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The construction of unramified abelian cubic extensions of a quadratic field

by

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Introduction. Consider the following problem: Given a quadratic field $Q(\sqrt{m})$ with class number $h(m) \equiv 0 \pmod 3$, construct an unramified abelian cubic extension K of $Q(\sqrt{m})$. That is, if $K = Q(\sqrt{m}, \xi)$ give a polynomial with coefficients in $Q(\sqrt{m})$ which has ξ as a root.

This problem could, of course, be regarded as an aspect of the theory of non-Galois cubic extensions of Q, and their normal closures. See, for example, [1] and [7]. Also, it is well known that the all-powerful class field theory solves the problem. However, it is not practical to apply these theories to solving a specific case. It is the main purpose of this paper to give a simple and concise method for the construction described above. One needs some knowledge of the ideal classes in $Q(\sqrt{m})$ and $Q(\sqrt{-3m})$; given this, the procedure is straightforward. It may be summarized as follows:

Let $F = Q(\sqrt{k})$ be any quadratic field, and let $\gamma \in F$. Define the ideal T in F by: if $3 \nmid k$, then $T = (9)$, and if $3 \mid k$ and $(3) = P^2$, then $T = P^3$.

We say that γ is a *semi-cube* in F provided γ is not a perfect cube, and γ is an integer, and

(i) the principal ideal (γ) is an ideal cube,
(ii) for some integer $x \in F$, $\gamma \equiv x^3 \pmod T$,
(iii) $3 \nmid N(\gamma)$.

We show that if γ is a semi-cube in $Q(\sqrt{-3m})$ and $\xi = \sqrt[3]{\gamma} + \sqrt[3]{\gamma'}$ (where γ' is the conjugate of γ), then $Q(\sqrt{m}, \xi)$ is an unramified abelian cubic extension of $Q(\sqrt{m})$. On the other hand, if $Q(\sqrt{m}, \xi)$ is an unramified abelian cubic extension of $Q(\sqrt{m})$, then we can find a semi-cube $\gamma \in Q(\sqrt{-3m})$ such that for $\eta = \sqrt[3]{\gamma} + \sqrt[3]{\gamma'}$ we have $Q(\sqrt{m}, \xi) = Q(\sqrt{m}, \eta)$.

Generally speaking, the proof may be described as a combination of standard field theory (à la Van der Waerden), and of the work done in [11] and [12], which allows the precise determination of the discriminant of any quadratic or cubic extension of a given field.

Section 1 contains notation and statements of known results. In Section 2, we establish some easy preliminary results, for example, if K is an unramified abelian cubic extension of $Q(\sqrt{m})$, then $J = K(\sqrt{-3})$ is a normal field, with dihedral Galois group, unramified over $Q(\sqrt{m}, \sqrt{-3})$ (Theorem 2.5).

In Section 3, we establish the pairing between unramified abelian cubic extensions of $Q(\sqrt{m})$ and pure cubic extensions $Q(\sqrt{-3m}, \sqrt[3]{\gamma})$ where γ is a semi-cube (Corollaries 3.4 and 3.5); we also give the discriminant of the pure cubic extension.

In Section 4 we state some known results, which are very easy consequences of the pairing. Theorems 4.1 and 4.2 are contained in the work of A. Scholz [9] and H. Reichardt [8].

In Section 5 we give some examples of the construction.

I would like to thank Professor R. Bölling for his helpful remarks, and for the references [8] and [9].

1. Preliminaries. If F is an algebraic field, and if $x \in F$, the *norm* of x (over Q) is the product of all the conjugates of x, denoted $N(x)$.

If n is an integer and t a non-negative integer, and p a prime, then $p^t \| n$ means $p^t \mid n$ and $p^{t+1} \nmid n$. We also use this notation for ideals.

If R is the ring of integers in F, write disc F = disc R (the discriminant of R over Q; see e.g. [6]). If $F \subseteq K$ where K is also algebraic, and if S is the ring of integers in K, write $\operatorname{disc}(K/F)$ for the discriminant of S over R. The Galois group of K over F is denoted $\operatorname{Gal}(K/F)$.

We will need the closed-form expression for the roots of a cubic polynomial. For details see [4] or [10]. Any polynomial ax^3+bx^2+cx+d can be rationally transformed into a polynomial of the form

$$f(x) = x^3+qx+r.$$

Put

$$D = -(4q^3+27r^2), \qquad L = (r/2)+(\sqrt{-3D})/18, \qquad M = (r/2)-(\sqrt{-3D})/18.$$

Then the roots of $f(x)$ are

$$-\sqrt[3]{L}-\sqrt[3]{M}, \quad -\omega\sqrt[3]{L}-\omega^2\sqrt[3]{M}, \quad -\omega^2\sqrt[3]{L}-\omega\sqrt[3]{M}$$

(Cardan's formulas), where $\omega = (-1+\sqrt{-3})/2$. L and M are the roots of the *auxiliary quadratic*

$$g(x) = x^2-ry-q^3/27$$

and D is the square of the difference product of the roots of $f(x)$. We may write $D = \operatorname{disc} f$, keeping in mind that this may not be an integer.

LEMMA 1.1. *Let $f(x) = x^3+qx+r$, with L, M, D defined as above, and suppose q and r are in a field F. Put $\gamma = -\sqrt[3]{L}-\sqrt[3]{M}$. Then $F(\sqrt{-3D}, \gamma) = F(\sqrt{-3D}, \sqrt[3]{L})$.*

Proof. This follows from the fact that L and M are in $F(\sqrt{-3D})$ and that $LM = -(q/3)^3$ is a cube in F. ∎

COROLLARY 1.2. *Let $F(\gamma)$ be any cubic extension of a field F, with discriminant D over F. Then $F(\sqrt{-3D}, \gamma)$ is a pure cubic extension of $F(\sqrt{-3D})$.* ∎

In [11] we discuss pure cubic extensions $F(\sqrt[3]{\gamma})$ of a field F. We say that $\gamma_1 \approx \gamma_2$ provided $F(\sqrt[3]{\gamma_1}) = F(\sqrt[3]{\gamma_2})$. Given $\gamma \in F$ and a prime p in Z, it is shown that there is a γ_1 in F so that $\gamma \approx \gamma_1$ and the principal ideal (γ_1) is not divisible by the cube of any prime ideal factor of (p). In particular, we may always assume without loss of generality, that γ is an integer and that (γ) is not divisible by the cube of any prime ideal factor of (3).

The lemmas stated below are extracted from [11].

LEMMA 1.3. *Let $F = Q(\sqrt{m})$ and $K = F(\sqrt[3]{\gamma})$ where γ is not a perfect cube in F, and (γ) is not divisible by the cube of any prime ideal factor of* (3) *in F. Let $D = \operatorname{disc}(K/F)$.*

(a) *Let $m \equiv 0 \pmod 3$. If $3 \nmid N(\gamma)$ and $\gamma \equiv x^3 \pmod T$ for some $x \in F$, then $3 \nmid D$. Otherwise $3^4 \mid D$.*

(b) *Let $m \not\equiv 0 \pmod 3$. If $3 \nmid N(\gamma)$ and $\gamma \equiv x^3 \pmod 9$, then $3^2 \| D$. Otherwise, $3^4 \mid D$.* ∎

LEMMA 1.4. *Let F be a field and $K = F(\sqrt[3]{\gamma})$ for some γ in F which is not a perfect cube. Put $D = \operatorname{disc}(K/F)$. Let p be a rational prime, $p \neq 3$. Then $p \mid D$ if and only if for some prime ideal factor P of (p) we have $P^{3t+r} \| (\gamma)$ where $r = 1$ or $r = 2$. In particular, if $p \nmid D$ for every prime p then (γ) must be an ideal cube. If (γ) is an ideal cube, and $p \neq 3$ is prime, then $p \nmid D$.* ∎

2. Second-order preliminaries. The main purpose of this section is to show that if K is an unramified abelian cubic extension of $Q(\sqrt{m})$, then $J = K(\sqrt{-3})$ is a normal field with dihedral Galois group, which is unramified over $Q(\sqrt{m}, \sqrt{-3})$ (Theorem 2.5). The proofs in this section are quite straightforward, and are included for the sake of completeness.

LEMMA 2.1. *Let J be a normal (over Q) cubic extension of $Q(\sqrt{m}, \sqrt{-3})$. Then $G = \operatorname{Gal}(J/Q)$ is either dihedral or $Z_2 \times Z_2 \times Z_3$.*

Proof. Since J contains three normal subfields of degree 2, then G has three normal subgroups of order 6. There are only five groups of order 12; only $Z_2 \times Z_2 \times Z_3$ and the dihedral group have three normal subgroups of order 6. ∎

LEMMA 2.2. *Let J be an unramified, normal (over Q) cubic extension of $Q(\sqrt{m}, \sqrt{-3})$. Then* $\mathrm{Gal}(J/Q)$ *is dihedral.*

Proof. We know that J contains at least one field $Q(\xi)$ of degree 3 over Q; $G = \mathrm{Gal}(J/Q)$ will be dihedral precisely when $Q(\xi)$ is not a normal field. We put $D_m = \mathrm{disc}(Q(\sqrt{m}))$; then disc J is $(3D_m)^6$ if $3 \nmid m$, and is D_m^6 if $3 \mid m$. Let $D = \mathrm{disc}(Q(\xi))$. Then $|D| \geqslant 23$, and $Q(\xi)$ is normal if and only if D is square in Z (see [5]). We also have $D^4 \mid \mathrm{disc}\, J$. Now suppose that $D = k^2$. Then k^8 divides disc J, and since m is squarefree, k can only be a power of 2. Since at most 2^{18} can divide disc J, then k is at most 4. But $4^2 = 16$ is too small for a cubic discriminant. Thus D is not square, $Q(\xi)$ is not normal, and G is dihedral. ∎

LEMMA 2.3. *Suppose that $F = Q(\sqrt{m})$ has an abelian extension of degree 3: $K = F(\gamma) = Q(\sqrt{m}, \gamma)$ where γ is a root of $f(x) = x^3 + qx + r$ $(q, r \in Q(\sqrt{m}))$. Define L, M as in Section 1, and assume $\gamma = -\sqrt[3]{L} - \sqrt[3]{M}$. Then*

$$J = Q(\sqrt{m}, \sqrt{-3}, \gamma) = Q(\sqrt{m}, \sqrt{-3}, \sqrt[3]{L})$$

and J is a normal field of degree 12 over Q.

Proof. Put $D = -(4q^3 + 27r^2)$. Since K is normal over F, then D must be square in F, so that $F(\sqrt{-3D}) = F(\sqrt{-3})$. By Lemma 1.1 we have $F(\sqrt{-3}, \gamma) = F(\sqrt{-3}, \sqrt[3]{L}) = J$. It is clear that $[J:Q] = 12$. Next, since $L \in Q(\sqrt{-3D}) \subseteq Q(\sqrt{m}, \sqrt{-3})$, then L has only two distinct conjugates in $Q(\sqrt{m}, \sqrt{-3})$, namely L and M. Since $LM = (-q/3)^3$, then $\sqrt[3]{M}$ is in J. Since J contains $\sqrt{-3}$ (and hence the cube roots of unity) it follows that J is the splitting field of the polynomial

$$(x^3 - M)(x^3 - L) \in Q[x]$$

(since this field also has degree 12 over Q and contains J). Thus J is normal over Q. ∎

LEMMA 2.4. *Let F be a field of degree n over Q, and put $K = F(\sqrt{-3})$.*
(a) *If 3 is not ramified in F, then* disc $K = (\mathrm{disc}\, F)^2 \cdot 3^{2n}$.
(b) *If (3) is an ideal square in F, then* disc $K = (\mathrm{disc}\, F)^2$.

Proof. This is an easy consequence of [12], § 3. ∎

THEOREM 2.5. *Assume the hypotheses and notation of Lemma 2.3, and assume also that K is unramified over F. Then J is unramified over $Q(\sqrt{m}, \sqrt{-3})$ and* $\mathrm{Gal}(J/Q)$ *is dihedral.*

Proof. Put $D_m = \mathrm{disc}\, Q(\sqrt{m})$, and $D = \mathrm{disc}\, Q(\sqrt{m}, \sqrt{-3})$. If $3 \nmid m$, then $D = (-3D_m)^2$, and if $3 \mid m$, then $D = D_m^2$. If $3 \nmid m$, then since disc $K = D_m^3$, 3 is not ramified in K, and by Lemma 2.4, disc $J = D_m^6 3^6 = D^3$. If $3 \mid m$, then (3) is an ideal square in K, and disc $J = D_m^6 = D^3$, again by Lemma 2.4. Now J is unramified over $Q(\sqrt{m}, \sqrt{-3})$ and $\mathrm{Gal}(J/Q)$ is dihedral by Lemma 2.2. ∎

3. The cubic connections. In this section we show how the unramified abelian cubic extensions of $Q(\sqrt{m})$ pair off with certain pure cubic extensions of $Q(\sqrt{-3m})$. We assume throughout that $D_m = \mathrm{disc}\, Q(\sqrt{m})$ and $|D_m| \geqslant 23$ (since it will be the discriminant of a cubic field also).

THEOREM 3.1. *Let L be any unramified cubic extension of $J = Q(\sqrt{m}, \sqrt{-3})$ such that L is normal over Q. Let $Q(\xi)$ be one of the cubic subfields of L. Then precisely one of the following two cases must occur:*

(a) *$K_1 = Q(\sqrt{m}, \xi)$ is an unramified abelian extension of $Q(\sqrt{m})$ and $K_2 = Q(\sqrt{-3m}, \xi)$ is a pure cubic extension of $Q(\sqrt{-3m})$;*

(b) *$K_2 = Q(\sqrt{-3m}, \xi)$ is an unramified abelian extension of $Q(\sqrt{-3m})$ and $K_1 = Q(\sqrt{m}, \xi)$ is a pure cubic extension of $Q(\sqrt{m})$.*

Proof. By Lemma 2.2, $G = \mathrm{Gal}(L/Q)$ is dihedral. Let C be the (only) cyclic normal subgroup of order 6, and H the (only) normal subgroup of order 2. H fixes a normal field of degree 6, which has just one quadratic subfield and three (conjugate) cubic subfields, one of which is $Q(\xi)$. Put $D = \mathrm{disc}\, Q(\xi)$; we know D is not square.

Suppose that $Q(\sqrt{m})$ is fixed by C. Then $K_1 = Q(\sqrt{m}, \xi)$ is fixed by H and is normal, so $\sqrt{D} \in K_1$. Since K_1 has only one quadratic subfield, we have $D = mk^2$ for some integer k. Since $D^4 \mid \mathrm{disc}\, L$, no odd prime can divide k, particularly 3. Now if $3 \nmid m$, then 3 is unramified in K_1 and disc $L = 3^6 D_m^6 = (\mathrm{disc}\, K_1)^2 3^6$ (Lemma 2.4) whence disc $K_1 = D_m^3$. If $3 \mid m$, then (3) is an ideal square in K_1 and disc $L = D_m^6 = (\mathrm{disc}\, K_1)^2$; again we find disc $K_1 = D_m^3$.

Now K_1 is unramified over $Q(\sqrt{m})$, and since disc $K_1 = D_m^3 = D^2 j$ (where $j = \mathrm{disc}(K_1/Q(\xi))$) it is not difficult to show $D = D_m$. Then by Corollary 1.2, $K_2 = Q(\sqrt{-3m}, \xi)$ is a pure cubic extension of $Q(\sqrt{-3m})$. The argument is similar in case $Q(\sqrt{-3m})$ is fixed by C.

It remains to show that $Q(\sqrt{-3})$ is never fixed by C. Suppose to the contrary that $Q(\sqrt{-3}, \xi)$ is normal. As before, we must have disc $Q(\xi) = D = -3k^2$ for some $k \in Z$ and (as in the proof of Lemma 2.2) k is a power of 2, and $k \leqslant 4$. Then $|D| = 1$ or 12 or 48 and D cannot be the discriminant of a cubic field ([5]). This contradiction completes the proof. ∎

COROLLARY 3.2. *Let L satisfy the hypotheses of Theorem 3.1. Then $L = Q(\sqrt{m}, \sqrt{-3}, \sqrt[3]{\gamma})$ where either $\gamma \in Q(\sqrt{m})$ or $\gamma \in Q(\sqrt{-3m})$.* ∎

Remark. One could equally well distinguish the two cases of Theorem 3.1 according to whether or not the quadratic subfield fixed by C has discriminant divisible by 3. We shall henceforth make the convenient assumption that $3 \nmid m$.

The next result fills in some of the details about the pure cubic extensions occurring in Theorem 3.1.

THEOREM 3.3. *Assume the hypotheses and notation of Theorem* 3.1, *and assume also that* $3 \nmid m$.

(a) *Suppose that* $Q(\sqrt{m})$ *is fixed by C, so that* $K_2 = Q(\sqrt{-3m}, \xi)$ *is a pure cubic extension of* $Q(\sqrt{-3m})$. *Then* $D(L/K_2) = 1$, $D(K_2/Q(\sqrt{-3m})) = 1$, *and* $K_2 = Q(\sqrt{-3m}, \sqrt[3]{\gamma})$ *where* γ *is a semi-cube in* $Q(\sqrt{-3m})$ (*that is,* γ *is not a perfect cube,* (γ) *is an ideal cube,* $3 \nmid N(\gamma)$, *and for some* $x \in Q(\sqrt{-3m})$, $\gamma \equiv x^3 \pmod{T}$).

(b) *Suppose that* $Q(\sqrt{-3m})$ *is fixed by C, so that* $K_1 = Q(\sqrt{m}, \xi)$ *is a pure cubic extension of* $Q(\sqrt{m})$. *Then* $D(L/K_1) = 3^2$, $D(K_1/Q(\sqrt{m})) = 3^2$, *and* $K_1 = Q(\sqrt{m}, \sqrt[3]{\gamma})$ *where* γ *is a semi-cube in* $Q(\sqrt{m})$.

Proof. (a) Since $3 \nmid m$, we know $\operatorname{disc} Q(\sqrt{-3m}) = -3D_m$. Then $\operatorname{disc} K_2 = (-3D_m)^3 D(K_2/Q(\sqrt{-3m}))$ and $\operatorname{disc} L = (-3D_m)^6 = (\operatorname{disc} K_2)^2 \times D(L/K_2)$. Then both relative discriminants are 1. Since $K_2 = Q(\sqrt{-3m}, \sqrt[3]{\alpha})$ for some $\alpha \in Q(\sqrt{-3m})$, it follows from Lemmas 1.3 and 1.4 and the preceding remarks, that $\alpha \approx \gamma$ for some γ, where γ is a semi-cube; $K_2 = Q(\sqrt{-3m}, \sqrt[3]{\gamma})$.

(b) We have $\operatorname{disc} K_1 = D_m^3 D(K_1/Q(\sqrt{m}))$, and $\operatorname{disc} L = 3^6 D_m^6 = (\operatorname{disc} K_1)^2 D(L/K_1)$. Clearly the two relative discriminants can only be powers of 3, and since $3^6 \| \operatorname{disc} L$, then $3^4 \nmid D(K_1/Q(\sqrt{m}))$. Then by Lemma 1.3 we must have $3^2 = D(K_1/Q(\sqrt{m}))$. It follows that $D(L/K_1) = 3^2$ also, and as before, $K_1 = Q(\sqrt{m}, \sqrt[3]{\gamma})$ where γ is a semi-cube. ∎

Putting all the pieces together, we have a constructive pairing between unramified abelian cubic extensions of $Q(\sqrt{m})$ and what might be called *minimally ramified* pure cubic extensions of $Q(\sqrt{-3m})$. This is summarized in the following two corollaries.

COROLLARY 3.4. *Let* $3 \nmid m$.

(a) *Let* $Q(\sqrt{m}, \xi)$ *be an unramified abelian cubic extension of* $Q(\sqrt{m})$. *Then we may assume* ξ *is a root of* x^3+qx+r *for some* $q, r \in Q$, *and if* γ *is a root of the auxiliary quadratic, then* $K_2 = Q(\sqrt{-3m}, \sqrt[3]{\gamma})$ *is an unramified pure cubic extension of* $Q(\sqrt{-3m})$. *We also have* $K_2 = Q(\sqrt{-3m}, \sqrt[3]{\alpha})$ *for a semi-cube* α *in* $Q(\sqrt{-3m})$.

(b) *Assume* $K_2 = Q(\sqrt{-3m}, \sqrt[3]{\alpha})$ *is an unramified extension of* $Q(\sqrt{-3m})$, *where* α *is a semi-cube in* $Q(\sqrt{-3m})$. *Then the minimum polynomial of* α *is the auxiliary quadratic for a cubic polynomial of the form* x^3+qx+r *with root* $\xi = -\sqrt[3]{\alpha}-\sqrt[3]{\alpha'}$, *and* $Q(\sqrt{m}, \xi)$ *is an unramified abelian extension of* $Q(\sqrt{m})$.

Proof. (a) By Lemma 2.3, $L = Q(\sqrt{m}, \sqrt{-3}, \xi)$ satisfies the hypotheses of Theorem 3.1. We also have $3 \nmid m$, so Theorem 3.3 applies.

(b) Put $L = Q(\sqrt{m}, \sqrt{-3}, \sqrt[3]{\alpha})$. As in the proof of Lemma 2.3 we find that L is normal over Q (we use the fact that $N(\alpha)$ is a cube in Q). By Lemma 2.4, L is unramified over $Q(\sqrt{m}, \sqrt{-3})$, and by Lemma 2.2, $\operatorname{Gal}(L/Q)$ is dihedral. The rest follows from Theorems 3.1 and 3.3. ∎

COROLLARY 3.5. *Let* $3 \nmid m$.

(a) *Suppose* $Q(\sqrt{-3m}, \xi)$ *is an unramified abelian cubic extension of* $Q(\sqrt{-3m})$. *We may assume* ξ *is a root of* x^3+qx+r *for some* $q, r \in Q$. *If* γ *is a root of the auxiliary quadratic, then* $K_1 = Q(\sqrt{m}, \sqrt[3]{\gamma})$ *has* $D(K_1/Q(\sqrt{m})) = 3^2$. *Furthermore,* $K_1 = Q(\sqrt{m}, \sqrt[3]{\alpha})$ *where* α *is a semi-cube.*

(b) *Assume* $K_1 = Q(\sqrt{m}, \sqrt[3]{\alpha})$ *has* $D(K_1/Q(\sqrt{m})) = 3^2$, *with* α *a semi-cube in* $Q(\sqrt{m})$. *Then the minimum polynomial for* α *is the auxiliary quadratic for a cubic polynomial* x^3+qx+r *with root* $\xi = -\sqrt[3]{\alpha}-\sqrt[3]{\alpha'}$, *and* $Q(\sqrt{-3m}, \xi)$ *is an unramified abelian extension of* $Q(\sqrt{-3m})$.

Proof. The proof for (a) is the same as for Corollary 3.4 (a). To see (b), put $L = Q(\sqrt{m}, \sqrt{-3}, \sqrt[3]{\alpha})$ and $D = \operatorname{disc}(L/Q(\sqrt{m}, \sqrt{-3}))$. By Lemma 1.3, if p is prime, $p \neq 3$, then $p \nmid D$. To see that $3 \nmid D$, consider the basis

$$B = \{1, (\sqrt{-3})(x-\sqrt[3]{\alpha})/3, (x-\sqrt[3]{\alpha})^2/3\}$$

for L over $Q(\sqrt{m}, \sqrt{-3})$. We have $3 \nmid N(\alpha)$ and $\alpha \equiv x^3 \pmod{T}$; with this choice of x, one checks that the members of the basis B are algebraic integers, and that $3 \nmid \operatorname{disc} B$. Then $3 \nmid D$. The rest of the proof is like Corollary 3.4 (b). ∎

4. Let $H^+(m)$ denote the maximal abelian unramified extension of $Q(\sqrt{m})$ with Galois group $G^+(m)$; $H(m)$ the Hilbert class field of $Q(\sqrt{m})$, with Galois group $G(m)$; $h^+(m) = |G^+(m)|$ and $h(m) = |G(m)| =$ the class number of $Q(\sqrt{m})$. When we take into account the nature of a semi-cube, the pairing established in Section 3 leads immediately to relations between $h^+(m)$ and $h^+(-3m)$, or between $G^+(m)$ and $G^+(-3m)$. In this section we give some of the more obvious consequences of the pairing.

Say that a pure cubic extension of $Q(\sqrt{m})$ is *minimally ramified* if its relative discriminant over $Q(\sqrt{m})$ is 1 when $3 \mid m$ and is 3^2 when $3 \nmid m$. A necessary condition for $Q(\sqrt{m})$ to have a minimally ramified pure cubic extension, is that it must contain an ideal cube, $(\gamma) = I^3$, with I not principal. If $m<0$, this would imply $3 \mid h(m)$, but not necessarily for $m>0$, since the fundamental unit of a real quadratic field may be a semi-cube (for example, if $m=87$, $h(m)=2$, the fundamental unit is a semi-cube and $3 \| h(-29)$). Since not every ideal cube yields a semi-cube, the number of minimally ramified pure cubic extensions of $Q(\sqrt{m})$ cannot exceed the number of distinct subgroups of order 3 in $G^+(m)$. Hence this latter number bounds the number of distinct subgroups of order 3 in $G^+(-3m)$, by the pairing. For instance, we have

THEOREM 4.1 ([8], [9]). *Suppose* $m<0$ *and* $3 \nmid h(m)$. *Then* $3 \nmid h^+(-3m)$. ■

THEOREM 4.2 ([8], [9]). *Suppose* $m>0$ *and* γ *is the fundamental unit of* $Q(\sqrt{m})$. *If* $\gamma \equiv x^3 \pmod{T}$ *for some* $x \in Q(\sqrt{m})$, *then* $3 \mid h(-3m)$. ■

QUESTION 4.3: For what integers $m>0$ do we have $3 \mid h(-3m)$ while the fundamental unit of $Q(\sqrt{m})$ is not a semi-cube? Are there any such m?

5. Examples.

5.1. Let $m=87$, and $\gamma = 28+3\sqrt{87}$. Since $3 \mid m$, we have $(3) = P^2$ and $T = P^3$; clearly $\gamma \equiv 1 \pmod{T}$. Since $N(\gamma)=1$, and $(\gamma) = (1)^3$, then γ is a semi-cube. The minimum polynomial for γ is $g(x) = x^2-56x+1$, which is the auxiliary quadratic for

$$f(x) = x^3-3x+56.$$

For ξ a root of $f(x)$, we have $Q(\sqrt{-29}, \xi)$ is an unramified abelian extension of $Q(\sqrt{-29})$. By the way, $(\xi-1)/3$ is a root of x^3+x^2+2, a polynomial with smaller coefficients and discriminant -4.29. This reduction seems to work fairly often.

5.2. Let $m=79$. The class number is 3, so there must be at least one unramified abelian cubic extensions of $Q(\sqrt{79})$, and a corresponding pure cubic extension of $Q\sqrt{-3\cdot 79})$ with discriminant 3^2. Looking around, we find $\gamma = 17^2+12\sqrt{-3\cdot 79}$. This γ is not a perfect cube, it has norm 7^6, so (γ) is an ideal cube, and we have $17^2 \equiv 1 \pmod 9$ and $12 \equiv 3 \pmod 9$. Then γ is a semi-cube. Now γ is a root of $g(x) = x^2-2\cdot 17^2 x+7^6$, which is the auxiliary quadratic for

$$f(x) = x^3-3\cdot 49x+2\cdot 17^2$$

and if ξ is a root of $f(x)$, then $Q(\sqrt{79}, \xi)$ is an unramified abelian extension of $Q(\sqrt{79})$. We also have $(\xi-1)/3$ is a root of $x^3+x^2-16x-16$.

5.3. From Example 5.2, we see that $Q(\sqrt{-3\cdot 79})$ has class number divisible by 3, so we seek a semi-cube in $Q(\sqrt{79})$. The fundamental unit $\gamma = 80+9\sqrt{79}$ is a semi-cube; it is a root of $g(x) = x^2-160x+1$, which is the auxiliary quadratic for $f(x) = x^3-3x+160$. If ξ is a root of $f(x)$, then $Q(\sqrt{-3\cdot 79}, \xi)$ is an unramified abelian extension of $Q(\sqrt{-3\cdot 79})$. Here we find $(\xi+1)/3$ is a root of x^3-x^2+6.

5.4. Let $m=109$. The fundamental unit is $\gamma = 118+25(1+\sqrt{109})/2$ which is congruent to $(\sqrt{109})^3 \pmod 9$, so γ is a semi-cube. It is a root of $x^2-261x-1$, which is the auxiliary quadratic for $f(x) = x^3+3x+261$. If ξ is a root of $f(x)$, then $Q(\sqrt{-3\cdot 109}, \xi)$ is an unramified abelian extension of $Q(\sqrt{-3\cdot 109})$.

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