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# An application of dihedral fields to representations of primes by binary quadratic forms

by

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**1. Introduction.** Let $H(m)$ denote the strict ideal class group of the quadratic field $Q(\sqrt{m})$ of discriminant $m$. We have

$$H(m) \simeq Z_{2^{n_1}} \times Z_{2^{n_2}} \times \ldots \times Z_{2^{n_k}} \times G, \tag{1.1}$$

where the order $g$ of the group $G$ is odd and $Z_{2^n}$ denotes the cyclic group of order $2^n$.

Let $p$ be a prime number such that $\left(\frac{m}{p}\right) = 1$. Then $p$ is represented by two inverse classes $C_p$, $C_p^{-1}$ (or one ambiguous class) of binary quadratic forms of discriminant $m$. Gauss's theory of genera determines $C_p$ modulo squares in the composition class group of discriminant $m$.

In this paper we determine the class $C_p$ modulo fourth powers in the simplest case, namely when

$$H(m) \simeq Z_{2^n} \times G, \quad n \geqslant 2, \tag{1.2}$$

and the class $C_p$ is a square, that is $p$ is a prime on which all the generic characters have the value $+1$. It is known (see for example [2]) that (1.2) occurs precisely for the following values of the discriminant $m$:

(I) $m = -4r$, $r(\text{prime}) \equiv 1 \pmod 8$;

(II) $m = -8r$, $r(\text{prime}) \equiv 1 \pmod 8$;

(III) $m = -8q$, $q(\text{prime}) \equiv 7 \pmod 8$;

(IV) $m = -qr$, $q(\text{prime}) \equiv 3 \pmod 4$, $r(\text{prime}) \equiv 1 \pmod 4$, $\left(\frac{q}{r}\right) = 1$;

(V) $m = 8r$, $r(\text{prime}) \equiv 1 \pmod 8$;

(VI) $m = qr$, $q(\text{prime}) \equiv r(\text{prime}) \equiv 1 \pmod 4$, $\left(\frac{q}{r}\right) = 1$.

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We define
$q = 1$ in case (I),
$q = 2$ in cases (II), (V),
$r = 2$ in case (III)
and

$$k_q = \begin{cases} Q(\sqrt{-q}) & \text{in cases (I), (II), (III), (IV);} \\ Q(\sqrt{q}) & \text{in cases (V), (VI);} \end{cases}$$

$$k_r = Q(\sqrt{r}); \quad k_m = Q(\sqrt{m})$$

$$K = Q(\sqrt{r}, \sqrt{m}) = \begin{cases} Q(\sqrt{r}, \sqrt{-q}) & \text{in cases (I) to (IV),} \\ Q(\sqrt{r}, \sqrt{q}) & \text{in cases (V), (VI).} \end{cases}$$

The strict class number of the quadratic field $Q(\sqrt{d})$ will be denoted by $h(d)$.

Throughout this paper the symbol $\left(\frac{x+y\sqrt{n}}{p}\right)$, where $n$ and $x^2-ny^2$ are quadratic residues of the odd prime $p$, will be used both as a Legendre symbol, in which case $\sqrt{n}$ is interpreted as a rational integer modulo $p$, as well as (equivalently) the quadratic residue symbol $\left[\frac{x+y\sqrt{n}}{P}\right]_2$ in the ring of integers of $Q(\sqrt{n})$, where $P$ is either of the two prime ideals dividing $p$.
We prove:

THEOREM 1. *Let $r$ be a prime $\equiv 1 \pmod 8$ and $p$ a prime satisfying $\left(\frac{-1}{p}\right) = \left(\frac{p}{r}\right) = 1$, so that $p$ is represented by the classes $C_p$ and $C_p^{-1}$ of discriminant $-4r$, and there exist integers $a$, $b$, $e$ and $f$ such that*

(1.3) $$p = a^2+b^2,$$

(1.4) $$p^{h(r)} = e^2-rf^2, \quad e > 0, \quad (e, f) = 1.$$

*Then the class $C_p$ is a fourth power if, and only if, for any solutions of* (1.3) *and* (1.4), $\left(\frac{a+b\sqrt{-1}}{r}\right) = 1$ *or, equivalently,* $e+f \equiv 1 \pmod 4$.

THEOREM 2. *Let $r$ be a prime $\equiv 1 \pmod 8$ and $p$ a prime satisfying $\left(\frac{-2}{p}\right) = \left(\frac{p}{r}\right) = 1$, so that $p$ is represented by the classes $C_p$ and $C_p^{-1}$ of discriminant $-8r$, and there exist integers $a$, $b$, $e$ and $f$ such that*



(1.5) $$p = a^2+2b^2,$$

(1.6) $$p^{h(r)} = e^2-rf^2, \quad e > 0, \quad (e, f) = 1.$$

*Then the class $C_p$ is a fourth power if, and only if, for any solutions of* (1.5) *and* (1.6), $\left(\frac{a+b\sqrt{-2}}{r}\right) = 1$ *or, equivalently,* $\left(\frac{2}{p}\right)^{(r-1)/8}\left(\frac{-2}{e+f}\right) = 1$.

THEOREM 3. *Let $q \equiv 7 \pmod 8$ be a prime. Let $p$ be a prime satisfying $\left(\frac{p}{q}\right) = \left(\frac{2}{p}\right) = 1$, so that $p$ is represented by the classes $C_p$ and $C_p^{-1}$ of discriminant $-8q$, and there exist integers $a$, $b$, $e$ and $f$ such that*

(1.7) $$p^{h(-q)} = a^2+qb^2, \quad (a, b) = 1, \ a \text{ or } b \equiv 1 \pmod 4,$$

(1.8) $$p = e^2-2f^2, \quad e > 0.$$

*Then the class $C_p$ is a fourth power if, and only if, for any solutions of* (1.7) *and* (1.8),

$$\left(\frac{-1}{p}\right)^{(q+1)/8}\left(\frac{2}{a+b}\right) = 1 \quad \text{or, equivalently,} \quad \left(\frac{e+f\sqrt{2}}{q}\right) = 1.$$

We note that Theorem 3 of [1] is part of the special case $q = 7$ of our Theorem 3.

THEOREM 4. *Let $q \equiv 3 \pmod 4$ and $r \equiv 1 \pmod 4$ be primes such that $\left(\frac{q}{r}\right) = 1$. Let $p$ be a prime satisfying $\left(\frac{p}{q}\right) = \left(\frac{p}{r}\right) = 1$, so that $p$ is represented by the classes $C_p$ and $C_p^{-1}$ of discriminant $-qr$ and there exist integers $a$, $b$, $e$ and $f$ such that*

(1.9) $$4p^{h(-q)} = a^2+qb^2, \quad (a, b) = 1 \text{ or } 2,$$

(1.10) $$4p^{h(r)} = e^2-rf^2, \quad (e, f) = 1 \text{ or } 2, \quad e > 0.$$

*Then the class $C_p$ is a fourth power if, and only if, for any solutions of* (1.9) *and* (1.10),

$$\left(\frac{(a+b\sqrt{-q})/2}{r}\right) = 1 \quad \text{or, equivalently,} \quad \left(\frac{(e+f\sqrt{r})/2}{q}\right) = 1.$$

We note that Theorems 6 and 7 of [1] can be deduced as special cases of our Theorem 4 with $q = 3$, $r = 13$ and $q = 11$, $r = 5$, respectively.

THEOREM 5. *Let $r$ be a prime $\equiv 1 \pmod 8$ and $p$ be a prime satisfying $\left(\frac{2}{p}\right)$*

$= \left(\frac{p}{r}\right) = 1$, *so that* $p$ *is represented by the classes* $C_p$ *and* $C_p^{-1}$ *of discriminant* $8p$, *and that there exist integers* $a$, $b$, $e$ *and* $f$ *such that*

$$p = a^2 - 2b^2, \quad (a, b) = 1, \quad a > 0; \tag{1.11}$$

$$p^{h(r)} = e^2 - rf^2, \quad (e, f) = 1, \quad e+f \equiv 1 \pmod 4. \tag{1.12}$$

*Then* $C_p$ *is a fourth power if, and only if, for any solutions of* (1.11) *and* (1.12), $\left(\frac{a+b\sqrt{2}}{r}\right) = 1$ *or, equivalently,* $e+f \equiv 1 \pmod 8$.

THEOREM 6. *Let* $q$ *and* $r$ *be primes* $\equiv 1 \pmod 4$ *such that* $\left(\frac{q}{r}\right) = 1$. *Let* $p$ *be a prime satisfying* $\left(\frac{p}{q}\right) = \left(\frac{p}{r}\right) = 1$, *so that* $p$ *is represented by the classes* $C_p$ *and* $C_p^{-1}$ *of discriminant* $qr$ *and that there exist integers* $a$, $b$, $e$ *and* $f$ *such that*

$$4p^{h(q)} = a^2 - qb^2, \quad (a, b) = 1 \text{ or } 2; \tag{1.13}$$

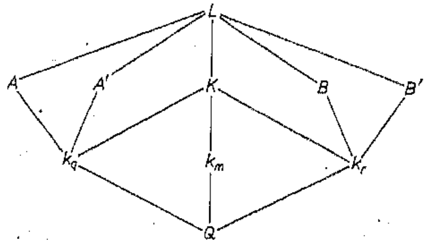
$$4p^{h(r)} = e^2 - rf^2, \quad (e, f) = 1 \text{ or } 2. \tag{1.14}$$

*Then* $C_p$ *is a fourth power if, and only if, for any solutions of* (1.13) *and* (1.14),

$$\left(\frac{(a+b\sqrt{q})/2}{r}\right) = 1 \quad \text{or, equivalently,} \quad \left(\frac{(e+f\sqrt{r})/2}{q}\right) = 1.$$

**2. Proof of the theorems.** The assumption (1.2) implies that the strict class group of $k_m$ contains exactly one subgroup of index 4. Let $L$ be the extension of $k_m$ corresponding to this subgroup by class field theory. Then $L$ is the cyclic extension of degree 4 of $k_m$, unramified at any finite prime.

It is known ([3]) that $L$ is a dihedral extension of $Q$ whose quadratic subfields are $k_m$, $k_q$ and $k_r$ and whose quartic subfields are the field $K$, two fields $A$ and $A'$ containing $k_q$ but neither $k_r$ nor $k_m$, and two fields $B$ and $B'$ containing $k_r$ but neither $k_q$ nor $k_m$.




Let $p$ be a prime on which all the generic characters of $k_m$ take the value $+1$. Then $p$ is completely decomposed in $K$, the genus field of $k_m$, and the classes $C_p$, $C_p^{-1}$ are squares. The classes $C_p$, $C_p^{-1}$ are fourth powers if, and only if $p$ is completely decomposed in $L$, that is if $p$ is completely decomposed in any of the four fields $A$, $A'$, $B$ or $B'$.



Consider for instance the extension $B/k_r$, of conductor $f_B$. There exists a character $\chi_B$ of order 2 on the group of ideals of $k_r$ prime to $f_B$ such that a prime ideal $i$ of $k_r$ is decomposed in $B$ if, and only if, $\chi_B(i) = 1$. The value $\chi_B(i)$ is equal to $\chi_B(i^{h(r)})$, as $h(r)$ is odd, and the value of $\chi_B$ on principal ideals prime to $f_B$ has been calculated in Propositions 2.6 to 2.11 of [4]. Applying this to either of the ideals $\bar{p}_1$, $\bar{p}_2$ such that $(p) = \bar{p}_1 \bar{p}_2$ in $k_r$ we shall obtain the results for those theorems involving the integers $e$ and $f$. The results involving the integers $a$ and $b$ will be obtained by considering the extension $A/k_q$. We give the details of the proof of Theorem 3, the other proofs are similar. In this case the decompositions of $p$, $q$ and $r = 2$ in the fields $k_q$ and $k_r$ are the following:

$$(p) = p_1 p_2, \quad (q) = (\sqrt{-q})^2, \quad (2) = r_1 r_2 \quad \text{in } k_q, \tag{2.1}$$

$$(p) = \bar{p}_1 \bar{p}_2, \quad (q) = \bar{q}_1 \bar{q}_2, \quad 2 = (\sqrt{2})^2 \quad \text{in } k_r. \tag{2.2}$$

We first consider the extension $A/k_q$. By Section 2 of [4] one of $r_1$, $r_2$ is ramified in $A/k_q$ and the other in $A'/k_q$; we choose the notation so that $r_1$ ramifies in $A/k_q$. By Proposition 2.9 of [4] the conductor of $A/k_q$ is $r_1^3$ and the value of the character $\chi_A$ on principal ideals is given by:

$$\chi_A((\lambda)) = \left(\frac{\lambda, 2}{r_1}\right) = \begin{cases} 1, & \text{if } \lambda \equiv \pm 1 \pmod{r_1^3}, \\ -1, & \text{if } \lambda \equiv \pm 3 \pmod{r_1^3}. \end{cases} \tag{2.3}$$

Let $(a, b)$ be a solution of $a^2 + b^2 q = p^{h(-q)}$. As the integers $a+b\sqrt{-q}$ and $a-b\sqrt{-q}$ are coprime we may set

$$(a+b\sqrt{-q}) = p_1^{h(-q)}, \quad (a-b\sqrt{-q}) = p_2^{h(-q)}. \tag{2.4}$$

Now from (2.3) we first see, as $p \equiv \pm 1 \pmod 8$, that:

$$\chi_A(p_1)\chi_A(p_2) = \chi_A((p)) = 1, \tag{2.5}$$

so that from (2.3) and the fact that $h(-q)$ is odd:

$$\chi_A(p_1) = \chi_A(p_2) = \begin{cases} 1, & \text{if } a+b\sqrt{-q} \equiv \pm 1 \pmod{r_1^3}, \\ -1, & \text{if } a+b\sqrt{-q} \equiv \pm 3 \pmod{r_1^3}. \end{cases} \tag{2.6}$$

Let $\beta = 1$ or $3$ be such that $q \equiv -\beta^2 \pmod{16}$. As $(\beta - \sqrt{-q})(\beta + \sqrt{-q}) \equiv 0 \pmod{r_1^4 r_2^4}$ and $(\beta - \sqrt{-q}, \beta + \sqrt{-q}) = 2$ there exists $\varepsilon = \pm 1$ such that $a+b\sqrt{-q} \equiv a + \varepsilon\beta b \pmod{r_1^3}$ and so

$$\chi_A(p_1) = \begin{cases} 1, & \text{if } a+\varepsilon\beta b \equiv \pm 1 \pmod 8, \\ -1, & \text{if } a+\varepsilon\beta b \equiv \pm 3 \pmod 8, \end{cases}$$

that is

$$\chi_A(p_1) = \chi_A(p_2) = \left(\frac{2}{a+\varepsilon\beta b}\right). \tag{2.7}$$

The integer $a$ is odd or divisible by 4 according as $p \equiv 1$ or $-1 \pmod 8$ so that when $q \equiv -9 \pmod{16}$ we have

$$\left(\frac{2}{a+3b}\right) = \left(\frac{-1}{p}\right)\left(\frac{2}{a+b}\right)$$

which together with (2.7) proves

$$\chi_A(p_1) = \chi_A(p_2) = \left(\frac{-1}{p}\right)^{(q+1)/8}\left(\frac{2}{a+b}\right). \tag{2.8}$$

We next consider the extension $B/k_r$. By (2.1) of [4] we can suppose that $q_1$ ramifies in $B$ and $q_2$ in $B'$. Then the character $\chi_B$ is given by

$$\chi_B((\lambda)) = \left[\frac{\lambda}{q_1}\right]_2 \times \operatorname{sgn}\lambda. \tag{2.9}$$

Let $(e, f)$ be any solution of $p = e^2 - 2f^2$ where $e > 0$. Then we may set $p_1 = (e+f\sqrt{2})$, $p_2 = (e-f\sqrt{2})$, and we deduce from (2.9) that

$$\chi_B(p_1) = \chi_B(p_2) = \left[\frac{e+f\sqrt{2}}{q_1}\right]_2 = \left(\frac{e+f\sqrt{2}}{q}\right), \tag{2.10}$$

which together with (2.8) completes the proof of Theorem 3.

Remark. The class $C_p$ of discriminant $m$ is a fourth power or not according as $p^{h(m)/4}$ is represented by the principal class $I$ or by the class $J$ of order 2. Using the well-known representative of $I$ and of $J$, and also the forms of discriminant $4m$ when $m$ is odd, we obtain:

| | $C_p$ fourth power | $C_p$ square, not fourth power |
|---|---|---|
| Theorem I | $p^{h(-r)/4} = X^2+rY^2$ | $2p^{h(-r)/4} = X^2+rY^2$ |
| Theorem II | $p^{h(-2r)/4} = X^2+2rY^2$ | $p^{h(-2r)/4} = 2X^2+rY^2$ |
| Theorem III | $p^{h(-2q)/4} = X^2+2qY^2$ | $p^{h(-2q)/4} = 2X^2+qY^2$ |
| Theorem IV | $p^{h(-qr)/4} = X^2+XY+\frac{qr+1}{4}Y^2$ | $p^{h(-qr)/4} = qX^2+qXY+\frac{q+r}{4}Y^2$ |
| | $4p^{h(-qr)/4} = X^2+qrY^2$ | $4p^{h(-qr)/4} = qX^2+rY^2$ |
| Theorem V | $p^{h(2r)/4} = X^2-2rY^2$ | $gp^{h(2r)/4} = X^2-2rY^2$ |
| Theorem VI | $p^{h(qr)/4} = X^2+XY+\frac{1-qr}{4}Y^2$ | $gp^{h(qr)/4} = X^2+XY+\frac{1-qr}{4}Y^2$ |
| | $4p^{h(qr)/4} = X^2-qrY^2$ | $4gp^{h(qr)/4} = X^2-qrY^2$ |



In the cases (V), (VI) when $m > 0$ the integer $g = -1$, $q$ or $r$ is such that the solvable non pellian equation is $X^2 - qrY^2 = g$.

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**References**

[1] Ezra Brown and Joseph B. Muskat, *Simultaneous representation of primes by binary quadratic forms*, Preprint.

[2] Pierre Kaplan, *Divisibilité par 8 du nombre des classes des corps quadratiques dont le 2-groupe des classes est cyclique, et réciprocité biquadratique*, J. Math. Soc. Japan 25 (1973), pp. 596–608.

[3] L. Rédei und H. Reichardt, *Die Anzahl der durch 4 teilbaren Invarianten der Klassengruppe eines beliebigen quadratischen Zahlkörpers*, J. Reine Angew. Math. 170 (1933), pp. 69–74.

[4] Yoshihiko Yamamoto, *Divisibility by 16 of class number of quadratic fields whose 2-class groups are cyclic*, Osaka Journal of Math. 21 (1984), pp. 1–22.

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