ACTA ARITHMETICA
XLV (1985)

# A Markov approach to the generalized Syracuse algorithm

by

K. R. MATTHEWS and A. M. WATTS (Brisbane, Australia)

**1. Introduction.** Let $d, m_0, \ldots, m_{d-1}$ be positive integers; $R$ is a complete set of residues mod $d$; $r_i \in R$ is defined for $i = 0, \ldots, d-1$ by $r_i \equiv im_i \pmod{d}$. Then $T\colon Z \to Z$ is defined by

$$T(x) = \frac{m_i x - r_i}{d} \quad \text{if} \quad x \equiv i \pmod{d}. \tag{1.1}$$

In this paper we extend some of the conjectures and results of our recent paper [4], which dealt with the special case where $\gcd(m_i, d) = 1$ for $i = 0, \ldots, d-1$.

We are interested in the distribution among the congruence classes mod $m$ of the sequence of iterates $T^K(n)$, $K \geqslant 0$, where the sequence is not eventually periodic. We call such sequences 'divergent trajectories', this being the nomenclature of a forth-coming survey [3] on the $3x+1$ problem by J. Lagarias. Only the case $m = d^\alpha$ was discussed in [4].

The $T$-invariant subsets of $Z$ (i.e. subsets $S$ of $Z$ satisfying $T(S) \subseteq S$, or equivalently $S \subseteq T^{-1}(S)$) are clearly relevant for if $n \in S$ then $T^K(n) \in S$ if $K \geqslant 1$. We are particularly interested in $T$-invariant sets which are unions of residue classes mod $m$. We call such sets $T$-invariant mod $m$. A non-empty $T$-invariant set mod $m$ contains a minimal $T$-invariant set mod $m$. The intersection of $T$-invariant sets mod $m$ is also $T$-invariant mod $m$, and hence distinct minimal $T$-invariant sets mod $m$ are disjoint.

Let $B(j, m) = \{x \in Z \mid x \equiv j \pmod{m}\}$. In Section 2 we observe that $T^{-1}(B(j, m))$ is a union of congruence classes mod $md$. If $p_{jk}(m)$ is the number of congruence classes mod $md$ contained in $T^{-1}(B(j, m)) \cap B(k, m)$ and $q_{jk}(m) = p_{jk}(m)/d$, then $Q(m) = [q_{jk}(m)]$ is a Markov matrix whose ergodic sets are in 1-1 correspondence with the minimal $T$-invariant sets mod $m$, the correspondence being

$$S' = \{B(i_1, m), \ldots, B(i_t, m)\} \leftrightarrow S = B(i_1, m) \cup \ldots \cup B(i_t, m).$$

If $\gcd(m_i, d) = 1$ for $i = 0, \ldots, d-1$, or if $\gcd(m_i, d^\alpha) = \gcd(m_i, d)$ for $\alpha \geqslant 1$, $i = 0, \ldots, d-1$ and $d \mid m$ (we call this condition C), then we can show that if

$p_{Kjk}(m)$ is the number of congruence classes mod $md^K$ contained in $T^{-K}(B(j, m)) \cap B(k, m)$, then

$$[p_{Kjk}(m)] = [p_{jk}(m)]^K. \tag{1.2}$$

Corresponding to conjecture (iv) of our paper [4], we have

CONJECTURE 4. If condition C holds, a divergent trajectory eventually enters a minimal $T$-invariant set $S$ mod $m$. If $B(j, m) \subseteq S$ the limiting frequency

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{K \leqslant N \mid T^K(n) \in B(j, m)\} \tag{1.3}$$

exists and equals $\mu_{S'}(B(j, m))$, the $B(j, m)$-component of the stationary vector of $Q(m)$ which corresponds to the ergodic set $S'$.

A standard theorem of Markov theory ([2], p. 78) gives an interpretation of $\mu_{S'}(B(j, m))$, namely

$$\mu_{S'}(B(j, m)) = \lim_{N\to\infty} \frac{1}{N} \sum_{K\leqslant N} \frac{\operatorname{card}_{md^K}(T^{-K}(B(j, m)) \cap S)}{\operatorname{card}_{md^K}(S)}, \tag{1.4}$$

where $\operatorname{card}_{md^K}(S)$ denotes the number of congruence classes mod $md^K$ contained in $S$, a union of congruence classes mod $md^K$. It appears that not just the Cesàro limit but the limit itself exists, but we are unable to prove this, except in the case where $\gcd(m_i, m) = 1$ for $i = 0, 1, \ldots, d-1$.

If condition C does not hold, the limiting frequencies still appear to exist, but the simple interpretation in terms of stationary vectors may not hold.

Corresponding to conjectures (i) and (ii) of our paper [4], we have the following more general conjectures.

CONJECTURE 1. Let condition C hold and let $S$ be a minimal $T$-invariant set mod $d$. Then if

$$\prod_{B(i,d)\in S'} \left(\frac{m_i}{d}\right)^{\mu_{S'}(B(i,d))} < 1, \tag{1.5}$$

all trajectories starting in $S$ will eventually be periodic.

CONJECTURE 2. If

$$\prod_{B(i,d)\in S'} \left(\frac{m_i}{d}\right)^{\mu_{S'}(B(i,d))} > 1, \tag{1.6}$$

almost all trajectories starting in $S$ will be divergent.

Lemma 2 of [4] remains valid and corresponding to Theorem 1 of [4] we have



THEOREM 1.1. *Suppose that the sequence $(T^K(n))_{K\geqslant 0}$ lies in $S$ (a minimal $T$-invariant set* mod $d$) *and is unbounded. Also suppose that for $B(i, d) \in S'$,*

$$\lim_{N\to\infty} \frac{1}{N} \operatorname{card}\{K \leqslant N \mid T^K(n) \in B(i, d)\} = \mu_{S'}(B(i, d)).$$

*Then*

$$\text{(a)} \quad \prod_{B(i,d)\in S'} \left(\frac{m_i}{d}\right)^{\mu_{S'}(B(i,d))} \geqslant 1, \tag{1.7}$$

$$\text{(b)} \quad \lim_{K\to\infty} |T^K(n)|^{1/K} = \prod_{B(i,d)\in S'} \left(\frac{m_i}{d}\right)^{\mu_{S'}(B(i,d))}. \tag{1.8}$$

In Section 3 we give some limited results on the ergodic sets of $Q(m)$. In Section 4 we give some numerical examples.

**2. The asymptotic behaviour of $T^{-K}(B(j, m))$.**

LEMMA 2.1. *$T^{-1}(B(j, m))$ is a disjoint union (possibly empty) of congruence classes* mod $md$, *the number of classes being*

$$\sum_{\substack{i=0 \\ \gcd(m_i,m)\mid j-M_i}}^{d-1} \gcd(m_i, m), \quad \textit{where} \quad M_i = \frac{m_i i - r_i}{d}. \tag{2.1}$$

Proof. The first of the assertion follows from the equation

$$T(B(x, md)) = B\left(\frac{m_i x - r_i}{d}, m_i m\right), \tag{2.2}$$

where $i \equiv x \pmod d$, $0 \leqslant i \leqslant d-1$.

To prove the second part of the assertion, we have to solve the congruences

$$x \equiv i \pmod d \quad \text{and} \quad \frac{m_i x - r_i}{d} \equiv j \pmod m$$

for $i = 0, \ldots, d-1$, where $0 \leqslant x \leqslant md-1$.

Let $x = i + dy$. Then we have to solve

$$\frac{m_i(i+dy) - r_i}{d} \equiv j \pmod m, \quad \text{or} \quad m_i y \equiv j - M_i \pmod m.$$

This congruence is soluble if and only if $\gcd(m_i, m) \mid j - M_i$, in which case there are $\gcd(m_i, m)$ solutions for $y$ mod $m$ and hence for $x$ mod $md$.

COROLLARY 2.2. *If* $\gcd(m_i, m) = 1$ *for* $i = 0, \ldots, d-1$, *then* $T^{-1}(B(j, m))$ *is a disjoint union of* $d$ *congruence classes* mod $md$.

The next lemma is the Chinese remainder theorem.

LEMMA 2.3. *Let $D = \gcd(m, d)$, $L = \operatorname{lcm}(m, d)$. Then the congruences*

$$x \equiv i \pmod d \quad \text{and} \quad x \equiv k \pmod m \tag{2.3}$$

*are soluble if and only if $i \equiv k \pmod D$ and the solution is unique* mod $L$.

LEMMA 2.4. *Let $m|n$ and $p_{jk}(n, m)$ be the number of congruence classes* mod $nd$ *contained in $T^{-1}(B(j, n)) \cap B(k, m)$. Then $p_{jk}(n, m)$ is given by the following formula:*

*Let $S_{k,D}$ consist of the integers $i \equiv k \pmod D$, $0 \leqslant i \leqslant d-1$. Also let $x_i$ denote the solution of (2.3) satisfying $0 \leqslant x \leqslant L-1$. Finally let $M_i = (m_i x_i - r_i)/d$ if $i \in S_{k,D}$. Then*

$$p_{jk}(n, m) = \sum_{i \in S_{k,D}}' \gcd\left(m_i, n, \frac{nd}{m}\right),$$

*where the dash denotes summation over those $i$ such that*

$$\frac{m}{D} \gcd\left(m_i, n, \frac{nd}{m}\right) | j - M_i.$$

Proof. We have

$$T^{-1}(B(j, n)) \cap B(k, m) = \{x \in Z |\ T(x) \equiv j \pmod n \text{ and } x \equiv k \pmod m\}.$$

Hence for each $i = 0, \ldots, d-1$, we have to solve the congruences

$$x \equiv i \pmod d, \quad \frac{m_i x - r_i}{d} \equiv j \pmod n, \quad x \equiv k \pmod m.$$

Write $x = x_i + Ly$, $i \in S_{k,D}$. Then we have to solve

$$\frac{m_i(x_i + Ly) - r_i}{d} \equiv j \pmod n, \quad \text{or} \quad \frac{m_i m}{D} y \equiv j - M_i \pmod n.$$

This congruence is soluble if and only if $\gcd\left(\frac{m_i m}{D}, n\right) | j - M_i$, in which case the solution for $y$ is unique mod $n/\gcd\left(\frac{m_i m}{D}, n\right)$. Now

$$\gcd\left(\frac{m_i m}{D}, n\right) = \frac{m}{D} \gcd\left(m_i, n, \frac{nd}{m}\right). \tag{2.4}$$

Hence, in the case of solubility, the solution for $x$ is unique mod

$$\frac{Ln}{\gcd\left(\frac{m_i m}{D}, n\right)} = \frac{LDn}{m \gcd\left(m_i, n, \frac{nd}{m}\right)} = \frac{nd}{\gcd\left(m_i, n, \frac{nd}{m}\right)}.$$

Hence there are $\gcd\left(m_i, n, \frac{nd}{m}\right)$ solutions for $x$ mod $nd$.



LEMMA 2.5. *$\exists \beta \geqslant 0$ such that*

$$\gcd(m_i, d^\alpha) = \gcd(m_i, d^\beta) \quad \text{if} \quad \alpha \geqslant \beta, \tag{2.5}$$

*for $i = 0, \ldots, d-1$. (If $\gcd(m_i, d) = 1$ for $i = 0, \ldots, d-1$, we can take $\beta = 0$.)*

LEMMA 2.6. *With $\beta$ as in Lemma 2.5, if $\delta_i = \gcd(m_i, d^\beta)$, then*

$$\gcd(m_i, d^\alpha d_1) = \delta_i \quad \text{if} \quad \alpha \geqslant \beta,$$

*provided that $d_1 | d$.*

Proof. Let $\Delta_i = \gcd(m_i, d^\alpha d_1)$, where $\alpha \geqslant \beta$. Then

$$\delta_i | d^\beta, \quad \text{so} \quad \delta_i | d^\alpha d_1.$$

Hence

$$\delta_i | \Delta_i \quad \text{as} \quad \delta_i | m_i.$$

Also $\Delta_i | d^\alpha d_1$ so $\Delta_i | d^{\alpha+1}$. Also $\Delta_i | m_i$. Hence $\Delta_i | \gcd(m_i, d^{\alpha+1}) = \delta_i$.

LEMMA 2.7.

$$p_{jk}(md^\alpha, m) = p_{jk}(md^\beta, m) \quad \text{if} \quad \alpha \geqslant \beta. \tag{2.6}$$

*Also if $\beta \geqslant 1$ in Lemma 2.5 and $d|m$, then*

$$p_{jk}(md^\alpha, m) = p_{jk}(md^{\beta-1}, m) \quad \text{if} \quad \alpha \geqslant \beta - 1. \tag{2.7}$$

*In particular, if $\beta = 0$ or $\beta = 1$ and $d|m$, then*

$$p_{jk}(md^\alpha, m) = p_{jk}(m) \quad \text{if} \quad \alpha \geqslant 0. \tag{2.8}$$

Proof. By Lemma 2.4,

$$p_{jk}(md^\alpha, m) = \sum_{i \in S_{k,D}}' \gcd(m_i, md^\alpha, d^{\alpha+1}),$$

where the summation is over those $i$ satisfying

$$\frac{m}{D} \gcd(m_i, md^\alpha, d^{\alpha+1}) | j - M_i.$$

However

$$\gcd(m_i, md^\alpha, d^{\alpha+1}) = \gcd(m_i, d^\alpha \cdot \gcd(m, d)),$$

and by Lemma 2.6, with $d_1 = D = \gcd(m, d)$, the right-hand side is constant if $\alpha \geqslant \beta$.

If $d|m$, then $\gcd(m_i, md^\alpha, d^{\alpha+1}) = \gcd(m_i, d^{\alpha+1})$. Also if $\beta \geqslant 1$, $\gcd(m_i, d^{\alpha+1})$ is constant if $\alpha \geqslant \beta - 1$.

LEMMA 2.8. *Let $\beta = 0$ or $\beta = 1$ and $d|m$. Then if $p_{Kjk}(m)$ is the number of congruence classes* mod $md^K$ *contained in $T^{-K}(B(j, m)) \cap B(k, m)$, we have*

$$[p_{Kjk}(m)] = [p_{jk}(m)]^K, \tag{2.9}$$

*where $p_{jk}(m) = p_{1jk}(m)$ is the number of congruence classes* mod $md$ *contained in $T^{-1}(B(j, m)) \cap B(k, m)$.*

Proof.

$$T^{-(K+1)}(B(j, m)) \cap B(k, m) = \bigcup_{l=0}^{m-1} T^{-(K+1)}(B(j, m)) \cap T^{-1}(B(l, m)) \cap B(k, m)$$

$$= \bigcup_{l=0}^{m-1} \{T^{-1}(T^{-K}(B(j, m)) \cap B(l, m))\} \cap B(k, m).$$

Now $T^{-K}(B(j, m)) \cap B(l, m)$ is a disjoint union of $p_{Kjl}(m)$ congruence classes mod $md^K$, each contained in $B(l, m)$. If $B(L, md^K)$ is such a class, then $T^{-1}(B(L, md^K)) \cap B(k, m)$ is a disjoint union of $p_{Lk}(md^K, m)$ congruence classes mod $md^{K+1}$. However by Lemma 2.7,

$$p_{Lk}(md^K, m) = p_{Lk}(m, m) = p_{lk}(m, m) = p_{lk}(m)$$

if condition C holds.

Hence $T^{-(K+1)}(B(j, m)) \cap B(k, m)$ is a disjoint union of $\sum_{l=0}^{m-1} p_{Kjl}(m)\, p_{lk}(m)$ congruence classes mod $md^{K+1}$, and

$$p_{K+1,jk}(m) = \sum_{l=0}^{m-1} p_{Kjl}(m)\, p_{lk}(m). \tag{2.10}$$

This gives (2.9).

LEMMA 2.9. *Let*

$$Q(m) = [q_{jk}(m)] = \left[\frac{p_{jk}(m)}{d}\right]. \tag{2.11}$$

*Then $Q(m)$ is a non-negative matrix, each of whose column sums equals* 1.

Proof.

$$B(k, m) = Z \cap B(k, m) = T^{-1}(Z) \cap B(k, m)$$

$$= \bigcup_{j=0}^{m-1} T^{-1}(B(j, m)) \cap B(k, m).$$

Hence $B(k, m)$ is a disjoint union of $\sum_{j=0}^{m-1} p_{jk}(m)$ congruence classes mod $md$. However

$$B(k, m) = \bigcup_{i=0}^{d-1} B(k+im, md),$$

a disjoint union of $d$ congruence classes mod $md$. Hence

$$\sum_{j=0}^{m-1} p_{jk}(m) = d,$$



which is equivalent to

$$\sum_{j=0}^{m-1} q_{jk}(m) = 1.$$

**3. Ergodic sets of $Q(m)$.** Let $Z_m = \{B(0, m), \ldots, B(m-1, m)\}$. Then the following definition of closed set of $Q(m)$ is standard for Markov matrices such as $Q(m)$. (See [2], p. 43.)

A subset of $S'$ of $Z_m$ is *closed* if

$$B(k, m) \in S' \quad \text{and} \quad B(j, m) \notin S' \Rightarrow q_{jk}(m) = 0. \tag{3.1}$$

A minimal closed set is called an *ergodic set of* $Q(m)$.

LEMMA 3.1. *Let $S$ be a union of congruence classes* mod $m$ *and let $S' \subseteq Z_m$ be defined by*

$$S' = \{B(j, m) |\ B(j, m) \subseteq S\}.$$

*Then the mapping $S \to S'$ is a 1-1 correspondence between the T-invariant sets* mod $m$ *and the closed sets of $Q(m)$, in which minimal T-invariant sets* mod $m$ *correspond to ergodic sets of $Q(m)$.*

Proof. Let $S'$ be closed. Also let $B(k, m) \subseteq S$. Then if $B(j, m) \subseteq Z-S$, $B(j, m) \notin S'$. Hence $T^{-1}(B(j, m)) \cap B(k, m) = \emptyset$ and so $T^{-1}(Z-S) \cap B(k, m) = \emptyset$. Hence $B(k, m) \subseteq T^{-1}(S)$ and $S \subseteq T^{-1}(S)$. The argument is reversible.

The second part of the lemma follows from the equivalence

$$S_1 \subseteq S_2 \Leftrightarrow S'_1 \subseteq S'_2.$$

LEMMA 3.2. *If* $\gcd(m_i, m) = 1$ *for $i = 0, \ldots, d-1$, then $Q(m)$ is doubly stochastic, i.e.*

$$\sum_{k=0}^{m-1} q_{jk} = 1. \tag{3.2}$$

Remark. There are hence no transient classes (see [2], p. 49) of $Q(m)$ and $Z$ is a disjoint union of minimal $T$-invariant sets mod $m$. Moreover the stationary vector of $Q(m)$ which corresponds to an ergodic set $S'$ of $Q(m)$ has its non-zero components equal to $1/n$, where $n = \text{card}\, S'$.

Proof.

$$T^{-1}(B(j, m)) = \bigcup_{k=0}^{m-1} T^{-1}(B(j, m)) \cap B(k, m).$$

Hence $T^{-1}(B(j, m))$ is a disjoint union of $\sum_{k=0}^{m-1} p_{jk}(m)$ congruence classes mod $md$. However from Corollary 2.1, the number of congruence classes equals $d$ if $\gcd(m_i, m) = 1$ for $i = 0, \ldots, d-1$. Hence

$$\sum_{k=0}^{m-1} p_{jk}(m) = d$$

and this is equivalent to (3.2).

Lemma 3.3. *If $m|m_i$, where $\gcd(m_i, d) = 1$, then $Q(m)$ has one ergodic set. Moreover if $\gcd(m_i, d) = 1$ for $i = 0, \ldots, d-1$, then the matrix corresponding to this ergodic set in the normal form of $Q(m)$ is primitive.*

Proof. Let $M_i = \dfrac{m_i i - r_i}{d}$, where $m|m_i$ and $\gcd(m_i, d) = 1$. Also let $x = amd + bd + i$. Then

$$T(x) = \frac{m_i(amd+bd+i) - r_i}{d} = amm_i + m_i b + M_i \equiv M_i \pmod{m}.$$

But if $0 \leqslant j \leqslant m-1$, $\exists b$ such that

$$amd + bd + i \equiv j \pmod{m}.$$

Hence $T^{-1}(B(M_i, m)) \cap B(j, m) \neq \emptyset$ and so $q_{M_i j}(m) > 0$. Thus all elements in the row corresponding to $M_i$ are non-zero and we see that $Q(m)$ has only one ergodic set.

For the second part, assume $\gcd(m_i, d) = 1$ for $i = 0, \ldots, d-1$. Let us write $Q_T(m)$ instead of the less precise $Q(m)$. Also let $A$ be the matrix corresponding to the ergodic set of $Q_T(m)$ in the normal form of $Q_T(m)$.

Then if $A$ is not primitive, $\exists t > 1$ such that $A^t$ has $t$ ergodic sets. However from (2.9)

$$Q_{T^t}(m) = (Q_T(m))^t.$$

Hence $Q_{T^t}(m)$ has $t$ ergodic sets.

However $T^t$ is a transformation with $d^t$ branches and associated moduli $m_{i_1} \ldots m_{i_t}$, where $0 \leqslant i_1, \ldots, i_t \leqslant d-1$. Each of these moduli is relatively prime to $d^t$. Hence by the part of Lemma 3.2 just proved, $Q_{T^t}(m)$ has one ergodic set, contradicting an earlier statement.

Corollary 3.4. *Let $m|m_i$, where $\gcd(m_i, d) = 1$ for $i = 0, \ldots, d-1$. Then $Q(m^k)$ has just one ergodic set for all $k \geqslant 1$. Moreover, the matrix corresponding to this ergodic set in the normal form of $Q(m^k)$, is primitive.*

Proof. Let $\gcd(m_i, d) = 1$ for $i = 0, \ldots, d-1$. Then it is easy to verify that for $k \geqslant 1$, the $d^k$ "cylinders"

$$B(i_0, d) \cap T^{-1}(B(i_1, d)) \cap \ldots \cap T^{-(k-1)}(B(i_{k-1}, d)), \quad 0 \leqslant i_j \leqslant d-1,$$

are precisely the $d^k$ residue classes mod $d^k$.



Now let $m|m_i$. Then

$$B(i, d) \cap T^{-1}(B(i, d)) \cap \ldots \cap T^{-(k-1)}(B(i, d)) = B(j, d^k)$$

for some $j$, $0 \leqslant j < d^k$. Hence

$$T^h(j) \equiv i \pmod{d} \quad \text{for all} \quad h, \quad 0 \leqslant h \leqslant k-1.$$

But if $0 \leqslant l < d^k$,

$$T^k(l + cd^k) = T^k(l) + m_l m_{T(l)} \ldots m_{T^{k-1}(l)} c,$$

where the indices are taken mod $d$.

In particular

$$T^k(j + cd^k) = T^k(j) + m_i^k c$$

for all $c \in \mathbf{Z}$.

We can now apply Lemma 3.3, with $T$ replaced by $T^k$, $d$ by $d^k$, $m_l$ by $m_l m_{T(l)} \ldots m_{T^{k-1}(l)}$, $0 \leqslant l < d^k$. Then if $m|m_i$, $m^k|m_i^k$ and we deduce that $Q_{T^k}(m^k)$ has precisely one ergodic set.

However, as

$$(3.3) \qquad Q_{T^k}(m^k) = [Q_T(m^k)]^k,$$

it follows that $Q_T(m^k)$ has precisely one ergodic set.

The proof of the second part of the theorem is the same as that of the second part of Lemma 3.3, with $m$ and $T$ replaced by $m^k$ and $T^k$, respectively.

Lemma 3.5. *Let $\gcd(m_i, m) = 1$ for $i = 0, \ldots, d-1$ and suppose that to an ergodic set $S'$ of $Q(m)$ there corresponds a periodic matrix $A$ in the normal form of $Q(m)$, i.e.*

$$(3.4) \qquad A = \begin{bmatrix} 0 & 0 & \ldots & 0 & B_t \\ B_1 & 0 & \ldots & 0 & 0 \\ 0 & B_2 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \ldots & B_{t-1} & 0 \end{bmatrix},$$

*where $B_1, \ldots, B_t$ are of sizes $n_2 \times n_1$, $n_3 \times n_2, \ldots, n_1 \times n_t$, respectively,*

$$(3.5) \qquad A^t = \operatorname{diag}[A_1, \ldots, A_t],$$

*where*

$$(3.6) \qquad A_1 = B_t B_{t-1} \ldots B_1, \; A_2 = B_1 B_t \ldots B_2, \; \ldots, \; A_t = B_{t-1} B_{t-2} \ldots B_t,$$

*and*

$$(3.7) \qquad A_1, \ldots, A_t \text{ are primitive.}$$

*Then $B_1, B_2, \ldots, B_t$ are each square and of the same size.*

Proof. Let $S'_1, \ldots, S'_t$ be the sets of residue classes mod $m$ which correspond to the columns of $B_1, \ldots, B_t$, respectively. Then as remarked after Lemma 3.2, $Q(m)$ has no transient classes if $\gcd(m_i, m) = 1$ for $i = 0, \ldots, d-1$. Hence

$$T^{-1}(S) = S = S_1 \cup \ldots \cup S_t.$$

Now from (3.4)

$$T^{-1}(S_1) \cap (S_1 \cup \ldots \cup S_{t-1}) = \emptyset,$$

so

$$T^{-1}(S_1) \subseteq S_t. \tag{3.8}$$

Similarly

$$T^{-1}(S_2) \subseteq S_1, \quad \ldots, \quad T^{-1}(S_t) \subseteq S_{t-1}. \tag{3.9}$$

Next by (3.3), (3.5), (3.6), it follows that $S'_1, \ldots, S'_t$ correspond to the columns of $A_1, \ldots, A_t$ in $Q_{T^t}(m)$ and that $S_1, \ldots, S_t$ are $T^t$-invariant. Thus

$$S_t \subseteq T^{-t}(S_t). \tag{3.10}$$

But from (3.9) we deduce that

$$T^{-t}(S_t) \subseteq T^{-1}(S_1) \tag{3.11}$$

and so from (3.8), (3.10), (3.11) we have

$$S_t = T^{-1}(S_1). \tag{3.12}$$

Similarly

$$S_1 = T^{-1}(S_2), \quad \ldots, \quad S_{t-1} = T^{-1}(S_t). \tag{3.13}$$

However, from Corollary 2.2,

$$\operatorname{card}_{md} T^{-1}(S_1) = d \operatorname{card}_m S_1. \tag{3.14}$$

Hence from (3.12) and (3.14),

$$\operatorname{card}_{md} S_t = d \operatorname{card}_m S_t = d \operatorname{card}_m S_1$$

and $\operatorname{card}_m S_t = \operatorname{card}_m S_1$. Similarly $\operatorname{card}_m S_1 = \operatorname{card}_m S_2, \ldots, \operatorname{card}_m S_{t-1} = \operatorname{card}_m S_t$. It follows that $A_1, \ldots, A_t$ have the same size and consequently $B_1, \ldots, B_t$ are each square and of the same size.

Corollary 3.6. *If $\gcd(m_i, m) = 1$ for $i = 0, \ldots, d-1$ and $S$ is a minimal $T$-invariant set* mod $m$, *then if $B(j, m) \subseteq S$,*

$$\lim_{K \to \infty} \frac{\operatorname{card}_{md^K}\left(T^{-K}(B(j, m)) \cap S\right)}{\operatorname{card}_{md^K}(S)} = \mu_{S'}(B(j, m)), \tag{3.15}$$

*the $B(j, m)$-component of stationary vector of $Q(m)$ which corresponds to $S'$.*



Proof. By standard Markov theory, we need only discuss the case where $A$, the matrix corresponding to $S$ in $Q(m)$, is periodic with period $t > 1$. We use the notation of Lemma 3.5.

Let $L_1, \ldots, L_t$ be the stationary vectors of the primitive matrices $A_1, \ldots, A_t$. Also let $n$ be the size of each $A_i$ and let $J$ be the $n$-dimensional row vector, all of whose components are equal to 1. Then standard theory gives

$$\lim_{N \to \infty} A^{Nt} = \begin{bmatrix} L_1 J & 0 & \ldots & 0 \\ 0 & L_2 J & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & L_t J \end{bmatrix}, \quad \lim_{N \to \infty} A^{Nt+1} = \begin{bmatrix} 0 & \ldots & 0 & L_1 J \\ L_2 J & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & L_{t-1} J & 0 \end{bmatrix},$$

ect.

Hence if $0 \leqslant k \leqslant t-1$,

$$\lim_{N \to \infty} A^{Nt+k} \begin{bmatrix} 1/nt \\ \vdots \\ 1/nt \end{bmatrix} = \frac{1}{t} \begin{bmatrix} L_1 \\ \vdots \\ L_t \end{bmatrix},$$

the stationary vector of $A$.

Hence

$$\lim_{K \to \infty} A^K \begin{bmatrix} 1/nt \\ \vdots \\ 1/nt \end{bmatrix} \text{ exists and equals } \frac{1}{t} \begin{bmatrix} L_1 \\ \vdots \\ L_t \end{bmatrix}. \tag{3.16}$$

Now

$$\frac{\operatorname{card}_{md^K}\left(T^{-K}(B(j, m)) \cap S\right)}{\operatorname{card}_{md^K}(S)} \tag{3.17}$$

$$= \sum_{B(k,m) \subseteq S} \frac{\operatorname{card}_{md^K}\left(T^{-K}(B(j, m)) \cap B(k, m)\right)}{d^K nt}.$$

Hence (3.15) follows from (3.16) and (3.17), together with the fact that the element of $A^K$ in the $B(j, m) - B(k, m)$ position is equal to

$$\frac{\operatorname{card}_{md^K}\left(T^{-K}(B(j, m)) \cap B(k, m)\right)}{d^K},$$

from (2.9).

## 4. Some examples.

1.

$$T(x) = \begin{cases} x/2 & \text{if } 2 \mid x, \\ (5x+1)/2 & \text{if } 2 \mid x-1. \end{cases}$$

There appears to be one ergodic set for each $m \geqslant 2$, though we cannot prove this in general. $n = 7$ appears to give the first divergent trajectory.

2.

$$T(x) = \begin{cases} x/2 & \text{if } 2|x, \\ (5x-3)/2 & \text{if } 2|x-1. \end{cases}$$

There appears to be one ergodic set if $3 \nmid m$, two ergodic sets if $3|m$.

3.

$$T(x) = \begin{cases} x/3 & \text{if } 3|x, \\ (2x-2)/3 & \text{if } 3|x-1, \\ (13x-2)/3 & \text{if } 3|x-2. \end{cases}$$

As expected, all trajectories appear to end up in cycles; there appear to be six cycles only. As a curiosity, the trajectory starting with $n = 338$ takes 7161 iterations to reach the cycle beginning with 2; also the maximum value of $T^K(338)$ is $T^{2726}(338)$, a number with 73 digits.

4.

$$T(x) = \begin{cases} x/3-1 & \text{if } 3|x, \\ (x+5)/3 & \text{if } 3|x-1, \\ 10x-5 & \text{if } 3|x-2. \end{cases}$$

Here

$$Q(3) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 1 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix},$$

which is primitive.

$S' = \boldsymbol{Z}_3$ is the only ergodic set and the stationary vector of $Q(3)$ is $[\frac{1}{2}, \frac{1}{4}, \frac{1}{4}]'$. Also

$$\prod_{\substack{i=0 \\ B(i,d) \in S'}}^{d-1} \left(\frac{m_i}{d}\right)^{\mu_{S'}(B(i,d))} = \left(\frac{1}{3}\right)^{\frac{1}{2}} \left(\frac{1}{3}\right)^{\frac{1}{4}} \left(\frac{30}{3}\right)^{\frac{1}{4}} < 1,$$

so we expect all trajectories to eventually cycle. There appear to be five cycles.

5.

$$T(x) = \begin{cases} 3x/2 & \text{if } 4|x, \\ (x+1)/2 & \text{if } 4|x-1, \\ x/2+1 & \text{if } 4|x-2, \\ (7x+1)/2 & \text{if } 4|x-3. \end{cases}$$



Here

$$Q(4) = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}, \quad \text{which has normal form} \quad \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

The stationary vectors corresponding to $S_1' = \{B(0, 4), B(2, 4)\}$ and $S_2' = \{B(1, 4), B(3, 4)\}$ are both equal to $[\frac{1}{2}, \frac{1}{2}]'$.

Also

$$\prod_{\substack{i=0 \\ B(i,d) \in S_1'}}^{d-1} \left(\frac{m_i}{d}\right)^{\mu_{S_1'}(B(i,d))} = \left(\frac{3}{2}\right)^{\frac{1}{2}} \left(\frac{1}{2}\right)^{\frac{1}{2}} < 1,$$

whereas

$$\prod_{\substack{i=0 \\ B(i,d) \in S_2'}}^{d-1} \left(\frac{m_i}{d}\right)^{\mu_{S_2'}(B(i,d))} = \left(\frac{1}{2}\right)^{\frac{1}{2}} \left(\frac{7}{2}\right)^{\frac{1}{2}} > 1.$$

Hence we expect all trajectories starting with an even integer to eventually cycle, while most trajectories starting with an odd integer should be divergent.

6.

$$T(x) = \begin{cases} 3x/2+1 & \text{if } 4|x, \\ (x-1)/2 & \text{if } 4|x-1, \\ x/2 & \text{if } 4|x-2, \\ (7x-1)/2 & \text{if } 4|x-3. \end{cases}$$

Here

$$Q(4) = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix} \quad \text{which has normal form} \quad \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}.$$

Hence $Q(4)$ is periodic with period 2 and has only one ergodic set, $\boldsymbol{Z}_4$. The stationary vector is $[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]'$ and we expect most trajectories to be divergent as

$$\prod_{\substack{i=0 \\ B(i,d) \in S'}}^{d-1} \left(\frac{m_i}{d}\right)^{\mu_{S'}(B(i,d))} = \left(\frac{3}{2}\right)^{\frac{1}{4}} \left(\frac{1}{2}\right)^{\frac{1}{4}} \left(\frac{1}{2}\right)^{\frac{1}{4}} \left(\frac{7}{2}\right)^{\frac{1}{4}} > 1.$$

7. (Conway [1], p. 121)

$$T(x) = \begin{cases} 3x/2 & \text{if } 4|x, \\ (3x+1)/4 & \text{if } 4|x-1, \\ 3x/2 & \text{if } 4|x-2, \\ (3x-1)/4 & \text{if } 4|x-3. \end{cases}$$

(This transformation is 1-1.)

$$Q(4) = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

and is primitive. The stationary vector is $[\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}]'$ and $S' = Z_4$ is the only ergodic set. Also

$$\prod_{\substack{i=0 \\ B(i,d)\in S'}}^{d-1} \left(\frac{m_i}{d}\right)^{\mu_{S'}(B(i,d))} = \left(\frac{3}{2}\right)^{\frac{1}{4}} \left(\frac{3}{4}\right)^{\frac{1}{4}} \left(\frac{3}{2}\right)^{\frac{1}{4}} \left(\frac{3}{4}\right)^{\frac{1}{4}} > 1.$$

Hence we expect most trajectories to be unbounded.

In conclusion, we wish to thank R. N. Buttsworth and G. Leigh for some useful comments.

### References

[1] R. K. Guy, *Unsolved problems in number theory*, Springer Verlag, New York 1981.

[2] D. L. Isaacson and R. W. Madsen, *Markov chains: theory and applications*, Wiley, New York 1976.

[3] J. Lagarias, *The '3x+1' problem and its generalizations*, to appear in Amer. Math. Monthly.

[4] K. R. Matthews and A. M. Watts, *A generalization of Hasse's generalization of the Syracuse algorithm*, Acta Arith. 43 (1984), pp. 167–175.

*Received on 6. 4. 1983*
*and in revised form on 17. 10. 1983* (1351)

ACTA ARITHMETICA
XLV (1985)

# Algebraic numbers near the unit circle

by

C. W. Lloyd-Smith (Belconnen, Australia)

**1. Introduction.** Let $\alpha$ be an algebraic number of degree $n \geqslant 2$ over the rationals, with conjugates $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$. In this paper we investigate conditions on $\alpha$ which imply that $\alpha$ has one or more conjugates on the unit circle. Our conditions will be expressed in terms of the following two functions:

$$\overline{|\alpha|} = \max_{1 \leqslant j \leqslant n} |\alpha_j|, \tag{1.1}$$

$$\Lambda(\alpha) = \prod_{j=1}^{n} \max\{1, |\alpha_j|\}. \tag{1.2}$$

Also we require the notion of denominator of an algebraic number.

Definition 1. Let $P(x) = qx^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ be the primitive minimal polynomial of $\alpha$ over the rationals, where $q > 0$. Then we say that $q$ is the *denominator of* $\alpha$ (cf. Blanksby [3]).

It is clear that $\alpha$ has denominator 1 if and only if $\alpha$ is an algebraic integer.

It is easy to see that there exist positive functions $\varphi(n, q)$ and $\psi(n, q)$ such that for all algebraic numbers $\alpha$ of degree $n$ and denominator $q$,

1) either $\overline{|\alpha|} \leqslant 1$ or

$$\overline{|\alpha|} \geqslant 1 + 1/\varphi(n, q), \tag{1.3}$$

2) either $\Lambda(\alpha) = 1$ or

$$\Lambda(\alpha) \geqslant 1 + 1/\psi(n, q). \tag{1.4}$$

There arises the problem of finding the best possible functions $\varphi$ and $\psi$ satisfying (1.3) and (1.4) respectively. There has been much recent work on this problem in the case $q = 1$ but very little in the general case.