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On consecutive Farey arcs

by

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1. Introduction. Let $F_N = \{x_r,\ 1 \leqslant r \leqslant R\}$, where $R = R(N) = \varphi(1) + \varphi(2) + \ldots + \varphi(N)$ and φ is Euler's totient function, denote the Farey series of order N, that is the sequence of positive, irreducible fractions not exceeding 1 with denominator not exceeding N arranged in increasing order. We write $l_r = x_r - x_{r-1}$, $2 \leqslant r \leqslant R$ and set $l_1 = x_1$, $l_{R+1} = l_1$.

Sums related to consecutive Farey fractions, i.e. involving the l_r have been studied by several authors [1], [2], [3], [4] and have been shewn to have applications to the evaluation of certain infinite series. Also, it has been known for a long time that the distribution of Farey arcs is closely related to the deepest problems in number theory — for the connection with the Riemann Hypothesis see [5], Chap. XIV.

In this note we are concerned with the joint distribution of the pairs (l_r, l_{r+1}), $1 \leqslant r \leqslant R$ which we approach via the moments

$$T_N(\alpha, \beta) := \sum_{r=1}^{R} l_r^\alpha l_{r+1}^\beta$$

which we evaluate in a region of the (α, β)-plane. Notice that because of the symmetry of Farey series we have $T_N(\alpha, \beta) = T_N(\beta, \alpha)$. We shall encounter several new features which did not appear in previous work, restricted to the case $\beta = 0$, on this topic [1]–[4]: a fuller set of references may be found in [3]. A point of interest is that the shape of the main term in our asymptotic formula for $T_N(\alpha, \beta)$ changes when $\gamma := \alpha + \beta$ passes through the value 2; when $\gamma = 2$ we obtain our weakest error term.

Interpreting $T_N(\alpha, \beta)$ as a two dimensional Laplace transform we may deduce the existence of the limiting distribution

$$\Delta(x, y) := \lim_{N \to \infty} R(N)^{-1} \operatorname{card}\{r:\ N^2 l_r \leqslant e^x,\ N^2 l_{r+1} \leqslant e^y\}.$$

An explicit formula for the transform of the measure $d\Delta(x, y)$ is given in Theorem 4 below. We also obtain the answer to a question raised in [3] (cf. Remark 2).

THEOREM 1. *Let α, β be real numbers such that* $\max(\alpha, \beta, \alpha+\beta) < 2$. *Then there exists* $\eta = \eta(\alpha, \beta) > 0$ *such that*

$$T_N(\alpha, \beta) = \left(\frac{6}{\pi^2}+O(N^{-\eta})\right) H(\alpha, \beta) N^{2-2\alpha-2\beta}$$

where

$$H(\alpha, \beta) := \int_1^\infty \int_0^1 \{(t-w)^{-\alpha}(t-\{2t-w\})^{-\beta}\} t^{2\alpha+2\beta-3}\, dw\,dt.$$

In particular

$$H(\alpha, 0) = \frac{1}{(1-\alpha)^2}\left\{1-\frac{\Gamma^2(2-\alpha)}{\Gamma(3-2\alpha)}\right\}, \quad \alpha \neq 1,$$

$$H(1, 0) = \Gamma''(1)-\Gamma'(1)^2 = \pi^2/6.$$

More precisely, if we set $a := \min(1, 2-\alpha, 2-\beta, 2-\alpha-\beta)$, $b := 1+\max(-1, \alpha, \beta, \alpha+\beta)$, $c := \min(1, 1-(\alpha+\beta)/2)$ *then we may replace* $N^{-\eta}$ *above by* $N^{-ac/(a+b)}\log^3 N$.

THEOREM 2. *We have*

$$T_N(\alpha, 2-\alpha) = (6/\pi^2+O(1/\log N))N^{-2}\log N$$

for each $\alpha \in (0, 2)$.

THEOREM 3. *Let* $\gamma = \alpha+\beta > 2$, $\alpha > 0$, $\beta > 0$. *Then*

$$T_N(\alpha, \beta) = \left(\frac{\zeta(\gamma-1)}{\zeta(\gamma)}+O(N^{-\alpha}\log^{\delta(\beta,2)} N+N^{-\beta}\log^{\delta(\alpha,2)} N+ +N^{-1}+N^{2-\gamma}\log^{\delta(\gamma,3)} N)\right)N^{-\gamma},$$

where $\delta(\gamma, \gamma') = 1$ $(\gamma = \gamma')$, $= 0$ $(\gamma \neq \gamma')$ *is Kronecker's symbol.*

When either α or β is zero the main terms given above no longer apply: in this case see the references cited. Usually in these papers it has been assumed that the exponent of l_r is an integer.

COROLLARY. *Let* $d_r = x_r - x_{r-2}$, $3 \leqslant r \leqslant R$ *and* $d_1 = 1+x_1-x_{R-1}$, $d_2 = x_2$. *Then we have*

$$D_m(N) = \sum_{r=1}^R d_r^m \sim (2^m+2)\frac{\zeta(m-1)}{\zeta(m)} N^{-m}$$

for integers $m \geqslant 3$, *also*

$$D_2(N) \sim \frac{36}{\pi^2} N^{-2}\log N.$$

Remark 1. It would be interesting to know whether the integral for $H(\alpha, \beta)$ may be evaluated in any case for which $\alpha\beta \neq 0$.

Remark 2. The problem of evaluating $D_2(N)$ was raised in [3], after a suggestion of one of us. It was shown in [3] that for large N,

$$\frac{24}{\pi^2} N^{-2}\log N \leqslant D_2(N) \leqslant \frac{48}{\pi^2} N^{-2}\log N.$$

We state the following result without proof.

THEOREM 4. *The limiting distribution*

$$\Delta(x, y) = \lim_{N\to\infty} R(N)^{-1}\,\mathrm{card}\,\{r\colon N^2 l_r \leqslant e^x,\ N^2 l_{r+1} \leqslant e^y\}$$

exists. Moreover for $\min(\mathrm{Re}\, z, \mathrm{Re}\,\zeta, \mathrm{Re}(z+\zeta)) > -2$ *we have*

$$\int_0^\infty\int_0^\infty e^{-zx-\zeta y}\, d\Delta(x, y) = 2H(-z, -\zeta)$$

where H is the function defined in the statement of Theorem 1.

Remark 3. The corollary to Theorem 3 is almost immediate: we write $d_r^m = (l_r+l_{r-1})^m$ and use the Binomial Theorem. Since this method requires that $m \in \mathbf{Z}^+$, it would be interesting to consider the function $D_m(N)$ for other values of m.

2. Proofs of the theorems.

LEMMA 1. *A necessary and sufficient condition that the positive integers r, s, t appear as successive denominators (in this order) in F_N is that*

$$r+s > N, \quad s+t > N, \quad (r, s) = (s, t) = 1, \quad r+t \equiv 0 \pmod s.$$

They are successive denominators at most once.

Proof. It was shown in [1] that r, s are adjacent denominators in F_N if and only if $r+s > N$ and $(r, s) = 1$. In any case r, s can be adjacent (in this order) at most once, hence r, s, t can be adjacent at most once and it is necessary that

$$(1) \qquad r+s > N, \quad s+t > N, \quad (r, s) = (s, t) = 1$$

for them to be so. Now let (1) hold, so that somewhere in F_N the denominators r, s and s, t are adjacent. A sufficient extra condition that in fact r, s, t should be adjacent is that the numerators of the fractions with denominator s in these occurrences should be the same. Accordingly let r'/r, u/s be adjacent fractions, similarly v/s, t'/t. Then $ru-sr' = st'-tv = 1$ (by a well-known property of Farey series) and so $ru \equiv 1 \pmod s$, $tv \equiv -1 \pmod s$. Since $0 < u, v < s$, the numbers u, v are precisely determined by these congruences, and $u = v$ if and only if $r+t \equiv 0 \pmod s$. This completes the proof.

Remark 4. The chain of denominators $r_1, r_2, \ldots, r_m$ will be adjacent (in this order) in F_N if and only if each triple, r_i, r_{i+1}, r_{i+2} $(1 \leqslant i \leqslant m-2)$ are adjacent. Hence there are no extra conditions beyond those given in the lemma for each separate triple.

LEMMA 2. *We have*

$$T_N(\alpha, \beta) = \sum_{s=1}^{N} \frac{1}{s^{\alpha+\beta}} \sum_{\substack{r=N-s+1 \\ (r,s)=1}}^{N} \frac{1}{r^\alpha t^\beta}, \quad \alpha, \beta \in C$$

where

$$t = t(r) = s\left[\frac{N+r}{s}\right] - r = N - s\left\{\frac{N+r}{s}\right\}, \tag{2}$$

and $[x]$, $\{x\}$ *denote as usual the integer and fractional parts of the real number* x.

Proof. We have

$$T_N(\alpha, \beta) = \sum (rs)^{-\alpha}(st)^{-\beta}$$

where the sum is over all sets of three adjacent fractions r'/r, s'/s, t'/t, in F_N. We just need a condition for the denominators r, s, t to be adjacent and we apply Lemma 1. Given r and s we can determine t for $N-s < t \leqslant N$, $t \equiv -r \pmod{s}$. The formula (2) will be useful in what follows.

We proceed to the proof of Theorems 2, 3. This is quite straightforward: the main term does not involve the awkward function $H(\alpha, \beta)$ indeed it depends on $\gamma = \alpha+\beta$ only. We have $\gamma \geqslant 2$.

We split the sum appearing in Lemma 2 into two parts $A(N)$ and $B(N)$ according as $s \leqslant M := [N/2]$ or not. We have r, $t \in (N-s, N]$ and so for $s \leqslant M$ we have $r^{-\alpha} = N^{-\alpha} + O(sN^{-\alpha-1})$, $t^{-\beta} = N^{-\beta} + O(sN^{-\beta-1})$ and so

$$r^{-\alpha} t^{-\beta} = N^{-\gamma} + O(sN^{-\gamma-1}).$$

The inner sum is therefore

$$\sum_{r=N-s+1}^{N} r^{-\alpha} t^{-\beta} = \varphi(s) N^{-\gamma} + O(s^2 N^{-\gamma-1}),$$

whence

$$A(N) = \frac{\zeta(\gamma-1)}{\zeta(\gamma)} N^{-\gamma} + O(N^{2-2\gamma} \log^{\delta(\gamma,3)} N + N^{-\gamma-1})$$

where $\delta(\gamma, \gamma') = 1$ if $\gamma = \gamma'$, $= 0$ if $\gamma \neq \gamma'$ and $\zeta(\gamma-1)$ is to be interpreted as $\log N$ if $\gamma = 2$. Next, we estimate $B(N)$ as an error term and we note that $r+t \geqslant s$ so that $\max(r, t) \gg N$. Let us assume for the moment that $\alpha \leqslant \beta$

so that $\beta \geqslant 1$ with equality implying $\gamma = 2$. Then

$$\sum r^{-\alpha} t^{-\beta} \ll N^{-\alpha} \sum t^{-\beta} + N^{-\beta} \sum t^{-\alpha} \ll N^{-\alpha} \sum t^{-\beta}$$
$$\ll N^{-\alpha}\left((N-s+1)^{1-\beta} + \delta(\beta, 1) \log \frac{1}{N-s+1}\right),$$

and so

$$B(N) \ll N^{2-2\gamma} + N^{-\alpha-\gamma} \log^{\delta(\beta,2)} N.$$

We drop the constraint $\alpha \leqslant \beta$ and put these results together to obtain

$$T_N(\alpha, \beta) = \left(\frac{\zeta(\gamma-1)}{\zeta(\gamma)} + O(N^{-\alpha} \log^{\delta(\beta,2)} N + N^{-\beta} \log^{\delta(\alpha,2)} N + N^{-1} + N^{2-\gamma} \log^{\delta(\gamma,3)} N)\right) N^{-\gamma},$$

which gives the results stated.

Proof of Theorem 1. In what follows we shall often make use of the inequality

$$\sum_{a<n\leqslant b} n^\lambda \ll (a^{1+\lambda} + b^{1+\lambda}) \log \frac{b}{a}$$

which is valid uniformly for $1 \leqslant a < b$, $-\lambda_0 \leqslant \lambda \leqslant \lambda_0$ provided we interpret $\log x$ as $\max(1, \log x)$. When $(1+\lambda)^{-1} = O(1)$ this factor may be omitted.

We apply Lemma 2, writing

$$T_N(\alpha, \beta) = A_1(N) + R_1(N)$$

where the s-summation is restricted by the condition $s \leqslant M$ in $A_1(N)$. Here $M = [(1-\varepsilon)N]$ and $\varepsilon = \varepsilon(N) \to 0$ in a manner to be determined. We estimate the remainder $R_1(N)$ first. We note that $t(r)$ is a permutation of r, and the inner sum is majorized by supposing $t(r) = r$ when $\alpha\beta \geqslant 0$, $t(r) = 2N-s-r+1$ if $\alpha\beta < 0$, and we distinguish these cases. In the case $\alpha\beta \geqslant 0$,

$$R_1(N) \ll N^{-\alpha-\beta} \sum_{r \leqslant N} r^{-\alpha-\beta} \sum_{\max(n-r, M) < s \leqslant N} 1 \ll N^{2-2\alpha-2\beta}(\varepsilon + \varepsilon^{2-\alpha-\beta}) \log \frac{1}{\varepsilon}.$$

When $\alpha\beta < 0$, we get (if $\beta < 0 < \alpha$),

$$R_1(N) \ll N^{-\alpha-\beta}\Big\{\sum_{r \leqslant \varepsilon N} r^{-\alpha} \sum_{N-r<s\leqslant N} (2N-s-r+1)^{-\beta} + \sum_{\varepsilon N < r \leqslant N} r^{-\alpha} \sum_{M<s\leqslant N} (2N-s-r+1)^{-\beta}\Big\}$$
$$\ll N^{2-2\alpha-2\beta}(\varepsilon + \varepsilon^{2-\alpha}) \log \frac{1}{\varepsilon}.$$

If $\alpha < 0 < \beta$, α is replaced by β in the exponent of ε. In any case,

$$R_1(N) \ll N^{2-2\alpha-2\beta}\varepsilon^a \log\frac{1}{\varepsilon}$$

where a was defined in the statement of the theorem.

We turn next to $A_1(N)$, taking account of the condition $(r, s) = 1$ by introducing the familiar sum over $\mu(d)$. We write $s = kd$, $r = hd$ and have

$$A_1(N) = \sum_{d \leqslant M} \frac{\mu(d)}{d^{2\alpha+2\beta}} \sum_{k \leqslant M/d} k^{-\alpha-\beta} \sum_{N/d-k<h\leqslant N/d} h^{-\alpha}\left(\frac{N}{d} - k\left\{\frac{N/d+h}{k}\right\}\right)^{-\beta}$$
$$= A_2(N) + R_2(N), \text{ say,}$$

where the h-summation is replaced by an integral in A_2 and R_2 is the remainder. To estimate R_2, set

$$D(k, y) = \sum_{y-k<h\leqslant y} h^{-\alpha}\left(y - k\left\{\frac{y+h}{k}\right\}\right)^{-\beta} - \int_{y-k}^{y} h^{-\alpha}\left(y - k\left\{\frac{y+h}{k}\right\}\right)^{-\beta} dh$$
$$= \sum_{l\leqslant k} \int_0^1 \left(f(l, \{y\}) - f(l, v)\right) dv$$

where

$$f(l, \theta) := (y-k+l-\theta)^{-\alpha}\left(y - k\left\{\frac{2y+l-\theta}{k}\right\}\right)^{-\beta}.$$

In the application, $y = N/d$, $k \leqslant M/d$: hence $k \leqslant (1-\varepsilon)y$. Now put $\eta := \{(2y+l)/k\}$ so that for $0 \leqslant \theta \leqslant 1$ we have

$$\left\{\frac{2y+l-\theta}{k}\right\} = \begin{cases} \eta - \theta/k & \text{if } \theta \leqslant \eta/k, \\ 1+\eta-\theta/k & \text{else.} \end{cases}$$

If l is such that $\eta k \geqslant 1$, we have

$$f(l, \{y\}) - f(l, v)$$
$$= (y-k+l-\{y\})^{-\alpha}(y-k\eta+\{y\})^{-\beta} - (y-k+l-v)^{-\alpha}(y-k\eta+v)^{-\beta}$$
$$\ll \varepsilon^{-b} y^{-\alpha-\beta-1}.$$

Thus the contribution of these l's to $R_2(N)$ is

$$\ll \varepsilon^{-b} \sum_{d\leqslant M} d^{-2\alpha-2\beta}(N/d)^{-\alpha-\beta-1} \ll \varepsilon^{-b} N^{1-2\alpha-2\beta} \log N.$$

If $\eta k < 1$ we observe that there exists an m such that

$$m \leqslant (2y+l)/k < m+1/k, \quad \text{i.e.,} \quad km \leqslant 2y+l < km+1.$$

Thus the number of such l's is bounded, and their contribution to $R_2(N)$ is

$$\ll \varepsilon^{-b'} \sum_{d\leqslant M} d^{-2\alpha-2\beta}(N/d)^{-\alpha-\beta} \sum_{k\leqslant M/d} k^{-\alpha-\beta}$$
$$\ll \varepsilon^{-b'}(\log N)^2(N^{-\alpha-\beta} + N^{1-2\alpha-2\beta})$$

where $b' := \max(0, \alpha, \beta, \alpha+\beta) \leqslant b$. Therefore

$$R_2(N) \ll N^{2-2\alpha-2\beta}(\log N)^2(\varepsilon^{-b}N^{-1} + \varepsilon^{-b'}N^{\alpha+\beta-2}).$$

Putting $h = N/d - k\omega$ in the integral and $m = kd$ in the summation, we see that

$$A_2(N) = N^{-\alpha-\beta} \sum_{m\leqslant M} \frac{\varphi(m)g(m)}{m^{\alpha+\beta}}$$

with

$$g(m) := \int_0^1 \left(1 - \frac{m}{N}\omega\right)^{-\alpha}\left(1 - \frac{m}{N}\left\{\frac{2N}{m} - \omega\right\}\right)^{-\beta} d\omega.$$

Let us write $\varphi(m) = 6m/\pi^2 + \psi(m)$ and $A_2(N) + A_3(N) + R_3(N)$ with

$$A_3(N) = \frac{6}{\pi^2} N^{-\alpha-\beta} \sum_{m\leqslant M} g(m) m^{1-\alpha-\beta}$$

and R_3 a similar sum involving ψ, to be regarded as a remainder term. We split R_3 into two sums $R_3^{(1)}$, $R_3^{(2)}$ according as $m \leqslant \sqrt{2N}$ or not. Noticing that $g(m) \ll \varepsilon^{-b'}$ for $m \leqslant M$ we have immediately

$$R_3^{(1)}(N) \ll \varepsilon^{-b'} N^{-\alpha-\beta} \sum_{m\leqslant\sqrt{2N}} m^{1-\alpha-\beta} \ll \varepsilon^{-b'} N^{1-3(\alpha+\beta)/2}.$$

Set

$$P(u) = \begin{cases} \sum_{m\leqslant u} \psi(m) m^{-\alpha-\beta} & \text{if } \alpha+\beta \leqslant 1, \\ -\sum_{m>u} \psi(m) m^{-\alpha-\beta} & \text{if } \alpha+\beta > 1, \end{cases}$$

it is easy to check that the infinite series converges if $\alpha+\beta > 1$, moreover that in either case

$$P(u) \ll u^{1-\alpha-\beta}(\log u)^2.$$

Now

$$R_3^{(2)}(N) = N^{-\alpha-\beta} \sum_{\sqrt{2N} < m \leqslant M} (P(m)-P(m-1))g(m)$$
$$= N^{-\alpha-\beta} \sum_{\sqrt{2N} < m \leqslant M} P(m)(g(m+1)-g(m)) + $$
$$+O\big(N^{2-2\alpha-2\beta}(\log N)^2(\varepsilon^{-b'}N^{-1}+N^{(\alpha+\beta-3)/2})\big).$$

For $m > \sqrt{2N}$ and $\omega \notin \left(\frac{-2N}{m(m+1)}, 0\right)$ (mod 1), we have

$$\left\{\frac{2N}{m} - \omega\right\} = \left\{\frac{2N}{m+1} - \omega\right\} + \frac{2N}{m(m+1)}.$$

We deduce that $g(m+1)-g(m) \ll m^{-1}\varepsilon^{-b}$ for $\sqrt{2N} < m \leqslant M$ so that

$$\sum_{\sqrt{2N} < m \leqslant M} P(m)\big(g(m+1)-g(m)\big) \ll \varepsilon^{-b}(\log N)^3(N^{1-\alpha-\beta}+N^{(1-\alpha-\beta)/2}),$$

and

$$R_3(N) \ll N^{2-2\alpha-2\beta}(\log N)^3(\varepsilon^{-b'}N^{(\alpha+\beta-2)/2}+\varepsilon^{-b}N^{(\alpha+\beta-3)/2}+\varepsilon^{-b}N^{-1}).$$

Next, we write $A_3(N) = A_4(N)+R_4(N)$ where

$$A_4(N) = \frac{6}{\pi^2} N^{-\alpha-\beta} \int_0^M g(v)v^{1-\alpha-\beta}\,dv$$

and

$$R_4(N) = \frac{6}{\pi^2} N^{-\alpha-\beta} \sum_{m=1}^{M} \int_{m-1}^{m} \big(g(m)m^{1-\alpha-\beta} - g(v)v^{1-\alpha-\beta}\big)\,dv.$$

It is not difficult to check that an estimate for $R_3(N)$ is equally valid for $R_4(N)$. We use the trivial bound $g(v) \ll \varepsilon^{-b'}$ when $m \leqslant \sqrt{2N}$; for $\sqrt{2N} < m \leqslant M$ we show that the integrand is $\ll \varepsilon^{-b}m^{-\alpha-\beta}$. Next, we have (substituting $v = N/t$),

$$N^{\alpha+\beta-2}\int_0^M g(v)v^{1-\alpha-\beta}\,dv$$
$$= H(\alpha, \beta) - \int_1^{N/M}\int_0^1 (t-\omega)^{-\alpha}(t-\{2t-\omega\})^{-\beta}t^{2\alpha+2\alpha-3\beta}\,d\omega dt.$$

We need an upper bound for the ω-integral when $t = 1+h$, h small. It is

equal to

$$O(h) + \int_{2h}^{1} (1+h-\omega)^{-\alpha}(\omega-h)^{-\beta}\,d\omega \ll h+(h^{1-\alpha}+h^{1-\beta}+1)\log(1/h)$$

so that we need $\max(\alpha, \beta) < 2$ for convergence at $t = 1$. The double integral above is then

$$\ll (\varepsilon+\varepsilon^{2-\alpha}+\varepsilon^{2-\beta})\log(1/\varepsilon) \ll \varepsilon^a \log(1/\varepsilon).$$

Provided $\log(1/\varepsilon) \ll (\log N)^3$, we may deduce from all the above that

$$T_N(\alpha, \beta) = N^{2-2\alpha-2\beta}H(\alpha, \beta)\Big(\frac{6}{\pi^2}+O\big((\log N)^3(\varepsilon^a+\varepsilon^{-b}N^{-1}+$$
$$+\varepsilon^{-b}N^{(\alpha+\beta-3)/2}+\varepsilon^{-b'}N^{(\alpha+\beta-2)/2})\big)\Big)$$

and we choose $\varepsilon = N^{-\gamma}$ and replace b' by b. The error term is

$$\ll (\log N)^3(N^{-\gamma a}+N^{\gamma b-c})$$

and we obtain our result on setting $\gamma = c/(a+b)$.

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