the case $q = 1$ but very little in the general case.

Dobrowolski [9] has shown that one can take ([1])

$$\psi(n, 1) = \frac{1}{1-\varepsilon}\left(\frac{\log n}{\log\log n}\right)^3$$

for any ε satisfying $0 < \varepsilon < 1$ and $n > n_0 = n_0(\varepsilon)$. As a corollary, he found that we can take

$$\varphi(n, 1) = \frac{n}{2-\varepsilon}\left(\frac{\log n}{\log\log n}\right)^3$$

for any $\varepsilon > 0$ and n sufficiently large.([1])

If α is a non-reciprocal algebraic integer, one can prove even stronger results. For instance, Smyth [20] obtained the best possible result $A(\alpha) \geqslant \theta_1$ where θ_1 is the real root of $x^3-x-1 = 0$. As a corollary, we get $\boxed{\alpha} > 1+(\log\theta_1)/n$.

The case $q = 1$ has been attacked by a number of authors, using a wide variety of methods. Unfortunately, most of the methods do not appear to generalize to algebraic numbers of denominator $q > 1$. However the techniques of Cassels used in [7] can be extended to handle arbitrary algebraic numbers. Smyth's technique is also capable of extension to algebraic numbers of norm ± 1 (cf. Pathiaux [14], Notari [13], Amara [2]), provided they are not reciprocal. Chowla [8] has given a result on reciprocal algebraic numbers which implies that one can take $\psi(n, q) = (qn^n)^{cn^2}$ where c is effectively computable.

In the general case nothing appears to have been published. We shall state and prove such results in the following sections of this paper.

2. Discussion of the results. In the sequel, let α denote an algebraic number of degree n and denominator $q > 0$ (as in Definition 1). Let the conjugates of α be $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n$. Denote the absolute norm of α by $N(\alpha) = \alpha_1 \ldots \alpha_n$. Recall that $n > 1$.

It is convenient to divide the results into two classes, according as $|N(\alpha)| \geqslant 1$ or $|N(\alpha)| < 1$. In the case $|N(\alpha)| \geqslant 1$, there is always a conjugate α_j, say, such that $|\alpha_j| \geqslant 1$. Further, if α is not reciprocal, then $\boxed{\alpha} > 1$ and so there exist functions φ and ψ satisfying (1.3) and (1.4) respectively. Also, if α is not reciprocal, one can adapt a technique used by Smyth [20], although it is much older, having been used by Pisot [15] and Siegel [19]. If α is reciprocal, then a technique of Cassels [7] can be applied. In the case $|N(\alpha)| < 1$, there need not be any conjugates on or outside the unit circle, but if there are any such conjugates, we can obtain results of the form (1.3) and (1.4).

([1]) The constants $1-\varepsilon$ and $2-\varepsilon$ in these inequalities have recently been slightly improved to $2-\varepsilon$ and $4-\varepsilon$ respectively, by V. Rausch, D. Cantor and E. Straus, and the first one to $9/4-\varepsilon$ by R. Louboutin (see R. Louboutin, *Sur la mésure de Mahler d'un nombre algébrique*, Comptes Rendus, t. 296, série I, No. 16 (1983), pp. 707–708.)

However the results for $|N(\alpha)| < 1$ take a different shape from those for the case $|N(\alpha)| \geqslant 1$. Hence it is best to treat the two cases separately.

Many of the results given in this paper do not appear to be as strong as is probably true.

The results for the case $|N(\alpha)| \geqslant 1$ will be expressed in terms of $A(\alpha)$ and the corresponding results for $\boxed{\alpha}$ will be derived as corollaries of those for $A(\alpha)$. We prove the following results.

THEOREM 1. *Suppose that* $|N(\alpha)| \geqslant 1$ *and*

$$A(\alpha) \leqslant 1+1/(4nq^2). \tag{2.1}$$

Then α *has at least two conjugates on the unit circle.*

As a consequence of Theorem 1 we get

COROLLARY 1. *Assume that* $|N(\alpha)| \geqslant 1$ *and*

$$\boxed{\alpha} \leqslant 1+1/(5n^2q^2). \tag{2.2}$$

Then α *has at least two conjugates on the unit circle.*

This corollary extends a theorem of Cassels [7], as slightly improved by Schinzel [17]. We can also generalize Theorem 1 as follows.

THEOREM 2. *Suppose that* $|N(\alpha)| \geqslant 1$ *and let* s *be the number of conjugates of* α *lying strictly outside the unit circle and suppose* $s \geqslant 1$. *Let* $A = 1+n/(2s)$ *and* $B = n/s$. *Then we have*

$$A(\alpha) > 1+s/(5^B n^A q^B). \tag{2.3}$$

As a consequence of Theorem 2 we get

COROLLARY 2. *Assuming the hypotheses of Theorem 2 we have*

$$\boxed{\alpha} > 1+1/(6^B n^A q^B). \tag{2.4}$$

Further it is possible to give conditions for all the conjugates of α to lie on the unit circle. Such results are of interest because of the following result of Blanksby and Loxton [5]:

An algebraic number field is a CM-field if and only if it is generated over the rationals by an element α *such that* $\boxed{\alpha} = 1$ $(\alpha \neq \pm 1)$.

Their result in a sense generalizes a theorem of Kronecker for algebraic integers (see [11]). Unfortunately our conditions for $\boxed{\alpha} = 1$ appear to be fairly weak. Thus our next theorem will, in effect, give a condition which assures that α is a generator of a CM-field.

THEOREM 3. *If* $|N(\alpha)| \geqslant 1$ *then either* $A(\alpha) = 1$ *or*

(2.5) $$\Lambda(\alpha) > 1+2/(5^{n/2} n^{1+n/4} q^{n/2}).$$

Similarly we have

COROLLARY 3. *If* $|N(\alpha)| \geqslant 1$ *then either* $\overline{|\alpha|} = 1$ *or*

(2.6) $$\overline{|\alpha|} > 1+1/(6^{n/2} n^{1+n/4} q^{n/2}).$$

Similarly the results for the case $|N(\alpha)| < 1$ will be expressed in terms of $\Lambda(\alpha)$. The corresponding results for $\overline{|\alpha|}$ are corollaries of those for $\Lambda(\alpha)$. It is necessary to assume that $n > 1$ and $q > 1$.

THEOREM 4. *Suppose that* $|N(\alpha)| < 1$. *Let* s *be the number of conjugates of* α *lying outside the unit circle, and suppose that* $s \geqslant 1$. *Let* $C = n/s$. *Then we have*

(2.7) $$\Lambda(\alpha) > 1+s/(2^{n+2} n^C q^{2C}).$$

COROLLARY 4. *Assuming the hypotheses of Theorem* 4, *we have*

(2.8) $$\overline{|\alpha|} > 1+1/(3\cdot 2^{n+1} n^C q^{2C}).$$

COROLLARY 5. *If* $|N(\alpha)| < 1$ *then either* $\overline{|\alpha|} < 1$ *or*

(2.9) $$\overline{|\alpha|} > 1+1/(3\cdot 2^{n+1} n^{n/2} q^n).$$

Finally we give a few results for the case where α belongs to a J-field.

We recall that an algebraic number field is called a *J-field* if either it is totally real or else it is a totally imaginary quadratic extension of a totally real field. A non-real J-field is usually called a *CM-field*. It is well known that a CM-field can also be characterized as a totally imaginary field which is closed under complex conjugation and is such that complex conjugation commutes with all its Q-monomorphisms into the field of complex numbers. This alternative characterization of CM-fields will be used in the sequel. See Blanksby and Loxton [5] for references to the literature.

As the reader might expect, the results for J-fields are much stronger than the general results previously given. We note if $\overline{|\alpha|} = 1$, then all conjugates of α lie on the unit circle. We confine our attention to lower bounds for $\overline{|\alpha|}$.

THEOREM 5. *Let* α *be an algebraic number of denominator* q *and degree* n, *belonging to a J-field. Suppose* $\overline{|\alpha|} > 1$ *and let* m *be the number of conjugates* α_j *satisfying the condition*

$$\overline{|\alpha|}^{-1} \leqslant |\alpha_j| \leqslant \overline{|\alpha|}.$$

Then

(2.10) $$\overline{|\alpha|}^2 \geqslant 1+1/q^{2/m}$$

with equality if and only if $m = n$ *and*

(2.11) $$|\alpha_j|^2 = 1+1/q^{2/n} \quad (1 \leqslant j \leqslant n).$$

If, in addition, α *is totally positive, then*

$$\overline{|\alpha|} \geqslant 1+1/q^{1/m}$$

with equality if and only if $\alpha = 1+1/q$.

Remark. It follows from the theorem of Blanksby and Loxton [5] that if α satisfies (2.10), then $Q(\alpha)$ is a J-field if and only if $1+1/q^{1/m}$ is rational.

As a consequence of Theorem 5, we get

COROLLARY 6. *Let* α *and* m *be as in Theorem* 5. *Then if* $m \geqslant 2$, *we have*

(2.12) $$\overline{|\alpha|}^2 \geqslant 1+1/q,$$

with equality only in the following cases:

1) *If* $q > 1$, *then* α *must be a zero of the quadratic* $qx^2-bx+(q+1)$, *where* b *is a rational integer satisfying* $b^2 < 4q(q+1)$, *or of* $qx^2-(q+1)$.

2) *If* $q = 1$, *then* α *is an algebraic integer of the form* $\gamma\sqrt{2}$ *where* γ *is an algebraic number satisfying* $\overline{|\gamma|} = 1$; *in particular, if* α *is quadratic, it is either a zero of* x^2-2 *or of* x^2-bx+2 *where* $b = 0$, ± 1 *or* ± 2.

Remark. The condition $m \geqslant 2$ of the Corollary is satisfied in any of the following cases:

(i) α is totally complex,
(ii) $-\alpha$ is a conjugate of α,
(iii) $|N(\alpha)| \geqslant 1$, $n > 1$.

3. Preliminaries. Here we give some lemmas which are required in the proofs of the results stated in the previous section. However, a fundamental result on non-reciprocal algebraic numbers α with $|N(\alpha)| \geqslant 1$ will be deferred till the next section.

The following lemma is basic in what follows.

LEMMA 1. *Let* α *be an algebraic number of denominator* q *and degree* n *with conjugates* $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$. *Then* q *times the product of distinct conjugates of* α *is an algebraic integer. In particular,* $q\Lambda(\alpha)$ *is an algebraic integer.*

Proof. The first part of the Lemma is proved in Hecke's book [10]. The second part is an immediate consequence of the first part of the Lemma. ∎

Next we give an elementary inequality due to Schinzel.

LEMMA 2. *Let* $y_1, y_2, \ldots, y_n$ *be real numbers satisfying* $y_j \geqslant 1$ *for* $1 \leqslant j \leqslant n$. *Then we have*

(3.1) $$\prod_{j=1}^{n}(y_j-1) \leqslant \left((y_1 y_2 \ldots y_n)^{1/n}-1\right)^n$$

with equality if and only if $y_1 = y_2 = \ldots = y_n$.

Proof. See Lemma 3 of Schinzel [18]. One can also use inequality

3.2.34 in D. S. Mitrinović, *Analytic inequalities*, Springer Verlag 1970, p. 208 putting there $a_k = y_k - 1$. ∎

We shall also need some extensions of the work of Cassels [7].

LEMMA 3. *Let $n > 1$ be an integer and let $\varrho \geq 1$ be a real number. Let $z_1, z_2, \ldots, z_n$ be complex numbers satisfying $|z_j| \leq \varrho$ for $1 \leq j \leq n$. Then we have*

$$\prod_{j \neq k} |z_j \bar{z}_k - 1| \leq n^n \varrho^{2n(n-1)}. \tag{3.2}$$

Proof. See Alexander [1]. ∎

Cassels proved (3.2) under the restriction

$$\cos(\pi/n) \leq \varrho^2/(\varrho^4 - \varrho^2 + 1).$$

LEMMA 4. *Let $m \geq 1$ be an integer and let $\varrho > 1$ be a real number. Let $w_1, w_2, \ldots, w_m$ be complex numbers satisfying*

$$\varrho^{-1} \leq |w_j| \leq \varrho \quad (1 \leq j \leq m). \tag{3.3}$$

Then we have

$$\prod_{j \neq k} |w_j \bar{w}_k - 1| \, |w_j w_k^{-1} - 1| \, |w_j^{-1} w_k - 1| \, |w_j^{-1} \bar{w}_k^{-1} - 1| \leq m^{4m} \varrho^{8m(m-1)}. \tag{3.4}$$

Proof. This is similar to Lemma 3 of Cassels [7], using our Lemma 3 which is valid for all $\varrho \geq 1$. The details are given in Lloyd-Smith [12]. ∎

We shall also need a specialized result analogous to Lemmas 3 and 4, due to Cassels [7].

LEMMA 5. *Let $n > 1$ be an integer and let $\theta_1, \theta_2, \ldots, \theta_n$ be real numbers such that $0 \leq \theta_j \leq 2\pi$ $(1 \leq j \leq n)$. Define ψ by*

$$2n\psi = \theta_1 + \theta_2 + \ldots + \theta_n.$$

Let $r > 1$ be a real number such that

$$|\cos \psi| < r/(r^2 - r + 1).$$

Then

$$\prod_{j=1}^{n} |re^{i\theta_j} - 1| \leq |re^{2i\psi} - 1|^n$$

with equality only when $\theta_1 = \theta_2 = \ldots = \theta_n = 2\psi$.

Proof. See Cassels [7]. ∎

4. A fundamental result on non-reciprocal algebraic numbers. In this section we shall give results of the type (1.3) and (1.4) in the case where α is non-reciprocal and $|N(\alpha)| \geq 1$. We shall also see why the condition $|N(\alpha)| \geq 1$ is necessary. In addition our results will permit the simplification of the proofs of Theorems 1, 2 and 3 as it will suffice to assume that α is reciprocal.

PROPOSITION 1. *Let α be a non-reciprocal algebraic number of degree $n > 1$ and denominator q, satisfying $|N(\alpha)| \geq 1$. Then*

$$\text{(i)} \quad \Lambda(\alpha) \geq 1 + 1/(4q), \tag{4.1}$$

$$\text{(ii)} \quad \overline{|\alpha|} > 1 + 1/(5nq). \tag{4.2}$$

This is not the best result of its kind. For instance, Notari [13] has shown that, if $|N(\alpha)| \geq 1$, then $\Lambda(\alpha) \geq \psi_q$, where ψ_q is the larger zero of $x^2 - \varrho x - 1$ where

$$\varrho^2 = \frac{16q^2 - 8q + 5}{8q^2(8q^2 - 4q + 1)}, \quad \varrho > 0.$$

Still assuming $|N(\alpha)| \geq 1$, Lloyd-Smith [12] independently obtained Notari's result by a somewhat different argument similar to that in Schinzel [18] and also obtained the slightly better result $\Lambda(\alpha) \geq \psi'_q$, where ψ'_q is the largest zero of

$$Ax^4 - Bx^2 + A = 0$$

where

$$A = \left(1 + \left(1 + \frac{1}{4q}\right)^2\right)\left(1 + \left(1 - \frac{1}{4q}\right)^2\right),$$

$$B = \left(1 + \left(1 + \frac{1}{4q}\right)^2\right)^2 + \left(1 + \left(1 - \frac{1}{4q}\right)^2\right)^2 + \frac{1}{4q^4}.$$

Proposition 1 has also been proved by Pathiaux [14].

The proof of Proposition 1 is a straightforward generalization of the argument in Smyth [20]. It rests on the following lemma concerning analytic functions.

LEMMA 6. *Let $f(z) = \sum_{k=0}^{\infty} e_k z^k$ be analytic in an open disc containing $|z| = 1$ and satisfy $|f(z)| \leq 1$ on $|z| = 1$. Then*

$$|e_k| \leq 1 - e_0^2 \quad (k = 1, 2, \ldots).$$

Proof. This result is due to Wiener but its proof first appeared in Bohr [6]. Also see Smyth [20]. ∎

The proof of Proposition 1 is omitted. See Pathiaux [14], Notari [13] or Lloyd-Smith [12].

It is not known whether it is true that $\Lambda(\alpha) \geq \theta_q$ if $|N(\alpha)| \geq 1$, where θ_q is the real zero of $qx^3 + (q-1)x^2 - qx - q$. This result, if true, is a natural generalization of Smyth's result. Also this problem is of interest as it is connected to the set S of PV numbers and its analogues for algebraic numbers of denominator q. Define S_q to be the set of all real algebraic numbers $\beta > 1$ such that β is a zero of an irreducible polynomial with integer coefficients and leading coefficient q where $|N(\beta)| \geqslant 1$ and all conjugates of β except itself lie inside the unit circle. It is known (see Pisot [15]) that θ_q is the least element of S_q. The fact that θ_1 is the least element of S_1 is a corollary of Smyth's work, but it was first proved by Siegel [19].

Also Notari [13] has made the following observation:

Let $P(z) = qz^n + a_1 z^{n-1} + \ldots + a_n$ *be the minimal polynomial of* α. *Let* $\varepsilon = \pm 1$ *be such that* $\varepsilon a_n > 0$. *Then* $\Lambda(\alpha) \geqslant \theta_q$ *if* $a_1 \neq \varepsilon a_{n-1}$.

This observation follows readily from the proof of Notari's result, cited just after Proposition 1.

There exist examples which show that the condition $|N(\alpha)| \geqslant 1$ is necessary for Proposition 1. The first example is due to Prenat [16]. The polynomial $qx^4 + (q-1)x^3 - x^2 - qx - (q-1)$ has a single zero outside the unit circle, say θ. It turns out that

$$1 + 1/(6q) < \theta < 1 + 1/(4q).$$

Thus the condition $|N(\alpha)| \geqslant 1$ is necessary for the first part of Proposition 1.

We give an example to show that it is possible to have

$$1 + 1/(5q^2) < \Lambda(\alpha) < 1 + 1/q^2.$$

Consider the polynomial $H(x) = qx^3 - qx^2 - (q-3)x + (q-1)$. It turns out that the discriminant of $H(x)$ is equal to $-2q(32q^2 - 72q + 54)$ which is negative for positive integers q. Hence $H(x)$ has exactly one real zero, say β. It is easily shown that β lies between $-1 + 1/q$ and $-1 + 1/q + 1/q^2$. In fact, since $q > 1$, we have

$$H\left(\frac{1}{5q^2} + \frac{1}{q} - 1\right) < 0 < H\left(\frac{2}{5q^2} + \frac{1}{q} - 1\right).$$

It is clear that $H(x)$ has a pair of complex conjugate zeros outside the unit circle. We deduce, after a straightforward calculation, that

$$1 + 1/(5q^2) < \Lambda(\beta) < 1 + 1/q^2.$$

Similarly, we can show that the condition $|N(\alpha)| \geqslant 1$ is necessary for the second part of Proposition 1. Prenat's example cannot be used for this purpose. Let β be the algebraic number specified in our example above. For any positive integer m, set $\alpha = \beta^{1/m}$. Then we have

$$\overline{|\alpha|} < \left(1 + \frac{1}{q^2}\right)^{1/(2m)} \leqslant \exp\left(\frac{1}{2mq^2}\right) < 1 + \frac{1}{mq^2}$$

and

$$\overline{|\alpha|} > \left(1 + \frac{1}{5q^2}\right)^{1/(2m)} > 1 + \frac{1}{20\,mq^2}.$$

Thus our claim is verified, as α is of degree $3m$.

5. Proof of the results for the case $|N(\alpha)| \geqslant 1$. In this section we give the proofs of Theorems 1, 2 and 3 and their corollaries as stated in Section 2. We commence by reducing the problem to the case where α is reciprocal (and so $|N(\alpha)| = 1$).

If α is non-reciprocal and $|N(\alpha)| \geqslant 1$, Theorems 1, 2 and 3 and their corollaries follow from Proposition 1. Hence we shall in the remainder of this section assume that α is reciprocal. It follows that we can assume $|N(\alpha)| = 1$.

Proof of Theorem 1. Suppose α satisfies (2.1). Since α is reciprocal, n is even and so we can write $n = 2m$ where m is a positive integer. Also we can write the conjugates of α in the shape

$$\beta_1, \beta_1^{-1}, \beta_2, \beta_2^{-1}, \ldots, \beta_m, \beta_m^{-1}.$$

We shall suppose that the theorem is false and ultimately derive a contradiction. We employ the product

$$P = \prod_{\alpha_j\alpha_k \neq 1} |\alpha_j \alpha_k - 1|.$$

It is clear that P is rational. A straightforward application of Lemma 1 shows that $P \geqslant q^{-(2n-2)}$. Since we are assuming Theorem 1 to be false, we can write $P = P_1 P_2$ where

$$P_1 = \prod_{j=1}^{n} |\alpha_j \bar{\alpha}_j - 1|$$

and

$$P_2 = \prod_{j \neq k} |\beta_j \bar{\beta}_k - 1|\,|\beta_j \bar{\beta}_k^{-1} - 1|\,|\beta_j^{-1} \bar{\beta}_k - 1|\,|\beta_j^{-1} \bar{\beta}_k^{-1} - 1|.$$

By Lemma 4 we get

$$P_2 \leqslant m^{4m} \varrho^{8m(m-1)} \leqslant 2^{-2n} n^{2n} \exp(n/(5q^2)).$$

Now set $c = \Lambda(\alpha)$. It is clear that α has exactly m conjugates outside the unit circle. By Lemma 2 we find

$$P_1 \leqslant \frac{1}{c^2}(c^{4/n} - 1)^n. \tag{5.1}$$

On the other hand we also find

$$P_1 = \frac{P}{P_2} \geqslant \frac{2^{2n}}{n^{2n} q^{2n} \exp(n/(5q^2))}. \tag{5.2}$$

From (5.1) and (5.2) we deduce

$$c^{4/n} - 1 > 2/(n^2 q^2)$$

and hence that

$$\Lambda(\alpha) \geqslant 1+1/(4nq^2), \qquad \text{since } n \geqslant 2.$$

This contradicts (2.1). Therefore α has a conjugate on the unit circle. Clearly it is not real and therefore there is another conjugate of α on the unit circle. This completes the proof of Theorem 1. ∎

Proof of Corollary 1. Suppose we had

$$\overline{|\alpha|} \leqslant 1+1/(5n^2q^2).$$

Then

$$\Lambda(\alpha) \leqslant \overline{|\alpha|}^n \leqslant \left(1+\frac{1}{5n^2q^2}\right)^n \leqslant \exp\left(\frac{1}{5nq^2}\right) \leqslant 1+\frac{1}{4nq^2},$$

where we have used the inequality $e^{\delta} \leqslant 1+\delta(1-\delta)^{-1}$ for $0 \leqslant \delta < 1$. The result follows from Theorem 1. ∎

Proof of Theorem 2. Without loss of generality, we may assume that

$$\overline{|\alpha|} < \varrho = 1+\frac{s}{4n^2q^2}.$$

Because α is reciprocal, we know that $\varrho^{-1} < |\alpha_j| < \varrho$ for $j = 1, 2, \ldots, n$. Also n is even. The case $n = 2$ is a trivial consequence of Theorem 1. Henceforth we assume $n \geqslant 4$.

Following an idea of Blanksby [4] we employ the product

$$P = \prod_{\alpha_j\alpha_k \neq 1} |\alpha_j \alpha_k - 1|.$$

Clearly P is rational and Lemma 1 shows that $P \geqslant q^{-(2n-2)}$. We may write P as a product $P = P'P''$ where

$$P' = \prod_{|\alpha_j| \neq 1} |\alpha_j \bar{\alpha}_j - 1|, \qquad P'' = \prod_{\substack{\alpha_j\bar{\alpha}_k \neq 1\\ j\neq k}} |\alpha_j \bar{\alpha}_k - 1|.$$

Without loss of generality, we may write the conjugates in the form $\alpha_j = r_j \exp(i\varphi_j)$ with $0 \leqslant \varphi_1 \leqslant \varphi_2 \leqslant \ldots \leqslant \varphi_n < 2\pi$, where if $|\alpha_j| < 1$, then $\alpha_{j+1} = \bar{\alpha}_j^{-1}$. Each factor of P has the form $|Re^{i\theta}-1|$ where $\varrho^{-2} < R < \varrho^2$. A simple geometric argument shows that

$$|Re^{i\theta}-1| < |\varrho^2 e^{i\theta}-1|.$$

The products P' and P'' contain, respectively, $2s$ and n^2-n-2s terms. Since $s \leqslant n/2$, it follows that

$$\overline{|\alpha|} < 1+1/(8nq^2) \leqslant 1+1/(8n).$$

Let $S = \{(j, k)\colon \alpha_j\bar{\alpha}_k \neq 1, j \neq k\}$. We write P'' as

$$P'' = \prod_{(j,k)\in S} |\alpha_j\bar{\alpha}_k - 1| \leqslant \prod_{t=1}^{n-1} P_t$$

where

$$P_t = \prod_{\substack{j-k \equiv t \,(\mathrm{mod}\, n)\\ (j,k)\in S}} |\varrho^2 e^{i(\varphi_j-\varphi_k)}-1|.$$

Now P_1 and P_{n-1} each contain $n-s$ factors while, for $2 \leqslant t \leqslant n-2$, P_t contains n factors. The factors $P_1, P_2, \ldots, P_{n-1}$ are estimated using Lemma 5. In that lemma we take $r = \varrho^2$ and

$$\theta_j = \begin{cases} \varphi_j-\varphi_k & \text{if } j > k,\\ \varphi_j-\varphi_k+2\pi & \text{if } j < k.\end{cases}$$

For $2 \leqslant t \leqslant n-2$, we take $\psi = t\pi/n$ and, by Lemma 5,

$$P_t \leqslant |\varrho^2 e^{2\pi i t/n}-1|^n.$$

Similarly we find

$$P_1 \leqslant |\varrho^2 e^{2\pi i/(n-s)}-1|^{n-s}$$

A straightforward calculation yields

$$P_1^2 < \left(\frac{5\pi\varrho}{n}\right)^{2(n-s)}.$$

Similarly we find

$$P_{n-1}^2 < \left(\frac{5\pi\varrho}{n}\right)^{2(n-s)}.$$

Also we have for $n \geqslant 4$,

$$|\varrho^2 e^{2\pi i/n}-1| \geqslant |e^{2\pi i/n}-1| \geqslant 4/n.$$

This leads to

$$P'' \leqslant \left(\frac{5\pi\varrho}{n}\right)^{2n-2s} \frac{\prod_{t=1}^{n-1} |\varrho^2 e^{2\pi i t/n}-1|^n}{|\varrho^2 e^{2\pi i/n}-1|^{2n}} < n^n \varrho^{2n^2} \left(\frac{n}{4}\right)^{2n} \left(\frac{5\pi\varrho}{n}\right)^{2n-2s}.$$

Let $c = \Lambda(\alpha)$. Recall that α has exactly s conjugates outside the unit circle. By Lemma 2, we find that

$$P' \leqslant c^{-2}(c^{2/s}-1)^{2s} \leqslant (c^{2/s}-1)^{2s}.$$

Also we have

$$P' = \frac{P}{P''} > \frac{1}{q^{2n} n^n \varrho^{2n^2}} \left(\frac{n}{4}\right)^{2n} \left(\frac{n}{5\pi\varrho}\right)^{2n-2s}.$$

A straightforward calculation leads to

$$c > 1 + \frac{s}{5^B n^A q^B}.$$

This completes the proof of Theorem 2. ∎

Proof of Corollary 2. Suppose we had

$$\overline{|\alpha|} \leqslant 1 + \frac{1}{6^B n^A q^B}.$$

By following the method of proof of Corollary 1, we obtain

$$\Lambda(\alpha) \leqslant 1 + \frac{s}{5^B n^A q^B}$$

which contradicts Theorem 2. The proof of Corollary 2 is complete. ∎

Proof of Theorem 3. Assume that $\Lambda(\alpha) \neq 1$. If $s \geqslant 2$ the result is an obvious consequence of Theorem 2. Henceforth, assume $s = 1$. Without loss of generality we assume that α is the only conjugate lying outside the unit circle, and that $\alpha > 1$ (since it is necessarily real).

We consider the product

$$P = \prod_{j=1}^{n} |\alpha_j - 1|.$$

Lemma 1 shows that $P \geqslant q^{-1}$. For $j \neq 1$, we have $|\alpha_j - 1| \leqslant 2$. This gives

$$1/q \leqslant P < 2^{n-1}(\alpha - 1)$$

and hence

$$\alpha > 1 + \frac{1}{2^{n-1} q} > 1 + \frac{2}{5^{n/2} n^{1+n/4} q^{n/2}} \quad \text{since } n \geqslant 2.$$

Thus the proof of Theorem 3 is complete. ∎

Proof of Corollary 3. This result follows easily by combining Corollary 2 with the argument used to complete the proof of Theorem 3. ∎

6. Proof of the results for the case $|N(\alpha)| < 1$. In this case, it is not necessarily true that α will have any conjugates outside the unit circle at all. For example, one need only take α to be a zero of $qx^n - 1$ with $q > 1$. However, if there are any conjugates of α outside the unit circle, then they are "bounded away" from the unit circle in the sense implied by Theorem 4 and its corollaries. Also there can be no conjugates on the unit circle, for otherwise α would be reciprocal, implying $|N(\alpha)| = 1$.

We shall assume that $q > 1$ and that α has at least one conjugate outside the unit circle.

Proof of Theorem 4. Without loss of generality, we may assume that $\overline{|\alpha|} \leqslant \varrho$ where $\varrho = 1 + s/(4nq^2)$. As usual, we employ the product

$$P = \prod_{j,k=1}^{n} |\alpha_j \alpha_k - 1|.$$

We may write $P = P' P''$ where

$$P' = \prod_{j=1}^{n} |\alpha_j \bar{\alpha}_j - 1|, \qquad P'' = \prod_{j \neq k} |\alpha_j \bar{\alpha}_k - 1|.$$

By Lemma 2 we get

$$P'' \leqslant n^n \varrho^{2n(n-1)}.$$

We can write P' in the shape

$$P' = \prod_{|\alpha_j| > 1} |\alpha_j \bar{\alpha}_j - 1| \prod_{|\alpha_j| < 1} |\alpha_j \bar{\alpha}_j - 1|.$$

Letting $c = \Lambda(\alpha)$, a straightforward application of Lemma 3 leads to

$$P' < (c^{2/s} - 1)^s.$$

Using arguments similar to those employed in the proof of Theorem 2, we find

$$c^{2/s} > 1 + \frac{1}{n^{n/s} q^{2n/s} \varrho^{2n(n-1)/s}}.$$

We have $\varrho^{2n(n-1)/s} < 2^n$ and a simple calculation yields

$$\Lambda(\alpha) > 1 + \frac{s}{2^{n+2} n^C q^{2C}}.$$

This completes the proof of Theorem 4. ∎

Proof of Corollary 4. Suppose we had

$$\overline{|\alpha|} \leqslant 1 + \frac{1}{3 \cdot 2^{n+1} n^C q^{2C}}.$$

A simple calculation shows that

$$\Lambda(\alpha) \leqslant 1 + \frac{s}{2^{n+2} n^C q^{2C}}$$

contrary to Theorem 4. ∎

Proof of Corollary 5. This follows from Corollary 4 and the method of proof of Theorem 3, noting that $\lceil\alpha\rceil \neq 1$, otherwise $|\alpha_j| = 1$ for $1 \leqslant j \leqslant n$, contradicting $|N(\alpha)| < 1$. ∎

7. Proof of the results for J-fields. The assumption that α belongs to a J-field allows us to modify Cassels' basic idea so as to obtain much stronger results than in the general case. Let σ be any Q-monomorphism of such a field into the field of complex numbers. Then it is easily seen that $|\sigma(\alpha)|^2 = \sigma(|\alpha|^2)$. It follows that the product

$$P = \prod_{j=1}^{n} (|\alpha_j|^2 - 1) \tag{7.1}$$

is rational.

Proof of Theorem 5. Since $\lceil\alpha\rceil > 1$, Lemma 1 and the comments above show that P is a non-zero rational number satisfying $|P| \geqslant q^{-2}$. We have

$$|P| = \prod_{|\alpha_j|>1} (|\alpha_j|^2 - 1) \prod_{\lceil\alpha\rceil^{-1} \leqslant |\alpha_j| < 1} |\alpha_j|^2 (|\alpha_j|^{-2} - 1) \prod_{|\alpha_j| < \lceil\alpha\rceil^{-1}} (1 - |\alpha_j|^2) \tag{7.2}$$

$$\leqslant (\lceil\alpha\rceil^2 - 1)^m.$$

This yields (2.10) as desired. By (7.2), equality can only hold in (2.10) if $m = n$ and $|\alpha_j| = \lceil\alpha\rceil$ for $j = 1, 2, \ldots, n$. This yields (2.11).

Now consider the case where α is totally positive. Then we employ the product

$$P^* = \prod_{j=1}^{n} (\alpha_j - 1).$$

An argument similar to the previous one soon yields the inequality

$$\lceil\alpha\rceil \geqslant 1 + \frac{1}{q^{1/m}}$$

with equality only if $m = n$ and $\alpha_j = \lceil\alpha\rceil$ for $j = 1, 2, \ldots, n$. Since α is totally positive, this implies $n = 1$ and $\alpha = 1 + 1/q$. This completes the proof of Theorem 5. ∎

Proof of Corollary 6. Inequality (2.12) follows immediately from (2.10) if $m \geqslant 2$. If $q > 1$ then equality holds in (2.12) only when $m = n = 2$ and $|\alpha_1|^2 = |\alpha_2|^2 = 1 + 1/q$. If α is real, it must be a zero of $qx^2 - (q+1)$. If α is complex, it must be a zero of $qx^2 - bx + (q+1)$ for some integer b satisfying $b^2 < 4q(q+1)$. If $q = 1$ and equality holds in (2.12), then it follows that $|\alpha_j|^2 = 2$ for $1 \leqslant j \leqslant n$. Setting $\gamma = \alpha/\sqrt{2}$, we obtain $\lceil\gamma\rceil = 1$. The final remark in part 2) of the corollary is proved in the same way as for the case $q > 1$. This completes the proof of Corollary 6. ∎

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