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# On sign-changes in the remainder-term of the prime-number formula, I

by

J. KACZOROWSKI (Poznań)

1. There are many interesting questions concerning the oscillatory nature of various remainder-terms in the prime-number theorem. If $\pi(x)$ stands for the number of primes not exceeding $x$, then the theorem in question shows that the average value of $\pi(x)$ is $\operatorname{li} x$; but there may appear wide deviations in either direction from this average.

It can be verified by the use of tables of primes that

$$\pi(x) < \operatorname{li} x, \tag{1.1}$$

for $x < 10^8$ (see [9], [18]).

There were not only numerical, but also theoretical grounds for believing that (1.1) holds generally for $x > 2$, and B. Riemann in his famous memoir from 1859 stated this as a conjecture.

It was proved, however, by J. E. Littlewood [1], [11] in 1914 that the function

$$\Delta_1(x) = \pi(x) - \operatorname{li} x \tag{1.2}$$

changes sign infinitely often as $x \to \infty$.

It can also be shown that if the Riemann hypothesis on the non-trivial zeros of $\zeta(s)$ is true then

$$\int_2^x \big(\pi(\xi) - \operatorname{li} \xi\big)\, d\xi < 0 \tag{1.3}$$

holds for large $x$. It means that although (1.1) fails in the general it is true on the average.

Unfortunately Littlewood's method was ineffective and it gave no numerical value of $x$ for which $\pi(x) > \operatorname{li} x$ and also no estimate from below for the number of sign-changes of $\Delta_1(x)$ in the interval $2 \leqslant x \leqslant T$. Also other theoretical reasonings provided no numerical solution of the inequality $\pi(x) > \operatorname{li} x$ until Skewes [21] in 1955 proved that there is at least one sign-change of $\Delta_1(x)$ in the interval

(1.4) $$2 \leqslant x \leqslant \exp\exp\exp\exp(7.705).$$

This was improved by S. Lehman [8] in 1966 who found smaller upper bound

(1.5) $$1.65 \cdot 10^{1165}$$

for the first sign-change.

Infinitely many sign changes of the related function

(1.6) $$\Delta_2(x) = \Pi(x) - \operatorname{li} x = \sum_{m \geqslant 1} \frac{1}{m} \pi(x^{1/m}) - \operatorname{li} x,$$

was exhibited in 1903 by E. Schmidt [19].

It is clear that methods used in the investigation of sign changes of $\Delta_1(x)$ and $\Delta_2(x)$ can be applied to similar questions concerning other remainders in the prime-number formula, namely

(1.7) $$\Delta_3(x) = \psi(x) - x = \sum_{n \leqslant x} \Lambda(n) - x,$$

and

(1.8) $$\Delta_4(x) = \vartheta(x) - x = \sum_{p \leqslant x} \log p - x.$$

Let $V_j(T)$, $1 \leqslant j \leqslant 4$ denote the number of sign-changes of $\Delta_j(x)$ in $[2, T]$.

One of the important problems in the analytic theory of numbers is to investigate the magnitude of the functions $V_j(T)$.

Let

(1.9) $$\Theta = \sup_{\zeta(\varrho)=0} \operatorname{Re} \varrho$$

and define $\gamma_1$ as follows. If $\zeta(s)$ has any zeros $\Theta + i\gamma$ on the line $\sigma = \Theta$, then $\gamma_1$ denotes the least positive $\gamma$ corresponding to these zeros; otherwise $\gamma_1 = +\infty$.

Pólya [14], [15] proved in 1930 that

(1.10) $$\overline{\lim_{T\to\infty}} \frac{V_3(T)}{\log T} \geqslant \frac{\gamma_1}{\pi}.$$

This interesting theorem, however, still gives no estimate of $V_3(T)$ from below.

Ingham attacked the problem in 1936. He proved, [2], that if there is a zero on the line $\sigma = \Theta$ ($\Theta$ defined by (1.9)), then for $T \geqslant T_0$, $\Delta_j(x)$, $1 \leqslant j \leqslant 4$, changes sign in every interval of the type $T \leqslant x \leqslant c_0 T$, where $c_0$ stands for positive constant. This implies that

(1.11) $$V_j(T) \geqslant c_1 \log T \quad \text{for} \quad T \geqslant T_0,\ 1 \leqslant j \leqslant 4.$$

Note that Ingham formulated his theorem only for $j = 1$ but the method of the proof works also in other cases.

Levinson [10] in 1974, using Ingham's theorem and modifying Pólya's method, proved that

(1.12) $$\overline{\lim_{T\to\infty}} \frac{V_1(T)}{\log T} > 0.$$

From Levinson's paper it follows that in the case $\Theta > 1/2$,

(1.13) $$\overline{\lim_{T\to\infty}} \frac{V_1(T)}{\log T} \geqslant \frac{\gamma_1}{\pi},$$

without the use of Ingham's theorem.

Using the estimates contained in Pólya's paper, and the result of Ingham, one can prove that

(1.14) $$\overline{\lim_{T\to\infty}} \frac{V_4(T)}{\log T} > 0,$$

and in the case $\Theta > 1/2$

(1.15) $$\overline{\lim_{T\to\infty}} \frac{V_4(T)}{\log T} \geqslant \frac{\gamma_1}{\pi}.$$

The last estimate is obtained without the use of Ingham's theorem.

Basing on the paper of Levinson, one can prove without the help of Ingham's theorem that

(1.16) $$\overline{\lim_{T\to\infty}} \frac{V_2(T)}{\log T} \geqslant \frac{\gamma_1}{\pi}.$$

The subsequent progress on the subject was achieved by the application of Turán's power sum method. In 1961 and 1962 S. Knapowski [4], [5] proved unconditionally the following ineffective estimate

(1.17) $$V_1(T) \geqslant c_2 \log\log T, \quad T \geqslant T_1,$$

and he also proved the effective estimate

(1.18) $$V_1(T) \geqslant e^{-35} \log\log\log\log T, \quad T \geqslant \exp\exp\exp\exp\exp(35).$$

These results were improved later by S. Knapowski and P. Turán [6],

[7] who obtained

(1.19) $$V_1(T) \geqslant c_3 \frac{(\log T)^{1/4}}{(\log\log T)^4}, \quad T \geqslant T_2,$$

(1.20) $$V_1(T) \geqslant c_4 \log\log\log T, \quad T \geqslant T_3,$$

where again (1.19) is ineffective while (1.20) is effective.

J. Pintz improved (1.19) and (1.20) by showing that ineffectively

(1.21) $$V_1(T) \geqslant c_5 \frac{\log T}{(\log\log T)^3}, \quad T \geqslant T_4,$$

and effectively

(1.22) $$V_1(T) \geqslant c_6 \frac{(\log T)^{1/2}}{\log\log T}, \quad T \geqslant T_5$$

(see [12], [13]).

He proved also similar estimates for $V_j(T)$, $j = 2, 3, 4$. For $V_2(T)$ and $V_3(T)$ the results are effective (compare [13]).

**2.** In this paper we are going to prove unconditionally the following stronger theorem.

THEOREM. *The inequality*

(2.1) $$V_j(T) \geqslant \frac{\gamma_0}{4\pi} \log T, \quad j = 2, 3$$

*holds for sufficiently large* $T$, *where* $\gamma_0 = 14.13\ldots$ *denotes the imaginary part of the "lowest" zero of the Riemann zeta function.*

The estimates (2.1) are stronger than any unconditional estimates given hitherto and are of the same strength as Ingham's conditional estimate (1.11).

The factor $\gamma_0/4\pi$ in (2.1) can be improved slightly and the best value which can be obtained by the use of the presented method seems to be $\gamma_0/\pi$, which would agree with Pólya's result (1.10).

Let us note that the estimates (2.1) are effective. It means that there is possible to compute numerically the constant $T_6$ such that (2.1) holds for $T \geqslant T_6$.

It is also possible to prove the inequality

(2.2) $$V_j(T) \geqslant c \log T,$$

for $j = 1, 4$, but it needs some additional modifications in the method of the proof. The proof of (2.2) will appear in the second paper of this series. The estimates (2.2) are ineffective.



**3.** For a complex valued function $f$ defined on the set of the positive real numbers, let us put

(3.1) $$\delta(f; x) = \delta_1(f; x) = \int_0^x f(\xi)\frac{d\xi}{\xi}, \quad x > 0,$$

and for $n \geqslant 2$

(3.2) $$\delta_n(f; x) = \int_0^x \delta_{n-1}(f; \xi)\frac{d\xi}{\xi}, \quad x > 0,$$

if these integrals do exist. It means that $\delta_n$ is the $n$-fold iteration of the operator $\delta$.

In the proof of (2.1) we shall use $\delta_n(f; x)$ with the function $f(\xi)$, which for $\xi \geqslant 2$ is equal to $\Delta_2(\xi)$ or $\Delta_3(\xi)$, respectively. As we shall see it is easier to deal with the smoother functions $\delta_n(f; x)$ that with $f(x)$ itself.

We shall give two separate proofs for $V_1(T)$ and $V_3(T)$.

**4.** Proof of the theorem in the case $j = 3$. First of all we are going to prove a certain 'explicit formula' for $\delta_n(\Delta_3; x)$.

Using Perron's summation formula we get

(4.1) $$\delta_1(\psi; x) = \sum_{n \leqslant x} \Lambda(n)\log(x/n) = \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty} \left\{-\frac{\zeta'}{\zeta}(s)\right\}\frac{x^s}{s^2}\,ds.$$

Let us choose a real number $\Gamma$ with the properties:

(4.2) $$\zeta(\sigma + i\Gamma) \neq 0 \quad \text{for} \quad 0 \leqslant \sigma \leqslant 1,$$

(4.3) $$10^6 < \Gamma < 10^6 + 1.$$

Let us define the curve $L$ consisting of the following five parts:

(4.4)
- $L_1$: the half-line: $s = 1 + it$, $-\infty < t < -\Gamma$,
- $L_2$: the line segment: $s = \sigma - i\Gamma$, $0.1 \leqslant \sigma \leqslant 1$,
- $L_3$: the line segment: $s = 0.1 + it$, $-\Gamma \leqslant t \leqslant \Gamma$,
- $L_4$: the line segment symmetrical to $L_2$ upon the real axis,
- $L_5$: the half-line symmetrical to $L_1$ upon the real axis.

Shifting the path of integration in (4.1) we get

(4.5) $$\delta_1(\psi; x) = x - \sum_{\substack{\varrho = 1/2 + i\gamma \\ |\gamma| < \Gamma}} \frac{x^\varrho}{\varrho^2} + \frac{1}{2\pi i}\int_L \left\{-\frac{\zeta'}{\zeta}(s)\right\}\frac{x^s}{s^2}\,ds.$$

We use here the known fact that all zeros of $\zeta(s)$ with $|\gamma| < \Gamma$ lie on the line $\sigma = 1/2$.

From the definition of the contour $L$ and from (4.5) it is easy to deduce that for $n \geqslant 2$

$$\delta_{n-1}(\Delta_3; x) = -\sum_{\substack{\varrho = 1/2+i\gamma \\ |\gamma| < \Gamma}} \frac{x^\varrho}{\varrho^n} + \frac{1}{2\pi i} \int_L \left\{-\frac{\zeta'}{\zeta}(s)\right\} \frac{x^s}{s^n} ds \tag{4.6}$$

$$= -\sum_{\substack{\varrho = 1/2+i\gamma \\ |\gamma| < \Gamma}} \frac{x^\varrho}{\varrho^n} + O(10^n x^{0.1} + x\Gamma^{-n}).$$

Let us denote by $\varrho_0 = 1/2 + i\gamma_0 = 1/2 + i \cdot 14.13\ldots$ the "lowest" zeta-zero and write

$$\varrho_0 = |\varrho_0| e^{i\varphi_0}, \quad 0 < \varphi_0 < \pi/2. \tag{4.7}$$

Since the only zeros $\varrho = 1/2 + i\gamma$ with $|\gamma| < 15$ are $\varrho_0$ and $\bar{\varrho}_0$, we have

$$\Bigg|\sum_{\substack{\varrho = 1/2+i\gamma \\ \gamma_0 < |\gamma| < \Gamma}} \frac{x^\varrho}{\varrho^n}\Bigg| \leqslant \frac{x^{1/2}}{15^{n-2}} \sum_\varrho \frac{1}{|\varrho|^2} \ll x^{1/2} \cdot 15^{-n} \tag{4.8}$$

and thus

$$\delta_{n-1}(\Delta_3; x) = -\frac{x^{\varrho_0}}{\varrho_0^n} - \frac{x^{\bar{\varrho}_0}}{\bar{\varrho}_0^n} + O(x^{0.5} \cdot 15^{-n} + 10^n \cdot x^{0.1} + x\Gamma^{-n}) \tag{4.9}$$

$$= -\frac{2x^{1/2}}{|\varrho_0|^n} \Bigg\{\cos(\gamma_0 \log x - n\varphi_0) + O\left(\left(\frac{|\varrho_0|}{15}\right)^n + \frac{(10|\varrho_0|)^n}{x^{0.4}} + \left(\frac{|\varrho_0|}{\Gamma}\right)^n x^{0.5}\right)\Bigg\}.$$

This is the needed 'explicit formula' for $\delta_{n-1}(\Delta_3; x)$.

**5.** We restrict the range for $x$ as follows:

$$T^{3/4-\varepsilon} < x < T, \quad 0 < \varepsilon < 3/4 \tag{5.1}$$

and put

$$n = \left[\frac{\log T}{1.9 \log(\Gamma/|\varrho_0|)}\right] + 1. \tag{5.2}$$

It is easy to verify that then

$$n < 0.29 \frac{\log T}{\log(10|\varrho_0|)}$$



and thus

$$\frac{(10|\varrho_0|)^n}{x^{0.4}} \ll T^{0.29} \cdot x^{-0.4} \ll T^{-0.01+\varepsilon'}, \tag{5.3}$$

$$\left(\frac{|\varrho_0|}{15}\right)^n \ll T^{-0.0015+\varepsilon'}, \tag{5.4}$$

$$x^{0.5}\left(\frac{|\varrho_0|}{\Gamma}\right)^n \ll T^{-0.02+\varepsilon'}, \tag{5.5}$$

where $\varepsilon' = \varepsilon'(\varepsilon)$ is positive and tends to zero as $\varepsilon$ does.

Hence for $x$ satisfying (5.1) and $n$ given by (5.2) we have

$$\delta_{n-1}(\Delta_3; x) = -\frac{2x^{1/2}}{|\varrho_0|^n} \{\cos(\gamma_0 \log x - n\varphi_0) + O(T^{-0.001})\} \tag{5.6}$$

for sufficiently small positive $\varepsilon$.

**6.** To finish the proof we must exhibit the connection between the number of sign-changes of $f(x)$ and $\delta_n(f; x)$.

In order to perform this we use the following known

LEMMA 1 (see [16], pp. 40–41). *Let $f\colon (0, \infty) \to R$ be a real valued, piecewise continuous function, non-constant in every interval $(a, b)$, $0 < a < b < \infty$ and such that the integral*

$$\int_0^A |f(\xi)| d\xi$$

*is finite for every $A > 0$. If $v(T)$ denotes the number of sign-changes of $f(x)$ in the interval $(0, T]$ and $V(T)$ denotes the number of sign-changes of the function $F(x)$, where*

$$F(x) = \int_0^x f(\xi) d\xi,$$

*then for $T > 0$,*

$$V(T) \leqslant v(T). \tag{6.1}$$

**7.** Let $V_3^{(n)}(T)$ denote the number of sign-changes of $\delta_n(\Delta_3; x)$ in the interval $0 \leqslant x \leqslant T$. From Lemma 1 it follows that

$$V_3(T) \geqslant V_3^{(n-1)}(T), \tag{7.1}$$

for every $n \geqslant 2$ and $T > 0$.

Since (5.6) evidently implies that

$$V_3^{(n-1)}(T) \geqslant V_3^{(n-1)}(T) - V_3^{(n-1)}(T^{3/4-\varepsilon}) \geqslant \frac{\gamma_0}{4\pi} \log T, \tag{7.2}$$

the proof of the case $j = 3$ is complete.

**8.** Before we turn to the proof of the case $j=2$, let us introduce the following notations. For every complex number $z$, with real part greater than 0.1, and any real number $\eta$, $0<\eta<\operatorname{Re} z-0.1$, let us denote by $C_z(\eta)$ the curve consisting of three parts:

$$\begin{aligned} &C_z^1(\eta)\colon \text{ the line segment: } s=\sigma+i\operatorname{Im} z,\ 0.1\leqslant\sigma\leqslant\operatorname{Re} z-\eta,\\ (8.1)\quad &C_z^2(\eta)\colon \text{ the circle: } s=z+\eta e^{i\varphi},\ -\pi\leqslant\varphi\leqslant\pi,\\ &C_z^3(\eta)\colon \text{ the line segment: } s=\sigma+i\operatorname{Im} z,\ \operatorname{Re} z-\eta\geqslant\sigma\geqslant 0.1.\end{aligned}$$

For $x>0$ and $0<\eta<0.75$ let

$$(8.2)\qquad l(x)=\begin{cases} -\dfrac{1}{2\pi i}\displaystyle\int_{C_1(\eta)}\log(s-1)\frac{x^s}{s}\,ds & \text{for } 0<x<2,\\ \operatorname{li}(x)=\displaystyle\int_0^x\frac{du}{\log u} & \text{for } x\geqslant 2.\end{cases}$$

With these notations we have the following

LEMMA 2. *For* $x>0$

$$(8.3)\qquad l(x)=-\frac{1}{2\pi i}\int_{C_1(\eta)}\log(s-1)\frac{x^s}{s}\,ds+O(x^{0.1}).$$

Proof. For $0<x<2$ the proof is obvious. Let $x\geqslant 2$ and let

$$(8.4)\qquad J(x)\overset{\mathrm{df}}{=}-\frac{1}{2\pi i}\int_{C_1(\eta)}\log(s-1)\frac{x^s}{s}\,ds.$$

Then

$$\begin{aligned}(8.5)\quad \frac{d}{dx}J(x)&=-\frac{1}{2\pi i}\int_{C_1(\eta)}\log(s-1)x^{s-1}\,ds\\ &=-\frac{1}{2\pi i}\left\{\int_{-\infty}^{0}(\log|\xi|-\pi i)x^{\xi}\,d\xi+\int_{0}^{-\infty}(\log|\xi|+\pi i)x^{\xi}\,d\xi\right\}+\\ &\qquad\qquad +O(x^{-0.9})\\ &=\int_0^\infty x^{-\xi}\,d\xi+O(x^{-0.9})=\frac{1}{\log x}+O(x^{-0.9})\end{aligned}$$

and (8.3) follows.

For the sake of convenience we shall introduce the following notation: if $f$ is a complex-valued function and $g$ is a positive-valued function, both



defined on the set $\Omega$ of complex numbers and satisfying the inequality

$$|f(x)|\leqslant g(x),\qquad x\in\Omega,$$

then we shall write

$$(8.6)\qquad f(x)=\bar{O}(g(x)),\qquad x\in\Omega.$$

LEMMA 3. *For any complex number* $z$ *such that* $\operatorname{Re} z=1/2$ *and* $|z|\geqslant 4$ *we have for* $0<\eta<0.4$, $x>e$ *and* $1\leqslant n\leqslant\frac{1}{2}\log x$

$$\begin{aligned}(8.7)\quad I_n(z,x)&\overset{\mathrm{df}}{=}\frac{1}{2\pi i}\int_{C_z(\eta)}\log(s-z)\frac{x^s}{s^n}\,ds\\ &=-\frac{x^z}{z^n\log x}+\bar{O}\left(\frac{x^{1/2}}{|z|^{n+1}\log x}\right)+O\left(\frac{x^{1/2}}{|z|^n\log^2 x}\right).\end{aligned}$$

Proof. Since

$$\int_{C_z^2(\eta)}\log(s-z)\frac{x^s}{s^n}\,ds=o(1)\quad\text{as}\quad\eta\to 0$$

we can write

$$\begin{aligned}(8.8)\quad I_n(z,x)&=\frac{1}{2\pi i}\int_{C_z(\eta)-z}\log s\frac{x^{s+z}}{(s+z)^n}\,ds=-x^z\int_0^{0.4}\frac{x^{-\xi}}{(z-\xi)^n}\,d\xi\\ &=-\frac{x^z}{z^n}\int_0^{0.4}x^{-\xi}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\xi^k z^{-k}\,d\xi\\ &=-\frac{x^z}{z^n\log x}+O\left(\frac{x^{0.1}}{|z|^n\log x}\right)-\\ &\qquad -\frac{x^z}{z^n}\sum_{k=1}^{\infty}\binom{n+k-1}{k}\frac{1}{z^k}\int_0^{0.4}x^{-\xi}\xi^k\,d\xi,\end{aligned}$$

$$\begin{aligned}(8.9)\quad &\left|\sum_{1\leqslant k\leqslant n}\binom{n+k-1}{k}\frac{1}{z^k}\int_0^{0.4}x^{-\xi}\xi^k\,d\xi\right|\\ &\leqslant\sum_{1\leqslant k\leqslant n}\binom{n+k-1}{k}\frac{1}{|z|^k}\cdot\frac{1}{(\log x)^{k+1}}\Gamma(k+1)\\ &=\frac{1}{|z|\log^2 x}\sum_{0\leqslant k\leqslant n-1}\frac{(n+k)!}{(n-1)!}\cdot\frac{1}{(|z|\log x)^k}.\end{aligned}$$

Since $n \leqslant (1/2)\log x$, we have

$$\frac{(n+k)!}{(n-1)!}\cdot\frac{1}{(\log x)^k} = n\,\frac{(n+1)\dots(n+k)}{(\log x)^k} \leqslant n\left(\frac{2n}{\log x}\right)^k \leqslant n;$$

thus (8.9) is smaller than

$$\frac{1}{|z|\log^2 x}\,2n \leqslant \frac{1}{|z|\log x}.$$

Moreover, since

$$\int\limits_0^{0.4} x^{-\xi}\,\xi\,d\xi < \frac{1}{\log^2 x}$$

and obviously $\binom{n}{k} < 2^n$, we have

$$\left|\sum_{k\geqslant n+1}\binom{n+k-1}{k}\frac{1}{z^k}\int\limits_0^{0.4} x^{-\xi}\xi^k\,d\xi\right| < \sum_{k\geqslant n+1}\frac{2^{n+k-1}}{|z|^k}\cdot\frac{1}{\log^2 x}$$

$$< \frac{2^{n-1}}{\log^2 x}\left(\frac{2}{|z|}\right)^{n+1}\sum_{k\geqslant 0}\left(\frac{2}{|z|}\right)^k = \frac{1}{|z|\log^2 x}\left(\frac{4}{|z|}\right)^n\frac{1}{1-2/|z|} < \frac{1}{\log^2 x}.$$

These estimates together with (8.8), give the assertion of Lemma 3.

**9.** Let us prove an 'explicit formula' for $\delta_{n-1}(\Pi - l;\, x)$ similar to (4.6). For $\sigma > 1$

(9.1) $$\log\zeta(s) = \sum_{n\geqslant 2}\frac{\Lambda(n)}{\log n\cdot n^s},$$

and hence by Perron's summation formula

(9.2) $$\delta_1(\Pi;\, x) = \frac{1}{2\pi i}\int\limits_{2-i\infty}^{2+i\infty}\log\zeta(s)\,\frac{x^s}{s^2}\,ds.$$

Let $\Gamma$ denote the number defined by (4.2) and (4.3), and let $L'$ be the curve consisting of the contour $L$ defined by (4.4), the curve $C_1(\eta)$, and the family of curves $C_\varrho(\eta)$, where $\varrho = 1/2 + i\gamma$ runs over all $\zeta$-zeros with $|\gamma| < \Gamma$.

By Lemma 2

(9.3) $$\frac{1}{2\pi i}\int\limits_{C_1(\eta)}\log\zeta(s)\,\frac{x^s}{s^2}\,ds = \delta_1(l;\, x) + O(x^{0.1}).$$



Shifting the line of integration in (9.2) and using (9.3) we get

(9.4) $$\delta_1(\Pi - l;\, x) = \frac{1}{2\pi i}\int\limits_{L\setminus C_1(\eta)}\log\zeta(s)\,\frac{x^s}{s^2}\,ds + O(x^{0.1})$$

$$= \sum_{\substack{\varrho = 1/2+i\gamma\\ |\gamma|<\Gamma}} I_2(\varrho,\, x) + \frac{1}{2\pi i}\int\limits_{L_1\cup L_2\cup L_4\cup L_5}\log\zeta(s)\,\frac{x^s}{s^2}\,ds + O(x^{0.1})$$

where $I_2(\varrho,\, x)$ are given by (8.7).

Applying the operator $\delta$, $n-2$ times we get

(9.5) $$\delta_{n-1}(\Pi - l;\, x) = \sum_{\substack{\varrho = 1/2+i\gamma\\ |\gamma|<\Gamma}} I_n(\varrho,\, x) + \frac{1}{2\pi i}\int\limits_{L_1\cup L_2\cup L_4\cup L_5}\log\zeta(s)\,\frac{x^s}{s^n}\,ds + O(10^n\cdot x^{0.1})$$

$$= \sum_{\substack{\varrho = 1/2+i\gamma\\ |\gamma|<\Gamma}} I_n(\varrho,\, x) + O(10^n\cdot x^{0.1} + x\Gamma^{-n}).$$

**10.** Proof of the theorem (2.1) in the case $j = 2$. Let us choose $n$ as in (5.2) and put the restriction (5.1) upon $x$. Then by Lemma 3 the contribution of the terms $I_n(\varrho,\, x)$ with $|\gamma| > 15$ in (9.5) can be estimated by

$$O\left(\frac{x^{1/2}}{15^n\log x}\right),$$

and thus still by Lemma 3 we get

$$\delta_{n-1}(\Pi - l;\, x) = -\frac{x^{\varrho_0}}{\varrho_0^n\log x} - \frac{x^{\bar\varrho_0}}{\bar\varrho_0^n\log x} + \bar O\left(\frac{2x^{1/2}}{|\varrho_0^{n+1}|\log x}\right) +$$

$$+ O\left(\frac{x^{1/2}}{|\varrho_0|^n\log^2 x} + \frac{x^{1/2}}{15^n\log x} + 10^n x^{0.1} + x\Gamma^{-n}\right)$$

$$= -\frac{2x^{1/2}}{|\varrho_0|^n\log x}\left\{\cos(\gamma_0\log x - n\varphi_0) + \bar O\left(\frac{3}{|\varrho_0|}\right) +\right.$$

$$\left. + O\left(\left[\left(\frac{|\varrho_0|}{15}\right)^n + \frac{(10|\varrho_0|)^n}{x^{0.4}} + \left(\frac{|\varrho_0|}{\Gamma}\right)^n x^{1/2}\right]\log T\right)\right\}$$

for sufficiently large $x$.

Using now (5.3)–(5.5) we get for sufficiently large $T$, satisfying (5.1) and $n$ given by (5.2)

$$\delta_{n-1}(\Pi - l;\, x)$$
$$= -\frac{2x^{1/2}}{|\varrho_0|^n \log x}\left\{\cos(\gamma_0 \log x - n\varphi_0) + \bar{O}\left(\frac{3}{|\varrho_0|}\right) + O(T^{-0.001} \log T)\right\}$$

and finally

$$\delta_{n-1}(\Pi - l;\, x) = -\frac{2x^{1/2}}{|\varrho_0|^n \log x}\{\cos(\gamma_0 \log x - n\varphi_0) + \bar{O}(0.5)\} \tag{10.1}$$

because $3/|\varrho_0| < 0.3$ and $T^{-0.001} \log T < 0.2$ for large $T$.

As previously (10.1) implies that the difference $\pi(x) - l(x)$ changes sign more than $(\gamma_0/4\pi) \log T$ times in the interval $(0, T)$. But in the interval $(0, 2)$

$$\pi(x) - l(x) = -l(x) = -\int_0^{0.4} \frac{x^{1-\xi}}{1-\xi}\, d\xi$$

is negative and so $\pi(x) - l(x)$ changes sign for $x \geqslant 2$ only. But for such $x$

$$\pi(x) - l(x) = \pi(x) - \mathrm{li}\, x$$

and thus the proof of the theorem is complete.

### References


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INSTITUTE OF MATHEMATICS OF THE ADAM MICKIEWICZ UNIVERSITY
Poznań, Poland



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