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On the discrepancy of $(n\alpha)$

by

JOHANNES SCHOIBENGEIER (Wien)

1. Notations. Let N be a natural number and for $0 < i \leqslant N$ let $0 \leqslant x_i < 1$. Then the *discrepancy* $D_N^*(x_1, \ldots, x_N)$ of $x_1, \ldots, x_N$ is given by the formula

$$D_N^*(x_1, \ldots, x_N) = \sup_{0 \leqslant x < 1} \left| \frac{1}{N} \sum_{n=1}^{N} c_{[0,x)}(x_n) - x \right|,$$

where c_A denotes the characteristic function of the set A.

PROPOSITION 1. *Let $\sigma \in \mathfrak{S}_N$ be any permutation of the set $\{1, \ldots, N\}$ for which $x_{\sigma(i)} \leqslant x_{\sigma(i+1)}$ for $1 \leqslant i < N$. Then*

$$ND_N^*(x_1, \ldots, x_N) = \tfrac{1}{2} + \max_{1 \leqslant i \leqslant N} |Nx_i - \sigma^{-1}(i) + \tfrac{1}{2}|.$$

For a proof see [5] (Theorem 1.4, p. 91).

Now let $\alpha \notin \mathbf{Q}$. Let us define $D_N^*(\alpha) = D_N^*(\{\alpha\}, \ldots, \{N\alpha\})$, where $\{x\}$ is the fractional part of the real number x. Then $\alpha \mapsto D_N^*(\alpha)$ is an even, periodic function (with period 1); so we may assume that $\alpha \in (0, \frac{1}{2})$.

We have to solve two problems: first of all we have to find a formula for σ^{-1}, where σ is the (uniquely determined) permutation of $\{1, \ldots, N\}$ for which $\{\alpha\sigma(i)\} < \{\alpha\sigma(i+1)\}$ $(1 \leqslant i < N)$. Then we have to find a k, $1 \leqslant k \leqslant N$, for which the maximum $\max\limits_{1 \leqslant i \leqslant N} |N\{\alpha i\} - \sigma^{-1}(i) + \frac{1}{2}|$ is attained.

Let $\alpha = [0; a_1, \ldots]$ be the continued fraction expansion of α with partial quotients a_i and convergents $p_n/q_n = r_n$. Note that $a_1 > 1$.

One can find, for N and α given, uniquely determined integers m and b_i, $0 \leqslant i \leqslant m$, with the following properties:

(1) $N = \sum\limits_{i=0}^{m} b_i q_i$.

(2) $b_m > 0$ and $0 \leqslant b_i \leqslant a_{i+1}$ for $0 \leqslant i \leqslant m$.

(3) If $0 < i \leqslant m$ and $b_i = a_{i+1}$, then $b_{i-1} = 0$. Furthermore $b_0 < a_1$.

One can find m by choosing it such that $q_m \leqslant N < q_{m+1}$, while the digits b_i are given by the following algorithm:

$$N = b_m q_m + N_{m-1}; \qquad 0 \leqslant N_{m-1} < q_m,$$
$$N_{m-1} = b_{m-1} q_{m-1} + N_{m-2}; \qquad 0 \leqslant N_{m-2} < q_{m-1},$$
$$\dots\dots\dots\dots\dots\dots\dots$$
$$N_0 = b_0 q_0$$

(see [7], p. 148). Note that $N_j = \sum_{i=0}^{j} b_i q_i$ for $0 \leqslant j \leqslant m$. We define $N_{-1} = 0$ and $N_m = N$.

Let $a, k, l \in \boldsymbol{Z}$, $k, l \geqslant 0$. In the following we are mainly concerned with the following sum:

$$S(a, k, l) = \sum_{a < n \leqslant a+k} c_{[0, \{l\alpha\})}(\{n\alpha\}).$$

2. $S(a, k, l)$ for special values of a, k and l.

Lemma 1. *Let $m \geqslant 0$, j, k integers such that* $\max(|j|, |k|, |j-k|) < q_{m+1}$. *Then*

(1) $[j\alpha] = [jr_{m+1}]$,

(2) $\{j\alpha\} < \{k\alpha\} \Leftrightarrow \{jr_{m+1}\} < \{kr_{m+1}\}$.

Proof. (1) First of all let $0 \leqslant j < q_{m+1}$. If m is even we get $r_{m+2} j - r_{m+1} j = -j/(q_{m+1} q_{m+2}) > -1/q_{m+2}$ and therefore $[r_{m+2} j] = [r_{m+1} j]$ (note that $\{r_{m+1} j\} \geqslant 1/q_{m+1} > 1/q_{m+2}$). Because of $r_{m+2} < \alpha < r_{m+1}$ the result follows. If m is odd the argument is similar.

Now let $-q_{m+1} < j \leqslant -1$. Then the result follows from the formula $[-x] = -[x]-1$ if $x \notin \boldsymbol{Z}$ and from $jr_{m+1} \notin \boldsymbol{Z}$, $j\alpha \notin \boldsymbol{Z}$.

(2) Let $\{j\alpha\} < \{k\alpha\}$. (1) implies $\{j\alpha\} - \{jr_{m+1}\} = j(\alpha - r_{m+1})$ and $\{k\alpha\} - \{kr_{m+1}\} = k(\alpha - r_{m+1})$. Therefore

$$\{jr_{m+1}\} - \{kr_{m+1}\}$$
$$= \{j\alpha\} - j(\alpha - r_{m+1}) + k(\alpha - r_{m+1}) - \{k\alpha\} < (k-j)(\alpha - r_{m+1}) < 1/q_{m+2}.$$

Because of $\{jr_{m+1}\} - \{kr_{m+1}\} \in (1/q_{m+1})\boldsymbol{Z}$ we get $\{jr_{m+1}\} - \{kr_{m+1}\} \leqslant 0$, where equality is impossible (note that $q_{m+1} \nmid j-k$). The proof of the converse is similar (and even simpler).

Proposition 1. *Let b, k, a and j be integers, $0 \leqslant j$, $1 \leqslant b \leqslant a_{j+1}$ and $b < a_1$ if $j = 0$. Let $0 \leqslant k < q_{j+1}$ and $bq_j - q_{j+1} \leqslant a < q_{j+1} - k$. Then*

(1) $a+k < 0$ *implies* $S(a, k, bq_j) = \frac{1}{2}(1-(-1)^j)k$.

(2) $a+1 \leqslant 0 \leqslant a+k$ *implies*

$$S(a, k, bq_j) = \tfrac{1}{2}(1-(-1)^j)k + (-1)^j \min\left(b, \left[\frac{a+k}{q_j}\right]\right) + \tfrac{1}{2}(1+(-1)^j) \sum_{a+k < bq_j} 1.$$

(3) $0 < a+1$ *implies*

$$S(a, k, bq_j) = \tfrac{1}{2}(1-(-1)^j)k + (-1)^j\left(\min\left(b, \left[\frac{a+k}{q_j}\right]\right) - \min\left(b, \left[\frac{a}{q_j}\right]\right)\right) - \tfrac{1}{2}(1+(-1)^j) \sum_{bq_j - k \leqslant a < bq_j} 1.$$

Remark. The sums in (2) and (3) must be understood to be 1 if the condition under the $\sum$ is fulfilled, and to be 0 else.

Proof. Note that $bq_j < q_{j+1}$. If $a+1 \leqslant n \leqslant a+k$ then $-q_{j+1} < a+1 \leqslant n < q_{j+1}$ and $-q_{j+1} < a+1-bq_j \leqslant n-bq_j < q_{j+1}$. Therefore Lemma 1 (2) implies

$$S(a, k, bq_j) = \sum_{a < n \leqslant a+k} c_{[0, \{bq_j r_{j+1}\})}(\{nr_{j+1}\}).$$

Let us prove the proposition if j is even; the argument is similar if j is odd. Then $\{bq_j r_{j+1}\} = b/q_{j+1}$. We have to find the number of n's with $a < n \leqslant a+k$ and $np_{j+1} \equiv 0, 1, \dots, b-1 \pmod{q_{j+1}}$, that is $n \equiv 0, q_j, \dots$ $\dots, (b-1)q_j \pmod{q_{j+1}}$.

(1) $a+k < 0$: then $n + q_{j+1} = sq_j$ for some s, $0 \leqslant s < b$. This implies $q_j | n + q_{j-1}$. Therefore for some $u \in \boldsymbol{Z}$ $n + q_{j-1} = uq_j$. We get $a+1 \leqslant uq_j - q_{j-1}$, that is $(a+q_{j-1})/q_j < u$ and $a_{j+1} + u = s$, that is $u < b - a_{j+1}$. Therefore $a + q_{j-1} < bq_j - a_{j+1} q_j$, a contradiction.

(2) $a+1 \leqslant 0 \leqslant a+k$. (1) implies

$$S(a, k, bq_j) = \sum_{n=0}^{a+k} c_{[0, b/q_{j+1})}(\{nr_{j+1}\}).$$

The number of n's with $0 \leqslant n \leqslant a+k$ and $0 \leqslant n \leqslant (b-1)q_j$, $q_j | n$, equals $1 + \min(b-1, [(a+k)/q_j])$.

(3) $0 < a+1$. Here we use that

$$S(a, k, bq_j) = \sum_{n=0}^{a+k} c_{[0, bq_{j+1})}(\{nr_{j+1}\}) - \sum_{n=0}^{a} c_{[0, b/q_{j+1})}(\{nr_{j+1}\})$$

and (2) to get the result.

Theorem 1. *Let $a \in \boldsymbol{Z}$, $k, l \geqslant 0$. Then*

$$S(a, k, l) = S(-a-1, l, k) + k\{l\alpha\} - l\{k\alpha\} + \sum_{a < 0 \leqslant a+k < l} 1 - \sum_{0 \leqslant a < l \leqslant a+k} 1.$$

Proof. We use the well known fact that for $0 \leqslant x < 1$ and $y \in \boldsymbol{R}$

$$c_{[0,x)}(\{y\}) = x + \sum_{p \neq 0} \frac{1}{2\pi i p}(1 - e^{-2\pi i p x}) e^{2\pi i p y} + \tfrac{1}{2} c_{\boldsymbol{Z}}(y) - \tfrac{1}{2} c_{\boldsymbol{Z}+x}(y).$$

Therefore

$$S(a,k,l) = k\{l\alpha\} + \sum_{p\neq 0}\frac{1}{2\pi i p}(1-e^{-2\pi i p\alpha l})\sum_{n=a+1}^{a+k} e^{2\pi i p\alpha n} +$$

$$+\tfrac{1}{2}\sum_{n=a+1}^{a+k}\big(c_Z(n\alpha)-c_Z((n-l)\alpha)\big)$$

$$= k\{l\alpha\} + \sum_{p\neq 0}\frac{1}{2\pi i p}(e^{2\pi i k p\alpha}-1)\,e^{2\pi i p\alpha(a+1-l)}\frac{e^{2\pi i p l\alpha}-1}{e^{2\pi i p\alpha}-1}+$$

$$+\tfrac{1}{2}\sum_{a+1\leqslant 0\leqslant a+k} 1-\tfrac{1}{2}\sum_{a<l\leqslant a+k} 1$$

$$= k\{l\alpha\} + \sum_{n=0}^{l-1}\sum_{p\neq 0}\frac{1}{2\pi i p}(e^{2\pi i p k\alpha}-1)\,e^{2\pi i p(a-n)\alpha}+$$

$$+\tfrac{1}{2}\sum_{a<0\leqslant a+k} 1-\tfrac{1}{2}\sum_{a<l\leqslant a+k} 1.$$

Using the above Fourier-expansion again we get the result.

PROPOSITION 2. *For* $1\leqslant k\leqslant N$

$$\sigma^{-1}(k) = S(0,N,k)+1.$$

Proof. This is almost trivial: we have

$$\{1,\ldots,k-1\} = \sigma^{-1}(\{j \mid 1\leqslant j\leqslant N,\ \{\alpha j\}<\{\alpha\sigma(k)\}\}).$$

Therefore

$$k-1 = \operatorname{card}\{j \mid 1\leqslant j\leqslant N,\ \{\alpha j\}<\{\alpha\sigma(k)\}\}.$$

Now to compute $S(0,N,k)$ we use Theorem 1 like a reciprocity law and Proposition 1 like complementary laws.

3. The formula for σ^{-1}.

Notation. Let $1\leqslant k\leqslant N$. We know (see § 1) that there are uniquely determined integers c_j $(0\leqslant j\leqslant m)$ with the following properties:

(1) $k=\sum_{j=0}^{m} c_j q_j$.

(2) For $0\leqslant j\leqslant m$, $0\leqslant c_j\leqslant a_{j+1}$.

(3) For $0<j\leqslant m$, $c_j=a_{j+1}$ implies $c_{j-1}=0$. Furthermore $c_0<a_1$.

Let i_k be the first integer i such that $c_i\neq 0$. Let $k_j=\sum_{i=0}^{j} c_i q_i$ for $-1\leqslant j\leqslant m+1$ (where $c_{m+1}=0$).

Later we shall make extensive use of the following abbreviations:

$\mathfrak{A}_k=\{j \mid 0\leqslant j\leqslant m,\ j \text{ even},\ N_{j-1}<k_{j-1},\ k_j\leqslant N_j,\ k_{j+1}\leqslant N_{j+1}\}$,

$\mathfrak{B}_k=\{j \mid 0\leqslant j\leqslant m,\ j \text{ even},\ k_{j-1}\leqslant N_{j-1},\ k_j\leqslant N_j,\ N_{j+1}<k_{j+1}\}$,

$\mathfrak{C}_k=\{j \mid 0\leqslant j\leqslant m,\ j \text{ even},\ k_{j-1}\leqslant N_{j-1},\ k_j\leqslant N_j,\ k_{j+1}\leqslant N_{j+1}\}$,

$\mathfrak{D}_k=\{j \mid 0\leqslant j\leqslant m,\ j \text{ even},\ N_{j-1}<k_{j-1},\ k_j\leqslant N_j,\ N_{j+1}<k_{j+1}\}$,

$\mathfrak{E}_k^{(1)}=\{j \mid 0\leqslant j\leqslant m,\ j \text{ even},\ N_{j-1}<k_{j-1},\ N_j<k_j\}$,

$\mathfrak{E}_k^{(2)}=\{j \mid 0\leqslant j\leqslant m,\ j \text{ even},\ k_{j-1}\leqslant N_{j-1},\ N_j<k_j\}$,

$\mathfrak{F}_k^{(1)}=\{j \mid 0\leqslant j\leqslant m,\ j \text{ odd},\ N_{j-1}<k_{j-1},\ N_j<k_j\}$,

$\mathfrak{F}_k^{(2)}=\{j \mid 0\leqslant j\leqslant m,\ j \text{ odd},\ k_{j-1}\leqslant N_{j-1},\ N_j<k_j\}$,

$\mathfrak{G}_k=\{j \mid -1\leqslant j\leqslant m,\ j \text{ odd},\ k_j\leqslant N_j\}$.

Let $\mathfrak{E}_k=\mathfrak{E}_k^{(1)}\cup\mathfrak{E}_k^{(2)}$ and $\mathfrak{F}_k=\mathfrak{F}_k^{(1)}\cup\mathfrak{F}_k^{(2)}$. Obviously $\{\mathfrak{A}_k,\mathfrak{B}_k,\mathfrak{C}_k,\mathfrak{D}_k,\mathfrak{E}_k,\mathfrak{F}_k,\mathfrak{G}_k\}$ is a partition of $\{-1,0,1,\ldots,m\}$, where some of the sets may be empty.

LEMMA 1. *We have*

$$\sum_{j=0}^{m} S(-N_{j-1}-1,k_j,b_jq_j)$$

$$=\sum_{j=0}^{m}\big(\tfrac{1}{2}(1-(-1)^j)k_j\operatorname{sgn}b_j+(-1)^j\min(b_j,c_j)\big)+|\mathfrak{A}_k|+|\mathfrak{D}_k|+$$

$$+\sum_{\substack{j=0\\ k_{j-1}\leqslant N_{j-1}<k_j\leqslant N_j}}^{m}\tfrac{1}{2}(1-(-1)^j).$$

Proof. Note that $0\leqslant k_j<q_{j+1}$ and $b_jq_j-q_{j+1}\leqslant -N_{j-1}-1<q_{j+1}-k_j$ for $0\leqslant j\leqslant m$. Together with Proposition 1 (1) and (2) of § 2 we get:

$$\sum_{j=0}^{m} S(-N_{j-1}-1,k_j,b_jq_j)=\sum_{\substack{j=0\\ k_j\leqslant N_{j-1}}}^{m}\tfrac{1}{2}(1-(-1)^j)k_j\operatorname{sgn}b_j+$$

$$+\sum_{\substack{j=0\\ N_{j-1}<k_j}}^{m}\Big(\tfrac{1}{2}(1-(-1)^j)k_j\operatorname{sgn}b_j+(-1)^j\min\Big(b_j,\Big[\frac{k_j-N_{j-1}-1}{q_j}\Big]\Big)+$$

$$+\frac{1}{2}(1+(-1)^j)\sum_{k_j\leqslant N_j}1\Big).$$

Now

$$\Big[\frac{k_j-N_{j-1}-1}{q_j}\Big]=c_j-\sum_{k_{j-1}\leqslant N_{j-1}}1$$

and therefore

$$\min\Big(b_j,\Big[\frac{k_j-N_{j-1}-1}{q_j}\Big]\Big)=\min(b_j,c_j)-\sum_{k_{j-1}\leqslant N_{j-1},\,c_j\leqslant b_j}1.$$

Now we use that obviously "$k_{j-1}\leqslant N_{j-1}<k_j$ and $c_j\leqslant b_j$" is equivalent to

"$k_{j-1} \leqslant N_{j-1} < k_j \leqslant N_j$" and that $k_j \leqslant N_{j-1}$ implies $c_j = 0 = \min(b_j, c_j)$. In the last step use that

$$\sum_{\substack{j=0 \\ N_{j-1}<k_j\leqslant N_j}}^{m} \tfrac{1}{2}(1+(-1)^j) - \sum_{\substack{j=0 \\ k_{j-1}\leqslant N_{j-1}<k_j\leqslant N_j}}^{m} (-1)^j$$

$$= \sum_{\substack{j=0 \\ N_{j-1}<k_{j-1}\leqslant k_j\leqslant N_j}}^{m} \tfrac{1}{2}(1+(-1)^j) + \sum_{\substack{j=0 \\ k_{j-1}\leqslant N_{j-1}<k_j\leqslant N_j}}^{m} \tfrac{1}{2}(1-(-1)^j).$$

LEMMA 2.

$$\sum_{j=0}^{m} S(-N_{j-1}+k_j-1,\, k-k_j,\, b_j q_j)$$

$$= \sum_{j=0}^{m} \Big((k-k_j)\{b_j q_j \alpha\} - b_j q_j \sum_{t=j+1}^{m} c_t(q_t\alpha - p_t)\Big) - |\mathfrak{B}_k| - |\mathfrak{D}_k| +$$

$$+\tfrac{1}{2}(1-(-1)^{i_k}) - \sum_{\substack{j=0 \\ k_{j-1}\leqslant N_{j-1}<k_j\leqslant N_j}}^{m} \tfrac{1}{2}(1-(-1)^j).$$

Proof. By Theorem 1 of § 2

$$S(-N_{j-1}+k_j-1,\, k-k_j,\, b_j q_j)$$

$$= \sum_{t=j+1}^{m} S(k_{t-1}-N_{j-1}-1,\, c_t q_t,\, b_j q_j) = \sum_{t=j+1}^{m} (c_t q_t \{b_j q_j \alpha\} - b_j q_j \{c_t q_t \alpha\}) +$$

$$+ \sum_{\substack{t=j+1 \\ k_{t-1}\leqslant N_{j-1}<k_t\leqslant N_j}}^{m} 1 - \sum_{\substack{t=j+1 \\ N_{j-1}<k_{t-1}\leqslant N_j<k_t}}^{m} 1 + \sum_{t=j+1}^{m} S(N_{j-1}-k_{t-1},\, b_j q_j,\, c_t q_t).$$

Proposition 1 in § 2 tells us that

$$\sum_{t=j+1}^{m} S(N_{j-1}-k_{t-1},\, b_j q_j,\, c_t q_t) = \sum_{t=j+1}^{m} \tfrac{1}{2}(1-(-1)^t) b_j q_j \operatorname{sgn} c_t +$$

$$+ \sum_{\substack{t=j+1 \\ N_{j-1}<k_{t-1}\leqslant N_j<k_t}}^{m} \tfrac{1}{2}(1+(-1)^t) - \sum_{\substack{t=j+1 \\ k_{t-1}\leqslant N_{j-1}<k_t\leqslant N_j}}^{m} \tfrac{1}{2}(1+(-1)^t) -$$

$$- \sum_{\substack{t=j+1 \\ k_{t-1}\leqslant N_{j-1}}}^{m} (-1)^t \min\Big(c_t, \Big[\frac{N_{j-1}-k_{t-1}}{q_t}\Big]\Big) + \sum_{\substack{t=j+1 \\ k_{t-1}\leqslant N_j}}^{m} (-1)^t \min\Big(c_t, \Big[\frac{N_j-k_{t-1}}{q_t}\Big]\Big).$$

Now $t > j$ and $k_{t-1} \leqslant N_j$ implies $0 \leqslant N_j - k_{t-1} < q_t$; therefore the two last sums vanish. Furthermore $t > j$ and $k_{t-1} \leqslant N_{j-1} < k_t \leqslant N_j$ implies $k_t < q_{j+1}$

and therefore $c_t = 0$. This contradiction proves that the third sum is empty and that $\sum_{\substack{t=j+1 \\ k_{t-1}\leqslant N_{j-1}<k_t\leqslant N_j}}^{m} 1 = 0$. We get

$$\sum_{j=0}^{m} S(-N_{j-1}+k_j-1,\, k-k_j,\, b_j q_j)$$

$$= \sum_{j=0}^{m} \sum_{t=j+1}^{m} (c_t q_t \{b_j q_j \alpha\} - b_j q_j \{c_t q_t \alpha\}) + \sum_{j=0}^{m} \sum_{t=j+1}^{m} \tfrac{1}{2}(1-(-1)^t) b_j q_j \operatorname{sgn} c_t +$$

$$+ \sum_{j=0}^{m} \sum_{\substack{t=j+1 \\ N_{j-1}<k_{t-1}\leqslant N_j<k_t}}^{m} \tfrac{1}{2}((-1)^t - 1).$$

Now use that $\big(\tfrac{1}{2}(1-(-1)^t) - \{c_t q_t \alpha\}\big) \operatorname{sgn} c_t = -c_t(q_t\alpha - p_t)$ and that the last sum is equal $\sum_{t=0}^{m} \sum_{\substack{j=0 \\ N_{j-1}<k_{t-1}\leqslant N_j<k_t}}^{t-1} \tfrac{1}{2}((-1)^t - 1)$. The existence of a j with $0 \leqslant j < t$ and $N_{j-1} < k_{t-1} \leqslant N_j < k_t$ is equivalent to $0 < k_{t-1} \leqslant N_{t-1} < k_t$. Therefore this sum is equal to

$$\sum_{\substack{j=0 \\ 0<k_{j-1}\leqslant N_{j-1}<k_j}}^{m} \tfrac{1}{2}((-1)^j - 1)$$

$$= \sum_{\substack{j=0 \\ k_{j-1}\leqslant N_{j-1}<k_j}}^{m} \tfrac{1}{2}((-1)^j - 1) - \sum_{\substack{j=0 \\ k_{j-1}=0<k_j}}^{m} \tfrac{1}{2}((-1)^j - 1)$$

$$= \sum_{\substack{j=0 \\ k_{j-1}\leqslant N_{j-1}<k_j\leqslant N_j}}^{m} \tfrac{1}{2}((-1)^j - 1) + \sum_{\substack{j=0 \\ k_{j-1}\leqslant N_{j-1}\leqslant N_j<k_j}}^{m} \tfrac{1}{2}((-1)^j - 1) + \tfrac{1}{2}(1-(-1)^{i_k}).$$

The second sum is equal to

$$-\sum_{\substack{j=0 \\ k_j\leqslant N_j\leqslant N_{j+1}<k_{j+1},\, 2|j}}^{m-1} 1 = -|\mathfrak{B}_k| - |\mathfrak{D}_k|.$$

Notation. For $0 \leqslant j \leqslant m$ let $A_j = \sum_{t=j}^{m} b_t(q_t\alpha - p_t) + N_{j-1}(\alpha - r_j)$. We note that the numbers A_j depend on N and α only. If $0 \leqslant j < m$ we get

$$A_j = A_{j+1} + (-1)^j \frac{N_j}{q_j q_{j+1}}.$$

THEOREM 1. *Let* $1 \leqslant k \leqslant N$. *Then*

$$\sigma^{-1}(k) = N\{k\alpha\} + \sum_{j=0}^{m} \big((-1)^j \min(b_j, c_j) - c_j q_j A_j\big) + |\mathfrak{A}_k| - |\mathfrak{B}_k| + \tfrac{1}{2}(1-(-1)^{i_k}).$$

Proof. Proposition 2 and Theorem 1 of § 2 and Lemma 1 and Lemma 2 tells us that

$$\sigma^{-1}(k) = 1+\sum_{j=0}^{m} S(N_{j-1},\, b_j q_j,\, k)$$

$$= 1+\sum_{j=0}^{m}\big(S(-N_{j-1}-1,\, k,\, b_j q_j)+b_j q_j\{k\alpha\}-k\{b_j q_j \alpha\}\big)-\sum_{\substack{j=0\\ N_{j-1}<k\leqslant N_j}}^{m} 1$$

$$= N\{k\alpha\}-k\sum_{j=0}^{m}\{b_j q_j\alpha\}+\sum_{j=0}^{m}\big(S(-N_{j-1}-1,\, k_j,\, b_j q_j)+$$
$$+S(-N_{j-1}-1+k_j,\, k-k_j,\, b_j q_j)\big)$$

$$= N\{k\alpha\}+\sum_{j=0}^{m}\Big(\tfrac{1}{2}(1-(-1)^j)k_j \operatorname{sgn} b_j+(-1)^j \min(b_j, c_j)-k_j\{b_j q_j\alpha\}-$$
$$-b_j q_j \sum_{t=j+1}^{m} c_t(q_t\alpha-p_t)\Big)+\tfrac{1}{2}(1-(-1)^{i_k})+|\mathfrak{A}_k|-|\mathfrak{B}_k|.$$

Now we use that $\frac{1}{2}(1-(-1)^j)\operatorname{sgn} b_j-\{b_j q_j\alpha\} = -b_j(q_j\alpha-p_j)$ and that

$$\sum_{j=0}^{m}\Big(k_j b_j(q_j\alpha-p_j)+b_j q_j\sum_{t=j+1}^{m} c_t(q_t\alpha-p_t)\Big)$$
$$= \sum_{t=0}^{m} c_t q_t \sum_{j=t}^{m} b_j(q_j\alpha-p_j)+\sum_{t=1}^{m} c_t(q_t\alpha-p_t)N_{t-1}.$$

Later the following proposition will be useful:

Proposition 1. (1) *For* $0\leqslant j\leqslant m$ *we have*

$$-|q_j\alpha-p_j| < (-1)^j\sum_{t=j}^{m} b_t(q_t\alpha-p_t) < |q_{j-1}\alpha-p_{j-1}|.$$

(2) *If* $0\leqslant j\leqslant m$ *and* $b_j\neq 0$, *then* $\operatorname{sgn}\sum_{t=j}^{m} b_t(q_t\alpha-p_t) = (-1)^j$.

(3) *If* $0\leqslant j\leqslant m$ *then* $-1/q_{j+1} < (-1)^j A_j < 1/q_j$.

Proof. We assume that j is even. It is similar if j is odd.

(1).

$$-(q_j\alpha-p_j) = \sum_{i=1}^{\infty}\big((q_{j+2i}-q_{j+2i-2})\alpha-(p_{j+2i}-p_{j+2i-2})\big)$$

$$= \sum_{i=1}^{\infty} a_{j+2i}(q_{j+2i-1}\alpha-p_{j+2i-1})$$

$$< \sum_{1\leqslant i\leqslant (m+1)/2} b_{j+2i-1}(q_{j+2i-1}\alpha-p_{j+2i-1})$$

$$\leqslant \sum_{t=j}^{m} b_t(q_t\alpha-p_t) < \sum_{i=0}^{\infty} a_{j+2i+1}(q_{j+2i}\alpha-p_{j+2i})$$

$$= \sum_{i=0}^{\infty}\big((q_{j+2i+1}-q_{j+2i-1})\alpha-(p_{j+2i+1}-p_{j+2i-1})\big)$$

$$= -(q_{j-1}\alpha-p_{j-1}).$$

(2). (1) implies

$$\sum_{t=j}^{m} b_t(q_t\alpha-p_t) > -(q_j\alpha-p_j)+b_j(q_j\alpha-p_j) > 0.$$

(3). (1) implies

$$A_j < b_j(q_j\alpha-p_j)+p_{j+1}-q_{j+1}\alpha+N_{j-1}(\alpha-r_j).$$

Assume first that $b_j < a_{j+1}$. Then

$$A_j < (a_{j+1}-1)(q_j\alpha-p_j)+p_{j+1}-q_{j+1}\alpha+(q_j\alpha-p_j) = -(q_{j-1}\alpha-p_{j-1}) < 1/q_j.$$

If $b_j = a_{j+1}$ then $b_{j-1} = 0$ and therefore $N_{j-1} < q_{j-1}$. We get

$$A_j < (q_{j+1}-q_{j-1})\alpha-(p_{j+1}-p_{j-1})+p_{j+1}-q_{j+1}\alpha+q_{j-1}(\alpha-r_j) = 1/q_j.$$

On the other hand

$$A_j > \sum_{t=j}^{m} b_t(q_t\alpha-p_t) > -(q_j\alpha-p_j) > -1/q_{j+1}.$$

Remarks. (1) Let $n = \sum_{j=0}^{m} n_j q_j$, where $n_0 < a_1$, $n_j\leqslant a_{j+1}$, and where $n_j = a_{j+1}$ implies $n_{j-1} = 0$ for $1\leqslant j\leqslant m$. Then

$$\{n\alpha\} = \tfrac{1}{2}(1-(-1)^{i_n})+\sum_{j=0}^{m} n_j(q_j\alpha-p_j).$$

For a proof note that equality is valid mod 1. Now use Proposition 1 to show that the right hand side is in $[0, 1)$.

(2) Let $n = \sum_{j=0}^{m} n_j q_j$, $k = \sum_{j=0}^{m} c_j q_j$ like above, $n\neq k$. Let t be the first integer $\geqslant 0$ with $n_j\neq c_j$. Then

$$\{n\alpha\} < \{k\alpha\} \Leftrightarrow (2\,|\,i_n \text{ and } 2\nmid i_k) \text{ or } (i_n\equiv i_k \ (\mathrm{mod}\ 2) \text{ and } (c_t-n_t)(-1)^t > 0).$$

This follows from (1) and an argument like in Proposition 1.

4. Results about $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}$ and i_k. By Proposition 1 of § 1 we have to determine integers k and k', $1\leqslant k, k'\leqslant N$, such that $\max_{1\leqslant n\leqslant N}(\sigma^{-1}(n)-$

$-N\{n\alpha\})$ is attained at k and $\min_{1\leqslant n\leqslant N}(\sigma^{-1}(n)-N\{n\alpha\})$ is attained at k'. Fortunately it is enough to determine one of these values:

Proposition 1. *Let* $1\leqslant k\leqslant N$. *Then*

$$\sigma^{-1}(k)-N\{k\alpha\}+\sigma^{-1}(N-k+1)-N\{(N-k+1)\alpha\} = N+1-2\sum_{n=1}^{N}\{n\alpha\}.$$

Proof. The following formula is valid for $x, y\in\boldsymbol{R}$:

$$c_{[0,\{x-y\})}(\{x\}) = \{x-y\}-\{x\}+\{y\}.$$

Taking $x = n\alpha$ and $y = (n-k)\alpha$ we get

$$c_{[0,\{k\alpha\})}(\{n\alpha\}) = \{k\alpha\}-\{n\alpha\}+\{(n-k)\alpha\}$$

and therefore by Proposition 2 of § 2:

$$\sigma^{-1}(k)-N\{k\alpha\}+\sigma^{-1}(N-k+1)-N\{(N-k+1)\alpha\}$$
$$= 2-2\sum_{n=1}^{N}\{n\alpha\}+\sum_{n=1}^{N}\{(n-k)\alpha\}+\sum_{n=1}^{N}\{(n-N+k-1)\alpha\}.$$

The last sum is equal to

$$\sum_{n=-N+k}^{k-1}\{n\alpha\} = \sum_{\substack{n=-N+k\\ n\neq 0}}^{k-1}(1-\{-n\alpha\})$$
$$= N-1-\sum_{n=1-k}^{N-k}\{n\alpha\} = N-1-\sum_{n=1}^{N}\{(n-k)\alpha\}.$$

Proposition 1 implies that the above k' is equal to $N-k+1$.

Corollary 1. *Let* $\alpha\notin\boldsymbol{Q}$ *and* $N\in N$. *Then*

$$\sum_{n=1}^{N}\{n\alpha\}-\tfrac{1}{2}N = \tfrac{1}{2}\Big(A_0-\sum_{j=0}^{m}b_j\big((-1)^j-q_jA_j\big)\Big).$$

Proof. We take $k = 1$ in Proposition 1. Note that $\mathfrak{A}_1 = \{j|\ j$ even, $0<j\leqslant m,\ N_{j-1} = 0<N_j\}$ and therefore $|\mathfrak{A}_1| = \frac{1}{2}(1+(-1)^{i_N})$ if $i_N>0$, and $|\mathfrak{A}_1| = 0$ if $i_N = 0$. Obviously $\mathfrak{B}_1 = \mathfrak{A}_N = \mathfrak{B}_N = \emptyset$. We get

$$\sigma^{-1}(1)-N\alpha = \min(b_0, 1)-q_0A_0+|\mathfrak{A}_1|$$
$$= \tfrac{1}{2}(1+(-1)^{i_N})\operatorname{sgn} i_N+1-\operatorname{sgn} i_N-q_0A_0 = 1-q_0A_0-\tfrac{1}{2}(1-(-1)^{i_N})$$

and

$$\sigma^{-1}(N)-N\{N\alpha\} = \sum_{j=0}^{m}b_j\big((-1)^j-q_jA_j\big)+\tfrac{1}{2}(1-(-1)^{i_N}).$$

From now on we are mainly interested in the number k, chosen such that $\max_{1\leqslant n\leqslant N}(\sigma^{-1}(n)-N\{n\alpha\})$ is attained at k. In the following, unless otherwise stated, we assume that k is chosen in this way.

Proposition 2. (1) $\{(k-1)\alpha\}<\{N\alpha\}\leqslant\{k\alpha\}$.

(2) *If* i_N *is odd, then* i_k *is odd.*

(3) *If* i_k *is even, then* $i_k = 0$.

Proof. (1) Note that Theorem 1 of § 2 implies

$$\sigma^{-1}(n)-N\{n\alpha\} = \sum_{j=0}^{n-1}c_{[0,\{N\alpha\})}(\{j\alpha\})-n\{N\alpha\}.$$

If $\varepsilon\in\{-1, 1\}$ we get

$$\sigma^{-1}(k)-N\{k\alpha\}>\sigma^{-1}(k+\varepsilon)-N\{(k+\varepsilon)\alpha\}.$$

Therefore

$$c_{[0,\{N\alpha\})}(\{k\alpha\})\leqslant\{N\alpha\}\quad\text{ and }\quad c_{[0,\{N\alpha\})}(\{(k-1)\alpha\})\geqslant\{N\alpha\}.$$

This implies $\{(k-1)\alpha\}<\{N\alpha\}\leqslant\{k\alpha\}$.

(2) follows from Remark (2) of § 3 and from (1).

(3) If $i_k\geqslant 2$ and if i_k is even, we get $i_{k-1} = 1$, contrary to Remark (2) of § 3.

The following two propositions are valid for all k, $1\leqslant k\leqslant N$.

Proposition 3. *Let* $1\leqslant k\leqslant N$. *Then*

(1) $j\in\mathfrak{A}_k\cup\mathfrak{D}_k$ *implies* $c_j<b_j$.

(2) $j\in\mathfrak{B}_k\cup\mathfrak{C}_k$ *implies* $c_j\leqslant b_j$.

(3) $j\in\mathfrak{E}_k^{(1)}\cup\mathfrak{F}_k^{(1)}$ *implies* $c_j\geqslant b_j$.

(4) $j\in\mathfrak{E}_k^{(2)}\cup\mathfrak{F}_k^{(2)}$ *implies* $c_j>b_j$.

(5) $j\in\mathfrak{G}_k$ *implies* $c_j\leqslant b_j$. *If* $j\in\mathfrak{G}_k\cap(\mathfrak{E}_k+1)$, *then* $c_j<b_j$.

The proof is trivial; so we omit it. We shall often use Proposition 3 without mentioning it.

Proposition 4. *Let* $1\leqslant k\leqslant N$. *Then*

(1) $\{\mathfrak{F}_k^{(1)}, \mathfrak{F}_k^{(2)}, \mathfrak{G}_k\}$ *is a partition of* $\{j|\ -1\leqslant j\leqslant m, j$ *odd*$\}$.

(2) *If* $j\in((\mathfrak{F}_k^{(2)}\cup\mathfrak{G}_k)+1)\cap(\mathfrak{F}_k^{(1)}-1)$, *then* $b_j<a_{j+1}$.

(3) *If* $j\in(\mathfrak{F}_k^{(1)}\cap(\mathfrak{G}_k+2))\cup\mathfrak{F}_k^{(2)}$, *then* $b_j<a_{j+1}$.

(4) $-1\in\mathfrak{G}_k$.

(5) $1\in\mathfrak{F}_k^{(1)}$ *implies* $b_0<a_1-1$.

(6) *If* $j\in(\mathfrak{F}_k+1)\cap(\mathfrak{F}_k^{(2)}-1)$, *then* $b_j\neq 0$.

(7) *If* $j\in(\mathfrak{F}_k+2)\cap\mathfrak{G}_k$ *and* $b_{j-1} = 0$, *then* $b_j\neq 0$.

(8) *If* m *is odd, then* $m\in\mathfrak{G}_k$.

Proof. (1) is trivial.

(2) Let $j-1\in\mathfrak{F}_k^{(2)}$, $j+1\in\mathfrak{F}_k^{(1)}$ and $b_j=a_{j+1}$. Then $N_{j-1}<k_{j-1}$, $N_{j+1}<k_{j+1}$ and $N_j<k_j$. Therefore $j\in\mathfrak{E}_k^{(1)}$, $c_j=a_{j+1}$ and $c_{j-1}=0>b_{j-1}$, a contradiction.

Let $j-1\in\mathfrak{G}_k$, $j+1\in\mathfrak{F}_k^{(1)}$ and $b_j=a_{j+1}$. Then $N_j<k_j$, $k_{j-1}\leqslant N_{j-1}$ and $c_{j-1}=b_{j-1}=0$. Therefore $N_{j-1}<k_{j-1}$, a contradiction.

(3) If $j\in\mathfrak{F}_k^{(2)}$ there is nothing to prove (Proposition 3 (3)). Otherwise, if $j\in\mathfrak{F}_k^{(1)}$, $j-2\in\mathfrak{G}_k$, then $N_{j-1}<k_{j-1}$ and therefore $b_j=a_{j+1}$ would imply $c_{j-1}=b_{j-1}=0$. Then $N_{j-2}<k_{j-2}$, a contradiction to $j-2\in\mathfrak{G}_k$.

(4) is trivial.

(5) If $b_0=a_1-1$ and $1\in\mathfrak{F}_k^{(1)}$, then $c_1\geqslant b_1$, $N_0<k_0$, a contradiction to $c_0<a_1$.

(6) If $b_j=0$, $j-1\in\mathfrak{F}_k$, $j+1\in\mathfrak{F}_k^{(2)}$, then $N_{j-1}<k_{j-1}$, $k_j\leqslant N_j$ and therefore $c_j<b_j=0$, a contradiction.

(7) Assume that $j-2\in\mathfrak{F}_k$, $j\in\mathfrak{G}_k$ and $b_{j-1}=b_j=0$. Then $N_{j-2}<k_{j-2}$, $k_j\leqslant N_j$ and $k_{j-1}\leqslant N_{j-1}$. Therefore $c_{j-1}<b_{j-1}=0$, a contradiction.

(8) is trivial.

Notation. From now on we omit the index k if k is chosen such that $\sigma^{-1}(k)-N\{k\alpha\}$ is maximal.

Remark. (1) Note that

$$\sigma^{-1}(k)-N\{k\alpha\}=\sum_{j\in\mathfrak{A}\cup\mathfrak{B}\cup\mathfrak{C}\cup\mathfrak{D}}c_j(1-q_jA_j)+\sum_{j\in\mathfrak{E}}(b_j-c_jq_jA_j)-$$
$$-\sum_{j\in\mathfrak{F}}(b_j+c_jq_jA_j)-\sum_{j\in\mathfrak{G}}c_j(1+q_jA_j)+|\mathfrak{A}|-|\mathfrak{B}|+\tfrac{1}{2}(1-(-1)^{i_k}).$$

From this we deduce immediately that $j\in\mathfrak{G}$ and $j\neq i_k$ implies $c_j=0$. If $i_k\in\mathfrak{G}$, then $c_{i_k}=1$.

Proposition 5. (1) $\mathfrak{A}=\{j\in(\mathfrak{F}+1)\cap(\mathfrak{G}-1)|\ b_j\neq 0\}$.

(2) $\mathfrak{B}=(\mathfrak{G}+1)\cap(\mathfrak{F}^{(2)}-1)$.

(3) $\mathfrak{C}=\{j\in(\mathfrak{G}-1)\cap(\mathfrak{G}+1)|\ j<i_k$ *or* $(j=i_k+1$ *and* $a_{j+1}=1)$ *or* $A_j>0$ *or* $b_{j+1}=0\}$.

(4) $\mathfrak{D}=(\mathfrak{F}+1)\cap(\mathfrak{F}^{(2)}-1)$.

(5) $\mathfrak{E}^{(1)}=((\mathfrak{F}+1)\cap(\mathfrak{F}^{(1)}-1))\cup\{j\in(\mathfrak{G}+1)\cap(\mathfrak{G}-1)|\ b_j=0\}$.

(6) $\mathfrak{E}^{(2)}=((\mathfrak{G}+1)\cap(\mathfrak{F}^{(1)}-1))\cup\{j\in(\mathfrak{G}-1)\cap(\mathfrak{G}+1)|\ j\geqslant i_k$ *and* $(j=i_k+1\Rightarrow a_{j+1}>1)$ *and* $A_j<0$ *and* $b_{j+1}\neq 0\}$.

Proof. (1) If $j\in\mathfrak{A}$, then $j\in(\mathfrak{G}+1)\cap(\mathfrak{G}-1)$ and $b_j\neq 0$, of course. Conversely, let $j\in(\mathfrak{F}+1)\cap(\mathfrak{G}-1)$, $b_j\neq 0$. Then $N_{j-1}<k_{j-1}$, $k_{j+1}<N_{j+1}$ and therefore $j\in\mathfrak{A}\cup\mathfrak{E}^{(1)}$. If $j\in\mathfrak{E}^{(1)}$, let $c_t'=c_t$ for $t\neq j$, $c_j'=b_j-1$, $k'=\sum_{t=0}^{m}c_t'q_t$. It is easily seen that $1\leqslant k'\leqslant N$ and $i_{k'}=i_k$. Note that $\mathfrak{E}^{(1)\prime}=\mathfrak{E}^{(1)}\setminus\{j\}$, $\mathfrak{A}'=\mathfrak{A}\cup\{j\}$ and that the other sets remain unaltered. This implies $0<b_j-c_jq_jA_j-(b_j-1)(1-q_jA_j)-1=(b_j-1-c_j)q_jA_j<0$, a contradiction.

(2) is trivial.

(3) Let $j\in\mathfrak{C}$. Then $j-1\in\mathfrak{G}$, $j+1\in\mathfrak{G}$. Assume that $j\geqslant i_k$, $j=i_k+1\Rightarrow a_{j+1}>1$, and $b_{j+1}\neq 0$. We have to prove that $A_j>0$. Assume the contrary and note that $k_{j-1}\leqslant N_{j-1}$, $k_{j+1}\leqslant N_{j+1}$. Let $c_t'=c_t$ for $t\neq j$, $c_j'=1$. Note that $b_j=c_j=0$. Let $k'=\sum_{t=0}^{m}c_t'q_t$. Because of $c_{j+1}=0$ we get $k'\leqslant N$. Because of $c_j=0$ we get $i_{k'}=i_k$. Note that $\mathfrak{E}^{(2)\prime}=\mathfrak{E}^{(2)}\cup\{j\}$, $\mathfrak{C}'=\mathfrak{C}\setminus\{j\}$ and that the other sets remain unaltered. We get $0<q_jA_j$, a contradiction.

Conversely let $j\in(\mathfrak{G}+1)\cap(\mathfrak{G}-1)$. Then $j\in\mathfrak{C}\cup\mathfrak{E}^{(2)}$. If $j<i_k$ then $j\in\mathfrak{C}$, of course. If $b_{j+1}=0$, then $j\notin\mathfrak{E}$ by Proposition 3 (5). If $j=i_k+1$ and $a_{j+1}=1$, then $c_j>b_j$ would imply $c_j=a_{j+1}$ and $c_{j-1}=0$. Therefore $j\notin\mathfrak{E}^{(2)}$.

Now let $A_j>0$, $j\in\mathfrak{E}^{(2)}$, $b_{j+1}\neq 0$, $j\geqslant i_k$ and $(j=i_k+1\Rightarrow a_{j+1}>1)$. Let $c_t'=c_t$ for $t\neq j$, $c_j'=b_j<c_j$, $k'=\sum_{t=0}^{m}c_t'q_t$. Then $\mathfrak{E}^{(2)\prime}=\mathfrak{E}^{(2)}\setminus\{j\}$, $\mathfrak{C}'=\mathfrak{C}\cup\{j\}$ and the other sets remain unaltered. We get $k'\geqslant 1$ (because otherwise i_N is even, $b_0=0$, $i_N\geqslant 2$ and $b_1=0$, a contradiction). This gives us

$$0<-c_j'(1-q_jA_j)+b_j-c_jq_jA_j=(b_j-c_j)q_jA_j<0.$$

(4) is trivial.

(5) follows easily from (1).

(6) follows easily from (3).

Proposition 6. *Let* $\{\mathfrak{F}^{(1)},\mathfrak{F}^{(2)},\mathfrak{G}\}$ *be a partition of* $\{j|-1\leqslant j\leqslant m, j\ \text{odd}\}$ *such that* $-1\in\mathfrak{G}$. *Define* $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}^{(1)}$ *and* $\mathfrak{E}^{(2)}$ *by the relations of Proposition* 5 *and let* $1\leqslant k\leqslant N$ *be such that the conditions of Proposition* 3 *are fulfilled. Then* $\mathfrak{A}=\mathfrak{A}_k$, $\mathfrak{B}=\mathfrak{B}_k$, $\ldots$, $\mathfrak{G}=\mathfrak{G}_k$.

Proof. It runs in two steps, proved by induction on j.

(1) $j\in\mathfrak{E}^{(1)}\cup\mathfrak{F}^{(1)}\Rightarrow N_{j-1}<k_{j-1}$.

$j=0$: then $0\in\mathfrak{E}^{(1)}$. But $-1\in\mathfrak{G}$ and therefore $0\in\mathfrak{G}+1$; we get $0\notin\mathfrak{E}$. Proposition 5 (5), (6) implies that $j-1\in\mathfrak{E}\cup\mathfrak{F}$. If $j-1\in\mathfrak{E}^{(2)}\cup\mathfrak{F}^{(2)}$, then $c_{j-1}>b_{j-1}$ and we are through. Otherwise $N_{j-2}<k_{j-2}$ by the induction-hypothesis. $c_{j-1}\geqslant b_{j-1}$ gives us $N_{j-1}<k_{j-1}$.

(2) $j\in\mathfrak{B}\cup\mathfrak{C}$ implies $k_{j-1}\leqslant N_{j-1}$.

$j=0$ is a trivial case. Assume that $j\geqslant 2$. Note that $j-1\in\mathfrak{G}$ and we are through if $j-2\in\mathfrak{E}$. If $j-2\in\mathfrak{B}\cup\mathfrak{C}$, then $k_{j-3}\leqslant N_{j-3}$ by the induction-hypothesis, $c_{j-2}\leqslant b_{j-2}$ and $c_{j-1}\leqslant b_{j-1}$. We get $k_{j-1}\leqslant N_{j-1}$. If $j-2\in\mathfrak{A}\cup\mathfrak{D}$, then $c_{j-2}<b_{j-2}$. Therefore $k_{j-2}\leqslant N_{j-2}$. Because of $c_{j-1}\leqslant b_{j-1}$ we get $k_{j-1}\leqslant N_{j-1}$ again. Now it is easily seen that $\mathfrak{A}\subseteq\mathfrak{A}_k,\ldots,\mathfrak{G}\subseteq\mathfrak{G}_k$. This implies equality.

One can prove the following: let $\{\mathfrak{F}^{(1)},\mathfrak{F}^{(2)},\mathfrak{G}\}$ be a partition of $\{j|-1\leqslant j\leqslant m, j\ \text{odd}\}$ such that the conditions of Proposition 4 are fulfilled. Define $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, $\mathfrak{E}^{(1)}$ and $\mathfrak{E}^{(2)}$ by the conditions of Proposition 5. Then there exists a k, $1\leqslant k\leqslant N$, such that the conditions of Proposition 3 are fulfilled. We shall not need this fact.

Remarks. (2) $j \in \mathfrak{A}$ implies $c_j = b_j - 1$.

(3) $j \in \mathfrak{C}$ and $j \geqslant i_k$ implies $c_j = b_j$.

(4) $j \in \mathfrak{E}^{(1)} \cup \mathfrak{F}^{(1)}$ and $c_j > b_j$ implies $A_j < 0$.

(5) $j \in \mathfrak{E}^{(2)} \cup \mathfrak{F}^{(2)}$ and $c_j > b_j + 1$ implies $A_j < 0$.

The proofs are all trivial: they follow from the formula mentioned in Remark (1).

PROPOSITION 7. *Let $A_{j-1} < 0, j \in \mathfrak{F}^{(1)}$ and let t be the first integer $\geqslant j$ such that $b_t \neq 0$. Let $j \leqslant s \leqslant t$, s odd. Then*

(1) $s < t$ implies $s \in \mathfrak{F}^{(1)}$.

(2) $t \in \mathfrak{F}^{(1)} \cup \mathfrak{G}$.

(3) $t \in \mathfrak{G}$ implies $c_s = 0$.

(4) $j < s$ and $t \in \mathfrak{F}^{(1)}$ implies $c_s = a_{s+1}$.

Proof. (1) Of course, t is odd. Let s be the first odd integer, $j \leqslant s < t$, with $s \notin \mathfrak{F}^{(1)}$. If $s \in \mathfrak{G}$, then $s-2 \in \mathfrak{F}^{(1)}$, $b_{s-1} = b_s = 0$, and Proposition 4 (7) gives us a contradiction. If $s \in \mathfrak{F}^{(2)}$, then $s-2 \in \mathfrak{F}$ and $b_{s-1} \neq 0$ by Proposition 4 (6), a contradiction too.

(2) follows from (1) and Proposition 4 (6).

(3) Assume that there is a largest odd s with $j \leqslant s \leqslant t$ such that $c_s > 0$. Then $s < t$, $c_{s+1} < a_{s+2}$, $c_{s+2} = 0$. Let $c'_s = 0$, $c'_{s+1} = a_{s+2}$. We get

$$0 < -c_s q_s A_s - c_{s+1} q_{s+1} A_{s+1} + a_{s+2} q_{s+1} A_{s+1} \leqslant -a_{s+1} q_s A_s + q_{s+1} A_{s+1} = q_{s-1} A_{s-1} < 0.$$

(4) Assume that there is a largest odd s with $j < s \leqslant t$ such that $c_s < a_{s+1}$. Let $c'_{s-1} = 0$, $c'_s = a_{s+1}$ (note that $c_{t+1} < a_{t+2}$). Then we get

$$0 < -c_{s-1} q_{s-1} A_{s-1} - c_s q_s A_s + a_{s+1} q_s A_s \leqslant -a_s q_{s-1} A_{s-1} + q_s A_s = q_{s-2} A_{s-2} < 0.$$

Later we shall prove that $j \neq i_k$ implies $j \notin \mathfrak{F}^{(2)}$. The proof is by contradiction: first we prove that $j \neq i_k$ and $j \in \mathfrak{F}^{(2)}$ implies $A_j < 0$. For this we need two lemmas:

LEMMA 1. *Let j be odd, $c_j = 1$ and $c'_j = 0$, $c'_t = c_t$ for $t \neq j$, $b_j = 0$. Then*

$$|\mathfrak{A}' \setminus \mathfrak{A}| + |\mathfrak{B} \setminus \mathfrak{B}'| = \sum_{k_{j-1} \leqslant N_{j-1}} 1.$$

Proof. Note that $k'_t \leqslant k_t$ for $0 \leqslant t \leqslant m$. For the proof we need two sublemmas:

(1) $t \in \mathfrak{A}' \setminus \mathfrak{A} \Leftrightarrow t = j-1$, $N_{j-2} < k_{j-2} \leqslant k_{j-1} \leqslant N_{j-1}$.

$$t \in \mathfrak{A}' \setminus \mathfrak{A} \Leftrightarrow N_{t-1} < k'_{t-1} \wedge k'_t \leqslant N_t \wedge k'_{t+1} \leqslant N_{t+1} \wedge (k_{t-1} \leqslant N_{t-1} \vee N_t < k_t \vee N_{t+1} < k_{t+1}) \Leftrightarrow (N_{t-1} < k'_{t-1} \wedge k'_t \leqslant N_t \leqslant k_t \wedge k'_{t+1} \leqslant N_{t+1}) \vee (N_{t-1} < k'_{t-1} \wedge k'_t \leqslant N_t \wedge k'_{t+1} \leqslant N_{t+1} < k_{t+1}).$$

The first condition is impossible: it implies $t > j$ and $c_u = b_u$ for $j \leqslant u \leqslant t$. We would get $k_{j-1} = k_j - q_j \leqslant N_j < k_j - q_j$.

Therefore $t \in \mathfrak{A}' \setminus \mathfrak{A} \Leftrightarrow N_{t-1} < k'_{t-1} \wedge k'_t \leqslant N_t \wedge k'_{t+1} \leqslant N_{t+1} < k_{t+1}$, t even. If $t > j$, then $c_u = b_u$ for $j < u \leqslant t+1$, $N_j < k_j - q_j = k_{j-1} < N_j$, a contradiction. We get $t = j-1$, $N_{j-2} < k_{j-2}$, $k_{j-1} \leqslant N_{j-1}$ and $k_j - q_j \leqslant N_j < k_j$. The last condition is superfluous.

(2) $t \in \mathfrak{B} \setminus \mathfrak{B}' \Leftrightarrow t = j-1$, $k_{j-2} \leqslant N_{j-2}$ and $k_{j-1} \leqslant N_{j-1}$.

$$t \in \mathfrak{B} \setminus \mathfrak{B}' \Leftrightarrow k_{t-1} \leqslant N_{t-1} \wedge k_t \leqslant N_t \wedge N_{t+1} < k_{t+1} \wedge (N_{t-1} < k'_{t-1} \vee N_t < k'_t \vee k'_{t+1} \leqslant N_{t+1}) \Leftrightarrow k_{t-1} \leqslant N_{t-1} \wedge k_t \leqslant N_t \wedge k'_{t+1} \leqslant N_{t+1} < k_{t+1}$$

for even t. We get $t+1 \geqslant j$. If $t+1 > j$, then $c_u = b_u$ for $j < u \leqslant t+1$, $k_j \leqslant N_j$ and $k_j - q_j \leqslant N_j < k_j$. Therefore $t+1 = j$, $k_{j-2} \leqslant N_{j-2}$, $k_{j-1} \leqslant N_{j-1}$ and $k_j - q_j \leqslant N_{j-1} < k_j = k_{j-1} + q_j$. The last condition is superfluous.

LEMMA 2. *Let j be odd, $c_j = 1$ and $c'_j = 0$, $c'_t = c_t$ for $t \neq j$, $b_j = 0$. Then*

$$|\mathfrak{A}| - |\mathfrak{A}'| \leqslant |\mathfrak{B}| - |\mathfrak{B}'|.$$

Proof. Again we prove two sublemmas:

(1) $t \in \mathfrak{A} \setminus \mathfrak{A}' \Leftrightarrow t$ even, $j < l < t \Rightarrow c_l = b_l$, $j < t$, $c_t < b_t$, $c_{t+1} \leqslant b_{t+1}$, $k_{j-1} \leqslant N_{j-1}$.

For even t we get: $t \in \mathfrak{A} \setminus \mathfrak{A}'$

$$\Leftrightarrow N_{t-1} < k_{t-1} \wedge k_t \leqslant N_t \wedge k_{t+1} \leqslant N_{t+1} \wedge (k'_{t-1} \leqslant N_{t-1} \vee N_t < k'_t \vee N_{t+1} < k'_{t+1})$$
$$\Leftrightarrow k'_{t-1} \leqslant N_{t-1} < k_{t-1} \wedge k_t \leqslant N_t \wedge k_{t+1} \leqslant N_{t+1}$$
$$\Leftrightarrow t > j, j < l < t \Rightarrow c_l = b_l, k_j - q_j \leqslant N_j < k_j, c_t < b_t \text{ and } c_{t+1} \leqslant b_{t+1}.$$

The last but two condition is equivalent to $k_{j-1} \leqslant N_{j-1}$.

(2) $t \in \mathfrak{B}' \setminus \mathfrak{B} \Leftrightarrow t$ even, $j < l < t \Rightarrow c_l = b_l$, $j < t$, $c_t \leqslant b_t$, $b_{t+1} < c_{t+1}$, $k_{j-1} \leqslant N_{j-1}$.

For even t we get: $t \in \mathfrak{B}' \setminus \mathfrak{B}$

$$\Leftrightarrow k'_{t-1} \leqslant N_{t-1} \wedge k'_t \leqslant N_t \wedge N_{t+1} < k'_{t+1} \wedge (N_{t-1} < k_{t-1} \vee N_t < k_t \vee k_{t+1} \leqslant N_{t+1})$$
$$\Leftrightarrow (k'_{t-1} \leqslant N_{t-1} < k_{t-1} \wedge k'_t \leqslant N_t \wedge N_{t+1} < k'_{t+1}) \vee (k'_{t-1} \leqslant N_{t-1} \wedge k'_t \leqslant N_t < k_t \wedge N_{t+1} < k'_{t+1}).$$

Now the second condition implies the first: it implies $j > t$ and $c_t = b_t$ and therefore $N_{t-1} < k_{t-1}$. The first condition is equivalent to

$$j < t, j < l < t \Rightarrow c_l = b_l, \quad c_t \leqslant b_t, \quad b_{t+1} < c_{t+1}, \quad k_{j-1} \leqslant N_{j-1}.$$

From (1) and (2) we deduce that $|\mathfrak{A} \setminus \mathfrak{A}'| + |\mathfrak{B}' \setminus \mathfrak{B}| = 0$ if $c_t = b_t$ for all $t > j$. Otherwise let t be the first integer $> j$ with $c_t \neq b_t$. Then (1) implies

$$|\mathfrak{A} \setminus \mathfrak{A}'| = \sum_{\substack{k_{j-1} \leqslant N_{j-1}, c_{t+1} \leqslant b_{t+1} \\ c_t < b_t, 2|t}} 1$$

and (2) implies

$$|\mathfrak{B}'\setminus\mathfrak{B}| = \sum_{\substack{k_{j-1}\le N_{j-1},b_t+1<c_t+1\\ c_t<b_t,2|t}} 1 + \sum_{\substack{k_{j-1}\le N_{j-1},b_t<c_t\\ 2|t}} 1.$$

We get

$$|\mathfrak{A}\setminus\mathfrak{A}'|+|\mathfrak{B}'\setminus\mathfrak{B}| = \sum_{\substack{k_{j-1}\le N_{j-1},c_t<b_t\\ 2|t}} 1 + \sum_{\substack{N_{j-1}\le k_{j-1},b_t<c_t\\ 2|t}} 1 \le \sum_{k_{j-1}\le N_{j-1}} 1.$$

Lemma 1 implies

$$|\mathfrak{A}|-|\mathfrak{A}'|-|\mathfrak{B}|+|\mathfrak{B}'| = |\mathfrak{A}\setminus\mathfrak{A}'|-|\mathfrak{A}'\setminus\mathfrak{A}|-|\mathfrak{B}\setminus\mathfrak{B}'|+|\mathfrak{B}'\setminus\mathfrak{B}| \le 0.$$

LEMMA 3. *Let $j > i_k$, $j\in\mathfrak{F}^{(2)}$. Then $A_j < 0$.*

Proof. Let $A_j > 0$. Then $c_j = 1$. Let $c'_j = 0$, $k' = \sum_{t=0}^{m} c'_t q_t$, where $c'_t = c_t$ for $t \ne j$. Note that $i_k = i_{k'}$ and $b_j = 0$. We get $0 < -q_j A_j + |\mathfrak{A}|-|\mathfrak{A}'|-|\mathfrak{B}|+ +|\mathfrak{B}'| \le -q_j A_j$, by Lemma 2. This is a contradiction.

LEMMA 4. *Let $j \ge i_k$, $j\in\mathfrak{B}$. Then $b_j = a_{j+1}$ (resp. $b_0 = a_1 - 1$ if $j = 0$).*

Proof. Assume that $b_j < a_{j+1}$ (resp. $b_0 < a_1 - 1$ for $j = 0$). We consider two cases:

(1) $j = i_k+1$ and $b_j = a_{j+1}-1$. Note that $j-1\in\mathfrak{G}$, $j+1\in\mathfrak{F}^{(2)}$. Let $c'_{j-1} = c'_j = 0$, $c'_{j+1} = a_{j+2}$. Then $i_{k'} = i_k$ and

$$0 < -1 - q_{j-1}A_{j-1} + c_j(1-q_jA_j) + (a_{j+2}-c_{j+1})q_{j+1}A_{j+1}.$$

This implies $c_{j+1} < a_{j+2}$ and $c_j = b_j$. We get

$$0 < -1 - q_{j-1}A_{j-1} + b_j - a_{j+1}q_jA_j + q_jA_j + q_{j+1}A_{j+1} = -1 + q_jA_j < 0.$$

(2) $j > i_k+1$ or $b_j < a_{j+1}-1$. Let $\mathfrak{F}^{(1)'} = \mathfrak{F}^{(1)}\cup\{j+1\}$, $\mathfrak{F}^{(2)'} = \mathfrak{F}^{(2)}\setminus\{j+1\}$. Then the conditions of Proposition 4 are fulfilled. Let $c'_{j+1} < a_{j+2}$ and $c'_j = b_j+1$. Then

$$0 < (c'_{j+1}-c_{j+1})q_{j+1}A_{j+1} + c_j(1-q_jA_j) - 1 - b_j + (b_j+1)q_jA_j.$$

If $c_{j+1} = a_{j+2}$ let $c'_{j+1} = a_{j+2}-1$. Then $0 < -q_{j+1}A_{j+1} - 1 - b_j + (b_j+ +1)q_jA_j$ and therefore $A_j > 0$. Furthermore $0 < -(q_{j+1}-(b_j+1)q_j)A_j + +(N_{j-1}/q_j) - 1 < 0$, a contradiction.

If $c_{j+1} < a_{j+2}$, let $c'_{j+1} = a_{j+2}$. Then we get $0 < q_jA_j - 1$, a contradiction too.

LEMMA 5. *Let $j-2 > i_k$, j odd, $A_{j-1} < 0$ and $b_j = 0$. Then $j\in\mathfrak{F}^{(1)}$.*

Proof. Proposition 4 (6) and Lemma 4 imply $j\notin\mathfrak{F}^{(2)}$. Assume now that $j\in\mathfrak{G}$. Then $j-2\in\mathfrak{G}$ by Proposition 4 (7). Assume that there is a largest $j\in\mathfrak{G}$ such that $j-2\in\mathfrak{G}$, $j-2 > i_k$, $A_{j-1} < 0$ and $b_j = 0$. Note that $A_{j+1} < 0$ and therefore $b_{j+2} \ne 0$ or $j+2\in\mathfrak{F}^{(1)}$. Let t be the first integer $\ge j+2$ such that $b_t \ne 0$. Proposition 7 (2) implies $t\in\mathfrak{F}^{(1)}\cup\mathfrak{G}$. Note that $j-1\in\mathfrak{C}$ and $c_{j-1} = 0$. Let $\mathfrak{F}^{(1)'} = \mathfrak{F}^{(1)}\cup\{j\}$, $\mathfrak{G}' = \mathfrak{G}\setminus\{j\}$. Then $j+1\in\mathfrak{E}^{(1)'}$, $j-1\in\mathfrak{E}^{(2)'}$ and $c'_{j-1} = a_j$.

(1) $t\in\mathfrak{G}$. Note that $b_{j+2} = 0$ implies $j+2\in\mathfrak{F}^{(1)}$. From Proposition 7 (3) we deduce $c_{j+2} = 0$, $c_{j+1} = a_{j+2}$ and $c_j = c_{j-1} = 0$. We get $0 < a_jq_{j-1}A_{j-1}$, a contradiction.

(2) $t\in\mathfrak{F}^{(1)}$. Then $t > j+2$ implies $c_{j+3} = 0$ (Proposition 7 (4)) and therefore $c_{j+3} < a_{j+4}$ in any case. We have $c_{j+2} = a_{j+3}-1$, $c_{j+1} = a_{j+2}$ and $c_j = c_{j-1} = 0$. Let $c'_{j+2} = a_{j+3}$, $c'_{j+1} = 0$, $c'_j = a_{j+1}-1$ and $c'_{j-1} = a_j$. We get

$$\begin{aligned} 0 < {} & -a_{j+2}q_{j+1}A_{j+1} + a_jq_{j-1}A_{j-1} + (a_{j+1}-1)q_jA_j - b_{j+2} - \\ & -(a_{j+3}-1)q_{j+2}A_{j+2} + b_{j+2} + a_{j+3}q_{j+2}A_{j+2} \\ = {} & a_jq_{j-1}A_{j-1} + a_{j+1}q_jA_j < 0. \end{aligned}$$

Lemma 5 is even true in the case $j-2 = i_k$. But here much more is true:

LEMMA 6. *Let $j-2 = i_k\in\mathfrak{G}$. Then $A_{j-1} > 0$.*

Proof. Assume that $A_{j-1} < 0$. Then $j-1\in\mathfrak{C}\cup\mathfrak{E}$, $c_{j-1} < a_j$. If $j-2 = m$ there is nothing to prove ($A_{m+1} = N(\alpha - r_{m+1}) > 0$). Let t be the smallest even integer such that $j+1 \le t \le m+2$ and $c_t < a_{t+1}$. If s is even, $j < s < t$, then $s\in\mathfrak{C}\cup\mathfrak{E}$ and $c_{s-1} = 0$. We prove the statement of the lemma in five steps:

(1) Let $j < s < t$, s even. Then $s\in\mathfrak{E}$.

Assume that $s\in\mathfrak{C}$. Then $s-1\in\mathfrak{G}$, $s+1\in\mathfrak{G}$, $b_s = a_{s+1}$, $b_{s-1} = 0$. Let $c'_r = 0$ for $r < s-1$, $c'_{s-1} = 1$, $c'_s = a_{s+1}-1$, $\mathfrak{F}^{(2)'} = \mathfrak{F}^{(2)}\cup\{s-1\}$ and $\mathfrak{G}' = \mathfrak{G}\setminus\{s-1\}$. Then $\mathfrak{B}' = \mathfrak{B}\cup\{s-2\}$ and $\mathfrak{A}' = \mathfrak{A}\cup\{s\}$. We get

$$\begin{aligned} 0 < {} & -1 - q_{j-2}A_{j-2} - c_{j-1}q_{j-1}A_{j-1} + \sum_{\substack{r=j+1\\ 2|r}}^{s-2}(b_r - a_{r+1}q_rA_r) + \\ & + a_{s+1}(1-q_sA_s) + q_{s-1}A_{s-1} - (a_{s+1}-1)(1-q_sA_s) \\ = {} & (a_j - c_{j-1})q_{j-1}A_{j-1} - q_sA_s < 0. \end{aligned}$$

(2) $t-1\in\mathfrak{F}$. Let $\mathfrak{F}^{(2)'} = \mathfrak{F}^{(2)}\cup\{t-1\}$, $c'_s = 0$ for $s < t-1$. We have to look if the conditions of Proposition 4 are fulfilled:

$b_{t-1} = a_t$ would imply $c_{t-2} = 0$ and this is impossible if $t-2 > j+1$. But if $t-2 = j-1$, then $j-1\in\mathfrak{E}^{(2)}$. This gives us $c_{t-2} > 0$.

If $b_t = a_{t+1}$ and $t+1\in\mathfrak{F}^{(1)}$, then $t\in\mathfrak{E}^{(1)}$ and $c_t = a_{t+1}$, contrary to our choice of t.

This implies

$$\begin{aligned} 0 < {} & -1 - q_{j-2}A_{j-2} - c_{j-1}q_{j-1}A_{j-1} + \\ & + \sum_{\substack{s=j+1\\ 2|s}}^{t-2}(b_s - a_{s+1}q_sA_s) + 1 + (c'_{t-1} - c_{t-1})q_{t-1}A_{t-1} \\ = {} & (a_j - c_{j-1})q_{j-1}A_{j-1}, \end{aligned}$$

where we choose $c'_{t-1} = c_{t-1}+1$.

All this runs if $t-1 \in \mathfrak{F}^{(1)}$. But if $t-1 \in \mathfrak{F}^{(2)}$ let $c'_s = 0$ for $s < t-1$, $c'_{t-1} = c_{t-1}+1$. We get

$$0 < -1-q_{j-2}A_{j-2}-c_{j-1}q_{j-1}A_{j-1}+\sum_{\substack{s=j+1\\2|s}}^{t-2}(b_s-a_{s+1}q_sA_s)+q_{t-1}A_{t-1}$$
$$=(a_j-c_{j-1})q_{j-1}A_{j-1}-1<0.$$

(3) $t-1 \in \mathfrak{G}$, $b_{t-1} \neq 0$. Let $c'_s = 0$ for $s < t-1$ and $c'_{t-1} = 1$. Then we get

$$0 < -1-q_{j-2}A_{j-2}-c_{j-1}q_{j-1}A_{j-1}+\sum_{\substack{s=j+1\\2|s}}^{t-2}(b_s-a_{s+1}q_sA_s)+1+q_{t-1}A_{t-1}$$
$$=(a_j-c_{j-1})q_{j-1}A_{j-1}<0.$$

(4) $t-1 \in \mathfrak{G}$, $b_{t-1} = 0$, $t+1 \in \mathfrak{F}$. $c_t < a_{t+1}$ implies $t+1 \in \mathfrak{F}^{(1)}$ (Lemma 4). Note that $t \in \mathfrak{E}^{(2)}$, $A_t > 0$ and that $c_t = b_t+1$. Let $c'_s = 0$ for $s < t-1$ and $c'_{t-1} = 1$, $\mathfrak{F}^{(2)\prime} = \mathfrak{F}^{(2)} \cup \{t-1\}$, $\mathfrak{G}' = \mathfrak{G} \setminus \{t-1\}$, $c'_t = b_t$. Then $t \in \mathfrak{E}^{(1)\prime}$ and we get

$$0 < -1-q_{j-2}A_{j-2}-c_{j-1}q_{j-1}A_{j-1}+$$
$$+\sum_{\substack{s=j+1\\2|s}}^{t-2}(b_s-a_{s+1}q_sA_s)+1+q_{t-1}A_{t-1}+b_t-(b_t+1)q_tA_t-b_t(1-q_tA_t)$$
$$=(a_j-c_{j-1})q_{j-1}A_{j-1}-q_tA_t<0.$$

(5) $t-1 \in \mathfrak{G}$, $b_{t-1} = 0$, $t+1 \in \mathfrak{G}$. If $A_t < 0$, then $b_{t+1} \neq 0$ by Lemma 5. We would get $t \in \mathfrak{E}^{(2)}$ and $c_t = a_{t+1}$, contrary to our choice of t.

If $A_t > 0$, then $A_{t-2} > 0$ and therefore $t-3 \in \mathfrak{G}$ ($t-3 \in \mathfrak{F}$ would imply $t-2 \in \mathfrak{A}$ or $c_{t-2} = 0$). Therefore $t-2 \in \mathfrak{C}$, contrary to (1).

Proposition 8. *Let $j-2 \geqslant i_k$, j odd, $A_{j-1} < 0$ and $b_j = 0$. Then $j \in \mathfrak{F}^{(1)}$.*

Proof. If $j-2 > i_k$ use Lemma 5. Now let $j-2 = i_k$. Then Lemma 4 implies $j \notin \mathfrak{F}^{(2)}$. If $j \in \mathfrak{G}$, then $j-2 \in \mathfrak{G}$ by Proposition 4 (7). Lemma 6 implies $A_{j-1} > 0$.

Proposition 9. (1) $\mathfrak{C} = \{j \in (\mathfrak{G}-1)\cap(\mathfrak{G}+1) \mid j \geqslant i_k \Rightarrow A_j > 0\}$.

(2) $\mathfrak{E}^{(2)} = \big((\mathfrak{G}+1)\cap(\mathfrak{F}^{(1)}-1)\big) \cup \{j \in (\mathfrak{G}-1)\cap(\mathfrak{G}+1) \mid j > i_k$ *and* $A_j < 0\}$.

Proof. This follows from Proposition 5 (3), (6) and Proposition 8.

Later we shall prove that $A_{i_k+1} > 0$.

Proposition 10. *Let $j = i_k \in \mathfrak{G}$. Then $A_{j-1} < 0$.*

Proof. Assume that $A_{j-1} > 0$. Let t be the largest even integer $< j$ such that $b_t < a_{t+1}$ (note that $b_0 \leqslant a_1 - 1$). Then $A_t > 0$.

Assume first that $t = 0$. Let $c'_s = \frac{1}{2}(1+(-1)^s)a_{s+1}$ for $1 \leqslant s < j$, $c'_j = c_j - 1$, $c'_s = c_s$ for $s > j$. Then $k-1 = \sum_{s=0}^{m} c'_s q_s$. If $b_0 \neq 0$, then $i_N = 0$. Let s_0

be the first integer with $c'_{s_0} \neq b_{s_0}$. Because of $\{(k-1)\alpha\} < \{N\alpha\}$ (Proposition 2) we get $(b_{s_0}-c'_{s_0})(-1)^{s_0} > 0$. This implies $s_0 \geqslant j$. If $s_0 > j$, then $b_j = c'_j = c_j - 1 < c_j$, a contradiction to $j \in \mathfrak{G}$. If $s_0 = j$, then $b_j < c'_j = c_j - 1$, a contradiction too. But if $b_0 = 0$, then $s_0 = 0$; this is impossible.

Now assume that $t \geqslant 2$. We consider two cases:

(1) $b_{t-1} \neq 0$. Let $c'_{t-1} = 1$, $c'_t = b_t$ and $c'_s = a_{s+1}$ for even s, $t < s < j$. Then we get

$$0 < -1-q_jA_j+1+q_{t-1}A_{t-1}-b_t(1-q_tA_t)-\sum_{\substack{t<s<j\\2|s}}a_{s+1}(1-q_sA_s)$$
$$=-(q_{t+1}-b_tq_t-q_{t-1})A_t.$$

(2) $b_{t-1} = 0$. Let $c'_{t-1} = 1$, $c'_s = a_{s+1}$ for even s, $t < s < j$.

(2.1) $b_t \neq 0$. Let $c'_t = b_t - 1$, $\mathfrak{F}^{(2)\prime} = \mathfrak{F}^{(2)} \cup \{t-1\}$, $\mathfrak{B}' = \mathfrak{B} \cup \{t-2\}$, $\mathfrak{A}' = \mathfrak{A} \cup \{t\}$. We get

$$0 < -1-q_jA_j+q_{t-1}A_{t-1}-(b_t-1)(1-q_tA_t)-\sum_{\substack{t<s<j\\2|s}}a_{s+1}(1-q_sA_s)$$
$$=-\big(q_{t+1}-(b_t-1)q_t-q_{t-1}\big)A_t.$$

(2.2) $b_t = 0$. Let $c'_t = 0$ and $\mathfrak{F}^{(2)\prime} = \mathfrak{F}^{(2)} \cup \{t-1\}$. Then $\mathfrak{B}' = \mathfrak{B} \cup \{t-2\}$, $\mathfrak{E}^{(1)\prime} = \mathfrak{E}^{(1)} \cup \{t\}$ and we get

$$0 < -1-q_jA_j+1+q_{t-1}A_{t-1}-\sum_{\substack{t<s<j\\2|s}}a_{s+1}(1-q_sA_s) = -(q_{t+1}-q_{t-1})A_t < 0.$$

5. A sufficient condition for $j \in \mathfrak{F}^{(1)}$. In this section we prove that $A_j > 0$, $j > i_k$ and j odd implies $j \in \mathfrak{F}^{(1)}$. For the proof we need ten lemmas.

Lemma 1. *Let $j-2 \in \mathfrak{F}$, $j+2 \in \mathfrak{G}$ and $j \in \mathfrak{G}$. Then $b_{j+1} = 0$ or $A_j < 0$.*

Proof. If $b_{j-1} = 0$, then $b_j \neq 0$ by Proposition 4 (7) of § 4. We may assume that $b_{j-1} \neq 0$. Note that $j-1 \in \mathfrak{A}$, $j+1 \in \mathfrak{C}$. Assume that $b_{j+1} \neq 0$ and $A_j > 0$.

(1) $j-2 = i_k$ and $b_{j-1} = a_j$. Then $c_{j-1} = a_j - 1$, $c_{j+1} = b_{j+1}$, $A_{j-2} > 0$ and $c_{j-2} = 1$. Let $\mathfrak{F}^{(2)\prime} = (\mathfrak{F}^{(2)} \cup \{j\}) \setminus \{j-2\}$, $\mathfrak{G}' = (\mathfrak{G} \cup \{j-2\}) \setminus \{j\}$. Then $\mathfrak{A}' = (\mathfrak{A} \cup \{j+1\}) \setminus \{j-1\}$, $\mathfrak{B}' = (\mathfrak{B} \cup \{j-1\}) \setminus \{j-3\}$, $\mathfrak{C}' = (\mathfrak{C} \cup \{j-3\}) \setminus \{j+1\}$, $c'_{j-2} = c'_{j-1} = 0$, $c'_j = 1$ and $c'_{j+1} = b_{j+1} - 1$. We get

$$0 < -q_{j-2}A_{j-2}+(a_j-1)(1-q_{j-1}A_{j-1})+b_{j+1}(1-q_{j+1}A_{j+1})+$$
$$+q_jA_j-(b_{j+1}-1)(1-q_{j+1}A_{j+1})$$
$$=-(q_{j+1}-q_{j-1})A_j<0.$$

(2) $b_{j-1} = a_j \Rightarrow j-2 > i_k$. Let $\mathfrak{F}^{(1)\prime} = \mathfrak{F}^{(1)} \cup \{j\}$, $\mathfrak{G}' = \mathfrak{G} \setminus \{j\}$. Then the conditions of Proposition 4 of § 4 are fulfilled, $\mathfrak{A}' = (\mathfrak{A} \setminus \{j-1\}) \cup \{j+1\}$, $\mathfrak{C}' = \mathfrak{C} \setminus \{j+1\}$, $\mathfrak{E}^{(1)\prime} = \mathfrak{E}^{(1)} \cup \{j-1\}$, $c_{j-1} = b_{j-1} - 1$, $c'_{j-1} = b_{j-1}$, $c'_j = 0$, c'_{j+1}

$= b_{j+1}-1$ and ($b_{j-1}=a_j$ implies ($A_{j-2}>0$ and $c_{j-2}=0$)). We get

$$0 < -(b_{j+1}-1)(1-q_{j+1}A_{j+1})+(b_{j-1}-1)(1-q_{j-1}A_{j-1})+$$
$$+b_{j+1}(1-q_{j+1}A_{j+1})-b_{j-1}(1-q_{j-1}A_{j-1}) = -a_{j+1}q_jA_j<0.$$

LEMMA 2. *Let* $1\in\mathfrak{G}$, $3\in\mathfrak{G}$ *and* $i_k=0$. *Then* $b_2=0$ *or* $A_1<0$.

Proof. We note that $2|i_N$ (Proposition 2 (2) of § 4). Assume that $b_2\neq 0$ and $A_1>0$. Then $A_0>0$ and $0\in\mathfrak{C}$. We get $b_0=c_0>0$. Note that $c_2=b_2$.

(1) $b_0=a_1-1$. Let $\mathfrak{F}^{(2)\prime}=\mathfrak{F}^{(2)}\cup\{1\}$, $\mathfrak{G}'=\mathfrak{G}\setminus\{1\}$. Then $\mathfrak{B}'=\mathfrak{B}\cup\{0\}$, $\mathfrak{A}'=\mathfrak{A}\cup\{2\}$, $\mathfrak{C}'=\mathfrak{C}\setminus\{0,2\}$, $c_0'=0$, $c_1'=1$ and $c_2'=b_2-1$. We get $i_{k'}=1$ and therefore

$$0<(a_1-1)(1-q_0A_0)+q_1A_1-1+b_2(1-q_2A_2)-(b_2-1)(1-q_2A_2)$$
$$= -(q_2-q_0)A_1<0.$$

(2) $b_0<a_1-1$. Let $\mathfrak{F}^{(1)\prime}=\mathfrak{F}^{(1)}\cup\{1\}$, $\mathfrak{G}'=\mathfrak{G}\cup\{1\}$. Then $\mathfrak{E}^{(2)\prime}=\mathfrak{E}^{(2)}\cup\{0\}$, $\mathfrak{A}'=\mathfrak{A}\cup\{2\}$, $\mathfrak{C}'=\mathfrak{C}\setminus\{0,2\}$, $c_0'=b_0+1$, $c_1'=0$, $c_2'=b_2-1$ and we get

$$0<b_0(1-q_0A_0)-b_0+(b_0+1)q_0A_0+b_2(1-q_2A_2)-(b_2-1)(1-q_2A_2)-1$$
$$= -(q_2-q_0)A_1<0.$$

LEMMA 3. *Let* $i_k\leqslant j-2\in\mathfrak{G}$, $j\in\mathfrak{G}$ *and* $j+2\in\mathfrak{G}$. *Then* $b_{j+1}=0$ *or* $A_j<0$.

Proof. Assume that $b_{j+1}\neq 0$ and that $A_j>0$. Note that $j-1\in\mathfrak{C}$, $j+1\in\mathfrak{C}$, $c_{j-1}=b_{j-1}$ and $c_{j+1}=b_{j+1}$. First we prove that $b_{j-1}=a_j$. We consider two cases:

(1) $j-2=i_k$ and $b_{j-1}=a_j-1$. Let $\mathfrak{G}'=\mathfrak{G}\setminus\{j\}$, $\mathfrak{F}^{(2)\prime}=\mathfrak{F}^{(2)}\cup\{j\}$, $c_{j-2}'=c_{j-1}'=0$. Then $\mathfrak{A}'=\mathfrak{A}\cup\{j+1\}$, $\mathfrak{B}'=\mathfrak{B}\cup\{j-1\}$, $\mathfrak{C}'=\mathfrak{C}\setminus\{j-1,j+1\}$, $c_{j+1}'=b_{j+1}-1$. We get

$$0<-1-q_{j-2}A_{j-2}+b_{j-1}(1-q_{j-1}A_{j-1})+b_{j+1}(1-q_{j+1}A_{j+1})+q_jA_j-$$
$$-(b_{j+1}-1)(1-q_{j+1}A_{j+1}) = -a_{j+1}q_jA_j<0.$$

(2) $b_{j-1}<a_j$ or $i_k<j-2$. Let $\mathfrak{G}'=\mathfrak{G}\setminus\{j\}$, $\mathfrak{F}^{(1)\prime}=\mathfrak{F}^{(1)}\cup\{j\}$. We get $\mathfrak{A}'=\mathfrak{A}\cup\{j+1\}$, $\mathfrak{E}^{(2)\prime}=\mathfrak{E}^{(2)}\cup\{j-1\}$, $\mathfrak{C}'=\mathfrak{C}\setminus\{j-1,j+1\}$, $c_{j-1}'=b_{j-1}+1$, $c_j'=0$, $c_{j+1}'=b_{j+1}-1$ and

$$0<b_{j-1}(1-q_{j-1}A_{j-1})-b_{j-1}+(b_{j-1}+1)q_{j-1}A_{j-1}+b_{j+1}(1-q_{j+1}A_{j+1})-$$
$$-1-(b_{j+1}-1)(1-q_{j+1}A_{j+1}) = -a_{j+1}q_jA_j<0.$$

Now we prove that $A_j<0$ even in the case $b_{j-1}=a_j$. Choose an integer t such that $b_t<a_{t+1}$, $b_{t+1}=0$, $b_{t+2}=a_{t+3}$, $\ldots$, $b_{j-2}=0$, $b_{j-1}=a_j$. Then $t<j-2$. Note that t is even and that $A_{t+1}\geqslant A_{t+2}\geqslant\ldots\geqslant A_{j-2}\geqslant A_j>0$. Again we consider two cases:

(3) Assume that there is an s, $t<s<j-2$ with $s\in\mathfrak{F}$. Choose the

greatest such s. Then $s+2\in\mathfrak{G}$, $s+4\in\mathfrak{G}$, $s\in\mathfrak{F}$. We get $b_{s+1}=a_{s+2}$ and therefore Lemma 1 implies $A_{s+2}<0$, a contradiction.

(4) Assume that for $t<s<j-2$ $s\in\mathfrak{G}$. Then $i_k\leqslant t$.

(4.1) $t=0$. Then $i_k=0$, $1\in\mathfrak{G}$ and $3\in\mathfrak{G}$, $b_2=a_3$. Lemma 2 implies $A_1<0$, a contradiction.

(4.2) $t\neq 0$. Then $i_k<t$. If $t-1\in\mathfrak{F}$, then $t+1\in\mathfrak{G}$, $t+3\in\mathfrak{G}$, $b_{t+2}\neq 0$ and therefore $A_{t+1}<0$, a contradiction (Lemma 1).

If $t-1\in\mathfrak{G}$, then $b_{t+2}\neq 0$, $t+1\in\mathfrak{G}$ and $t+3\in\mathfrak{G}$. Because of $b_t<a_{t+1}$ we get $A_{t+1}<0$ by what was just proved in (1) and (2).

LEMMA 4. *Let* $i_k\leqslant j-2\in\mathfrak{G}$, $j\in\mathfrak{G}$ *and* $j+2\in\mathfrak{F}^{(1)}$. *Then* $A_j<0$.

Proof. Assume that $A_j>0$. Then $A_{j-1}>0$ and $A_{j+1}>0$, $j-1\in\mathfrak{C}$, $c_{j-1}=b_{j-1}$, $c_j=0$, $j+1\in\mathfrak{E}^{(2)}$ and $c_{j+1}=b_{j+1}+1$. If $b_{j+1}=0$, then $A_{j+2}>0$ and therefore $c_{j+2}=0$ in this case. We consider three cases:

(1) $j-2=i_k$ and $b_{j-1}=a_j-1$. Let $c_{j-2}'=c_{j-1}'=0$, $c_j'=1$, $c_{j+1}'=b_{j+1}$, $\mathfrak{G}'=\mathfrak{G}\setminus\{j\}$ and $\mathfrak{F}^{(2)\prime}=\mathfrak{F}^{(2)}\cup\{j\}$. Then $\mathfrak{B}'=\mathfrak{B}\cup\{j-1\}$, $\mathfrak{C}'=\mathfrak{C}\setminus\{j-1\}$, $\mathfrak{E}^{(1)\prime}=\mathfrak{E}^{(1)}\cup\{j+1\}$, $\mathfrak{E}^{(2)\prime}=\mathfrak{E}^{(2)}\setminus\{j+1\}$ and $i_{k'}=i_k+2$. We get

$$0<-1-q_{j-2}A_{j-2}+(a_j-1)(1-q_{j-1}A_{j-1})+b_{j+1}-(b_{j+1}+1)q_{j+1}A_{j+1}+$$
$$+1+q_jA_j-b_{j+1}(1-q_{j+1}A_{j+1}) = -a_{j+1}q_jA_j<0.$$

(2) $b_{j-1}=a_j$. Then $i_k<j-2$. If $j=3$ and $i_k=0$, then $b_2\neq 0$ and we get $A_1<0$ by Lemma 2. Therefore $A_3\leqslant A_1<0$.

If $j\neq 3$ or $i_k>1$ we get $j-4\geqslant i_k$. If $j-4\in\mathfrak{F}$, then $A_{j-2}<0$ by Lemma 1. This implies $A_j<0$. If $j-4\in\mathfrak{G}$, then $A_{j-2}<0$ by Lemma 3. Again we get $A_j<0$.

(3) $b_{j-1}<a_j$ and ($j-2=i_k\Rightarrow b_{j-1}<a_j-1$). Let $\mathfrak{F}^{(1)\prime}=\mathfrak{F}^{(1)}\cup\{j\}$, $\mathfrak{G}'=\mathfrak{G}\setminus\{j\}$. Because of $j>1$ the conditions of Proposition 4 of § 4 are fulfilled, $\mathfrak{E}^{(2)\prime}=(\mathfrak{E}^{(2)}\cup\{j-1\})\setminus\{j+1\}$, $\mathfrak{E}^{(1)\prime}=\mathfrak{E}^{(1)}\cup\{j+1\}$, $\mathfrak{C}'=\mathfrak{C}\setminus\{j-1\}$, $c_{j-1}'=b_{j-1}+1$, $c_j'=0$ and $c_{j+1}'=b_{j+1}$. We get

$$0<b_{j-1}(1-q_{j-1}A_{j-1})-b_{j-1}+(b_{j-1}+1)q_{j-1}A_{j-1}+$$
$$+b_{j+1}-(b_{j+1}+1)q_{j+1}A_{j+1}-b_{j+1}(1-q_{j+1}A_{j+1}) = -a_{j+1}q_jA_j<0,$$

a contradiction.

Remark 1. *Let* $j>i_k$ *and* $j\in\mathfrak{B}$, $A_{j-1}<0$. *Then* $c_{j+1}=a_{j+2}$.

Proof. Note that $j-1\in\mathfrak{G}$, $j+1\in\mathfrak{F}^{(2)}$ and $b_j=a_{j+1}$ (Lemma 4 of § 4). We have to prove that

$$a_{j+1}(1-q_jA_j)-b_{j+1}-(a_{j+2}-1)q_{j+1}A_{j+1}<-b_{j+1}-a_{j+2}q_{j+1}A_{j+1}.$$

This is equivalent to $A_{j-1}<0$.

LEMMA 5. *Let* $i_k\leqslant j-2\in\mathfrak{G}$ *and* $j\in\mathfrak{G}$. *Then* $j+2\notin\mathfrak{F}^{(2)}$.

Proof. Assume that $j+2\in\mathfrak{F}^{(2)}$. Then $j+1\in\mathfrak{B}$, $b_{j+1}=a_{j+2}$ by Lemma 4 of § 4. Because of $b_j = 0$ we get $A_{j-1} > 0$ and therefore $j-1 \in \mathfrak{C}$. We consider three cases:

(1) $j-2 = i_k$ and $b_{j-1} = a_j - 1$. Let $c'_{j-2} = c'_{j-1} = c'_{j+1} = 0$, $c'_j = 1$, $\mathfrak{F}^{(2)\prime} = \mathfrak{F}^{(2)} \cup \{j\}$, $\mathfrak{G}' = \mathfrak{G} \setminus \{j\}$. Then $\mathfrak{B}' = (\mathfrak{B} \setminus \{j+1\}) \cup \{j-1\}$, $\mathfrak{C}' = \mathfrak{C} \setminus \{j-1\}$, $\mathfrak{D}' = \mathfrak{D} \cup \{j+1\}$. Note that $c_{j-2} = 1$, $c_{j-1} = b_{j-1}$ and $c_j = 0$. We get

$$0 < -1 - q_{j-2} A_{j-2} + (a_j - 1)(1 - q_{j-1} A_{j-1}) + c_{j+1}(1 - q_{j+1} A_{j+1}) + \\ + (c'_{j+2} - c_{j+2}) q_{j+2} A_{j+2} + q_j A_j \\ = -1 + q_{j-1} A_{j-1} + c_{j+1}(1 - q_{j+1} A_{j+1}) + (c'_{j+2} - c_{j+2}) q_{j+2} A_{j+2}.$$

We choose $c'_{j+2} = a_{j+3}$. We get $c_{j+2} = a_{j+3} - 1$ and $c_{j+1} = a_{j+2}$. Remark 1 implies $A_j > 0$ and therefore

$$0 < -1 + q_{j-1} A_{j-1} + a_{j+2}(1 - q_{j+1} A_{j+1}) + q_{j+2} A_{j+2} = -a_{j+1} q_j A_j < 0.$$

(2) $b_{j-1} < a_j$ and $(j-2 = i_k \Rightarrow b_{j-1} < a_j - 1)$. Let $\mathfrak{F}^{(1)\prime} = \mathfrak{F}^{(1)} \cup \{j\}$, $\mathfrak{G}' = \mathfrak{G} \setminus \{j\}$. Then $\mathfrak{B}' = \mathfrak{B} \setminus \{j+1\}$, $\mathfrak{D}' = \mathfrak{D} \cup \{j+1\}$, $\mathfrak{E}^{(2)\prime} = \mathfrak{E}^{(2)} \cup \{j-1\}$, $\mathfrak{C}' = \mathfrak{C} \setminus \{j-1\}$, $c_{j-1} = b_{j-1}$, $c_j = 0$, $c'_{j-1} = b_{j-1} + 1$, $c'_{j+1} = 0$, $c'_{j+2} = a_{j+3}$. We get

$$0 < b_{j-1}(1 - q_{j-1} A_{j-1}) - b_{j-1} + (b_{j-1} + 1) q_{j-1} A_{j-1} + c'_j q_j A_j + \\ + c_{j+1}(1 - q_{j+1} A_{j+1}) + (a_{j+3} - c_{j+2}) q_{j+2} A_{j+2} - 1 \\ = -1 + q_{j-1} A_{j-1} + c'_j q_j A_j + c_{j+1}(1 - q_{j+1} A_{j+1}) + (a_{j+3} - c_{j+2}) q_{j+2} A_{j+2}.$$

If $c_{j+2} = a_{j+3}$ we get a contradiction (note that $c'_j = 0$ if $A_j > 0$). Therefore $A_j > 0$ by Remark 1, $c_{j+2} = a_{j+3} - 1$ and $c_{j+1} = a_{j+2}$. We get

$$0 < -1 + q_{j-1} A_{j-1} + a_{j+2}(1 - q_{j+1} A_{j+1}) + q_{j+2} A_{j+2} = -a_{j+1} q_j A_j < 0.$$

(3) $b_{j-1} = a_j$. Note that $j-2 > i_k$. We prove that $A_{j-2} < 0$. If $i_k = 0$ and $j = 3$, then $A_1 < 0$ from Lemma 2.

Assume now that $j-4 \geqslant i_k$. If $j-4 \in \mathfrak{F}$, then we get $A_{j-2} < 0$ by Lemma 1. If $j-4 \in \mathfrak{G}$ we get $A_{j-2} < 0$ by Lemma 3. From this we deduce that $A_j < 0$. Therefore $c_{j+1} = 0$ and $c_{j+2} = a_{j+3}$ by Remark 1. Let $\mathfrak{F}^{(2)\prime} = \mathfrak{F}^{(2)} \cup \{j\}$, $\mathfrak{G}' = \mathfrak{G} \setminus \{j\}$. We get $\mathfrak{B}' = (\mathfrak{B} \cup \{j-1\}) \setminus \{j+1\}$, $\mathfrak{D}' = \mathfrak{D} \cup \{j+1\}$, $\mathfrak{C}' = \mathfrak{C} \setminus \{j-1\}$. Let $c'_{j-1} = 0$, $c'_j = a_{j+2}$, $c'_{j+1} = 0$ and $c'_{j+2} = a_{j+3}$. Then

$$0 < a_j(1 - q_{j-1} A_{j-1}) + a_{j+1} q_j A_j = q_{j-2} A_{j-2} + (a_{j+1} - 1) q_j A_j < 0.$$

Lemma 6. *Let $j \geqslant i_k$, $j \in \mathfrak{G} \cap (\mathfrak{G} - 2)$. Then $A_j < 0$.*

Proof. Assume that $A_j > 0$. Lemmas 1, 2 and 3 imply $b_{j+1} = 0$. Let t be the first integer $> j$ such that $b_t \neq 0$ (note that t exists, because otherwise $j = m$). t is even, $t \geqslant j+3$ and $A_{t-1} \geqslant A_{t-3} \geqslant \ldots \geqslant A_{j+2} \geqslant A_j > 0$.

Assume first that there is an s, $j+2 < s < t$, with $s \in \mathfrak{F}$. Choose a smallest such s. Then $s-2 \in \mathfrak{G}$, $s-4 \in \mathfrak{G}$ and $s \in \mathfrak{F}^{(1)}$ by Lemma 3 of § 4.

Furthermore $A_{s-1} \geqslant A_s > 0$. Lemma 4 implies $A_{s-2} < 0$, a contradiction.

We get $j+4 \in \mathfrak{G}, \ldots, t-1 \in \mathfrak{G}$ and $b_t \neq 0$. If $t+1 \in \mathfrak{F}$, then $t+1 \in \mathfrak{F}^{(1)}$ by Lemma 5. Lemma 4 implies $A_{t-1} < 0$, a contradiction. If $t+1 \in \mathfrak{G}$, we get $A_{t-1} < 0$ by Lemma 3. ∎

Lemma 7. *Let $j-2 \in \mathfrak{F}$, $j \in \mathfrak{G}$ and $j+2 \in \mathfrak{F}^{(1)}$. Then $A_j < 0$.*

Proof. Assume that $A_j > 0$. Proposition 4 of § 4 implies $b_{j-1} \neq 0$. Therefore $j-1 \in \mathfrak{A}$, $j+1 \in \mathfrak{E}^{(2)}$, $A_{j+1} > 0$, $c_{j-1} = b_{j-1} - 1$, $c_{j+1} = b_{j+1} + 1$. We consider two cases:

(1) $j-2 = i_k$ and $b_{j-1} = a_j$. Then $A_{j-2} > 0$, $c_{j-2} = 1$ and $j-3 \in \mathfrak{B}$. Let $\mathfrak{F}^{(2)\prime} = (\mathfrak{F}^{(2)} \setminus \{j-2\}) \cup \{j\}$ and $\mathfrak{G}' = (\mathfrak{G} \setminus \{j\}) \cup \{j-2\}$. Then $\mathfrak{B}' = (\mathfrak{B} \setminus \{j-3\}) \cup \{j-1\}$, $\mathfrak{A}' = \mathfrak{A} \setminus \{j-1\}$, $\mathfrak{C}' = \mathfrak{C} \cup \{j-3\}$, $\mathfrak{E}^{(2)\prime} = \mathfrak{E}^{(2)} \setminus \{j+1\}$ and $\mathfrak{E}^{(1)\prime} = \mathfrak{E}^{(1)} \cup \{j+1\}$ (note that $b_{j+1} < a_{j+2}$). Let $c'_{j-2} = c'_{j-1} = 0$, $c'_j = 1$ and $c'_{j+1} = b_{j+1}$. We get

$$0 < -q_{j-2} A_{j-2} + (b_{j-1} - 1)(1 - q_{j-1} A_{j-1}) + 1 + b_{j+1} - \\ -(b_{j+1} + 1) q_{j+1} A_{j+1} + q_j A_j - b_{j+1}(1 - q_{j+1} A_{j+1}) = -a_{j+1} q_j A_j < 0.$$

(2) $j-2 > i_k$ or $b_{j-1} < a_j$. Let $\mathfrak{G}' = \mathfrak{G} \setminus \{j\}$ and $\mathfrak{F}^{(1)\prime} = \mathfrak{F}^{(1)} \cup \{j\}$. If $b_{j-1} = a_j$, note that $A_{j-2} > 0$, $j-2 \in \mathfrak{F}^{(1)}$ (§ 4, Lemma 4) and $c_{j-2} = 0$. In any case we get $\mathfrak{A}' = \mathfrak{A} \setminus \{j-1\}$, $\mathfrak{E}^{(2)\prime} = \mathfrak{E}^{(2)} \setminus \{j+1\}$, $\mathfrak{E}^{(1)\prime} = \mathfrak{E}^{(1)} \cup \{j-1, j+1\}$. Let $c'_{j-1} = b_{j-1}$, $c'_j = 0$ and $c'_{j+1} = b_{j+1}$. Then

$$0 < (b_{j-1} - 1)(1 - q_{j-1} A_{j-1}) + 1 - b_{j-1}(1 - q_{j-1} A_{j-1}) + \\ + b_{j+1} - (b_{j+1} + 1) q_{j+1} A_{j+1} - b_{j+1}(1 - q_{j+1} A_{j+1}) = -a_{j+1} q_j A_j < 0.$$

Lemma 8. *Let $i_k = 0$, $1 \in \mathfrak{G}$ and $3 \in \mathfrak{F}^{(1)}$. Then $A_1 < 0$.*

Proof. Assume that $A_1 > 0$. Then $A_0 > 0$, $A_2 > 0$, $0 \in \mathfrak{C}$, $2 \in \mathfrak{E}^{(2)}$, $c_0 = b_0 > 0$ and $c_2 = b_2 + 1$. We consider two cases:

(1) $b_0 = a_1 - 1$. Let $\mathfrak{F}^{(2)\prime} = \mathfrak{F}^{(2)} \cup \{1\}$ and $\mathfrak{G}' = \mathfrak{G} \setminus \{1\}$. Then $\mathfrak{B}' = \mathfrak{B} \cup \{0\}$, $\mathfrak{E}^{(1)\prime} = \mathfrak{E}^{(1)} \cup \{2\}$, $\mathfrak{E}^{(2)\prime} = \mathfrak{E}^{(2)} \setminus \{2\}$ and $\mathfrak{C}' = \mathfrak{C} \setminus \{0\}$. Let $c'_0 = 0$, $c'_1 = 1$ and $c'_2 = b_2$. We get

$$0 < b_0(1 - q_0 A_0) + b_2 - (b_2 + 1) q_2 A_2 + q_1 A_1 - b_2(1 - q_2 A_2) = -a_2 q_1 A_1 < 0.$$

(2) $b_0 < a_1 - 1$. Let $\mathfrak{F}^{(1)\prime} = \mathfrak{F}^{(1)} \cup \{1\}$ and $\mathfrak{G}' = \mathfrak{G} \setminus \{1\}$. Then $\mathfrak{C}' = \mathfrak{C} \setminus \{0\}$, $\mathfrak{E}^{(2)\prime} = (\mathfrak{E}^{(2)} \cup \{0\}) \setminus \{2\}$, $\mathfrak{E}^{(1)\prime} = \mathfrak{E}^{(1)} \cup \{2\}$. Let $c'_0 = b_0 + 1$, $c'_1 = 0$ and $c'_2 = b_2$. We get

$$0 < b_0(1 - q_0 A_0) - b_0 + (b_0 + 1) q_0 A_0 + b_2 - (b_2 + 1) q_2 A_2 - b_2(1 - q_2 A_2) \\ = -a_2 q_1 A_1 < 0.$$

Lemma 9. *Let $j-2 \in \mathfrak{F}$, $j \in \mathfrak{G}$ and $j+2 \in \mathfrak{F}^{(2)}$. Then $A_j < 0$.*

Proof. Assume that $A_j > 0$. Lemma 4 of § 4 implies $b_{j+1} = a_{j+2}$. Therefore $b_j = 0$ and $b_{j-1} \neq 0$ (Proposition 4 (7) of § 4). We get $j-1 \in \mathfrak{A}$ and consider two cases:

(1) $j-2=i_k$ and $b_{j-1}=a_j$. Then we get $A_{j-2}>0$ and $c_{j-2}=1$. Let $\mathfrak{F}^{(2)\prime}=(\mathfrak{F}^{(2)}\cup\{j\})\setminus\{j-2\}$, $\mathfrak{G}'=((\mathfrak{G}\cup\{j-2\})\setminus\{j\}$, $c'_{j-2}=c'_{j-1}=0$ and $c'_j=1$. Then $\mathfrak{A}'=\mathfrak{A}\setminus\{j-1\}$, $\mathfrak{B}'=(\mathfrak{B}\setminus\{j-3,\ j+1\})\cup\{j-1\}$, $\mathfrak{C}'=\mathfrak{C}\cup\{j-3\}$, $\mathfrak{D}'=\mathfrak{D}\cup\{j+1\}$. Take $c'_{j+1}=0$ and $c'_{j+2}=a_{j+3}$. We get

$$\begin{aligned}0 &< -q_{j-2}A_{j-2}+(a_j-1)(1-q_{j-1}A_{j-1})+\\ &\quad +c_{j+1}(1-q_{j+1}A_{j+1})+q_jA_j+(a_{j+3}-c_{j+2})q_{j+2}A_{j+2}\\ &= -1+q_{j-1}A_{j-1}+c_{j+1}(1-q_{j+1}A_{j+1})+(a_{j+3}-c_{j+2})q_{j+2}A_{j+2}.\end{aligned}$$

Therefore $c_{j+2}=a_{j+3}-1$, $c_{j+1}=a_{j+2}$ and

$$\begin{aligned}0 &< -1+q_{j-1}A_{j-1}+a_{j+2}(1-q_{j+1}A_{j+1})+q_{j+2}A_{j+2}\\ &= -1+\frac{N_{j-1}}{q_j}+(q_j+q_{j-1})A_{j-1}\\ &= -1+\frac{N_{j-1}}{q_j}\left(\frac{1}{q_j}-(q_j+q_{j-1})|\alpha-r_j|\right)+(q_j+q_{j-1})\sum_{t=j+1}^{m}b_t(q_t\alpha-p_t)\\ &< -1+q_j\left(\frac{1}{q_j}-(q_j+q_{j-1})|\alpha-r_j|\right)+(q_j+q_{j-1})|q_j\alpha-p_j|=0.\end{aligned}$$

(2) $b_{j-1}=a_j\Rightarrow j-2>i_k$. Note that $b_{j-1}=a_j$ implies $A_{j-2}>0$ and therefore $j-2\in\mathfrak{F}^{(1)}$ and $c_{j-2}=0$ in this case. Let $\mathfrak{F}^{(1)\prime}=\mathfrak{F}^{(1)}\cup\{j\}$ and $\mathfrak{G}'=\mathfrak{G}\setminus\{j\}$, $c'_{j-1}=b_{j-1}$, $c'_j=0$, $c'_{j+1}=0$ and $c'_{j+2}=a_{j+3}$. We get

$$\begin{aligned}0 &< (b_{j-1}-1)(1-q_{j-1}A_{j-1})+c_{j+1}(1-q_{j+1}A_{j+1})+\\ &\qquad +(a_{j+3}-c_{j+2})q_{j+2}A_{j+2}-b_{j-1}(1-q_{j-1}A_{j-1}).\end{aligned}$$

If $c_{j+2}=a_{j+3}$ we get $0<-1+q_{j-1}A_{j-1}$, a contradiction. If $c_{j+2}=a_{j+3}-1$, then $c_{j+1}=a_{j+2}$ and $0<-1+q_{j-1}A_{j-1}+q_jA_j<0$ like in (1).

Lemma 10. *Let* $1\in\mathfrak{G}$ *and* $i_k=0$. *Then* $3\notin\mathfrak{F}^{(2)}$.

Proof. Assume that $3\in\mathfrak{F}^{(2)}$. Then $2\in\mathfrak{B}$ and therefore $b_2=a_3$, $b_1=0$, $A_0>0$, $0\in\mathfrak{C}$ and $c_0=b_0>0$. Note that $\{N\alpha\}\leqslant\{k\alpha\}$ (Proposition 2 of § 4). Remark (2) of § 3 implies $c_2=a_3$ and $c_3\leqslant b_3$, contrary to $3\in\mathfrak{F}^{(2)}$.

Proposition 1. *Let* $j>i_k$, j *odd and* $A_j>0$. *Then* $j\in\mathfrak{F}^{(1)}$.

Proof. This follows immediately from Lemmas 4, 5, 6, 7, 8, 9 and 10 and from Lemma 3 of § 4.

Remark 2. It follows from Remark 1 and Proposition 1 that $j>i_k$ and $j\in\mathfrak{B}$ implies $c_{j+1}=a_{j+2}$, $c_j=0$.

6. Determination of $\mathfrak{F}^{(1)}$, $\mathfrak{F}^{(2)}$ and $\mathfrak{G}$. Let $M=\{j\geqslant i_k|\ j$ odd, $A_j>0$ or $(b_j=0$ and $A_{j-1}<0)\}$. Proposition 8 of § 4 and Proposition 1 of § 5 imply $M\subseteq\mathfrak{F}^{(1)}\cup\{i_k\}$. In this section we prove that $M=\mathfrak{F}$ and therefore $\mathfrak{F}^{(2)}\subseteq\{i_k\}$.

Notation. To prove the statement made above we shall show the relation

$$\Omega(j)\colon\ j\in\mathfrak{F}\Rightarrow b_j=0 \text{ and } (A_j>0 \text{ or } A_{j-1}<0).$$

The proof is by contradiction. Assume that there is a j such that $\Omega(j)$ is false. By t we denote the largest such j. There exists a $j\leqslant t$, j odd, such that $\Omega(j-2)$, e.g. $j=1$. By s we denote the largest such j. Note that if j is odd and $s\leqslant j\leqslant t$, then $\Omega(j)$ is false.

Remark 1. *Let* $j>t$ *and let* j *be odd. Then* $c_j=0$.

Proof. If $j\in\mathfrak{G}$ there is nothing to prove. If $j\in\mathfrak{F}^{(1)}$ and $A_j>0$, then clearly $c_j=0$. If $A_j<0$, then $A_{j-1}<0$. Let l be the first integer $>j-1$ with $b_l\neq 0$. Then $l\in\mathfrak{G}\cup\mathfrak{F}^{(1)}$ by Proposition 7 of § 4. Because of $b_l\neq 0$ and $l>t$ we get $l\in\mathfrak{G}$. Proposition 7 (3) of § 4 implies $c_j=0$.

Remark 2. $c_{t+1}<a_{t+2}$.

Proof. Assume that $c_{t+1}=a_{t+2}$. Note that $c_t=b_t=0$, $t\in\mathfrak{F}^{(1)}$, $t+1\in\mathfrak{E}^{(1)}$ and that $b_{t+1}=a_{t+2}$ or $A_{t+1}<0$. The relation "$\Omega(t)$ is false" implies $A_{t-1}>0$. $b_{t+1}=a_{t+2}$ and $t+2\in\mathfrak{F}^{(1)}$ would imply $A_{t+2}>0$ and therefore $A_t>0$, a contradiction to "$\Omega(t)$ is false". Therefore this case is impossible. We consider two cases:

(1) $t-2\in\mathfrak{G}$. Then $c_{t-1}=b_{t-1}+1$. Let $\mathfrak{G}'=\mathfrak{G}\cup\{t\}$ and $\mathfrak{F}^{(1)\prime}=\mathfrak{F}^{(1)}\setminus\{t\}$. In the case $t=1$ and $i_k=0$ note that $b_0>0$. Furthermore $t-1\in\mathfrak{E}^{(2)}$ and $t-1\in\mathfrak{C}'$. Let $c'_{t-1}=b_{t-1}$. We get

$$0<b_{t-1}-(b_{t-1}+1)q_{t-1}A_{t-1}-b_{t-1}(1-q_{t-1}A_{t-1})=-q_{t-1}A_{t-1}<0.$$

(2) $t-2\in\mathfrak{F}$. Then $c_{t-1}=b_{t-1}$. Assume first that $b_{t-1}\neq 0$. Let $\mathfrak{G}'=\mathfrak{G}\cup\{t\}$, $\mathfrak{F}^{(1)\prime}=\mathfrak{F}^{(1)}\setminus\{t\}$. Then $t+1\in\mathfrak{E}^{(2)\prime}$, $t-1\in\mathfrak{A}'$ and $t-1\in\mathfrak{E}^{(1)}$. Let $c'_{t-1}=b_{t-1}-1$. Again we get

$$0<b_{t-1}(1-q_{t-1}A_{t-1})-(b_{t-1}-1)(1-q_{t-1}A_{t-1})-1=-q_{t-1}A_{t-1}<0.$$

Now let $b_{t-1}=0$. Here we note that $-a_{t+2}q_{t+1}A_{t+1}<-a_{t+1}q_tA_t-(a_{t+2}-1)q_{t+1}A_{t+1}$, for this is equivalent to $A_{t-1}>0$.

Remark 3. *In the following with* $[a,b]$ *we denote an interval in* $2Z+1$. *With this notation we prove:*

Let u *be odd,* $s\leqslant u\leqslant t$, $b_u\neq 0$ *or* $u=s$. *Let* $\mathfrak{G}'=\mathfrak{G}\cup[u,t]$, $\mathfrak{F}^{(1)\prime}=\mathfrak{F}^{(1)}\setminus[u,t]$ *and* $\mathfrak{F}^{(2)\prime}=\mathfrak{F}^{(2)}\setminus[u,t]$. *Then* $\{\mathfrak{F}^{(1)\prime},\ \mathfrak{F}^{(2)\prime},\ \mathfrak{G}'\}$ *satisfies the conditions of Proposition* 4 *of* § 4.

Proof. It is enough to prove the points (2), (3) and (7) of Proposition 4 of § 4.

Point (2): Let $j\in[u,t]$, $j+2\in\mathfrak{F}^{(1)\prime}$ and $b_{j+1}=a_{j+2}$. Then $b_j=0$. Note that $j+2$ fulfills the condition Ω when we replace $\mathfrak{F}$ by $\mathfrak{F}'$. Therefore $A_{j+2}>0$. This implies $A_j>0$.

Point (3): Let $j\in[u,t]$, $j+2\in\mathfrak{F}^{(1)\prime}$. Then $b_{j+2}=0<a_{j+3}$.

Point (7): Let $j\in[u,t]$, $j-2\in\mathfrak{F}'$ and $b_{j-1}=b_j=0$. Then $j\neq u$ if $u>s$. Furthermore $j-2\notin[u,t]$. This implies $u=s=j$ and therefore $\Omega(j-2)$. Note that $A_j<0$ and $A_{j-1}>0$. Furthermore $A_{j-3}<0$ or $A_{j-2}>0$. If $A_{j-3}<0$, then $b_{j-2}=0$ and therefore $A_{j-1}<0$, a contradiction. If $A_{j-2}>0$, then $A_j>0$, a contradiction too.

Remark 4. *$j\in\mathfrak{D}$ implies $c_{j+1}=a_{j+2}$ and $c_j=0$.*

Proof. Assume that $c_j=b_j-1>0$. We consider three cases and note that $j-1\in\mathfrak{F}$, $j+1\in\mathfrak{F}^{(2)}$.

(1) $b_j<a_{j+1}$ or ($j-1>i_k$ and $A_{j-1}>0$). Let $\mathfrak{F}^{(1)\prime}=\mathfrak{F}^{(1)}\cup\{j+1\}$, $\mathfrak{F}^{(2)\prime}=\mathfrak{F}^{(2)}\setminus\{j+1\}$. Then $j\in\mathfrak{E}^{(1)\prime}$. Let $c_j'=b_j$ and note that in the case $b_j=a_{j+1}$ we get $c_{j-1}=0$. Therefore $0<(b_j-1)(1-q_jA_j)-b_j(1-q_jA_j)=-1+q_jA_j<0$.

(2) $A_{j-1}<0$. Here it is enough to prove that $(a_{j+1}-1)(1-q_jA_j)-b_{j+1}-(a_{j+2}-1)q_{j+1}A_{j+1}<-b_{j-1}-a_{j+2}q_{j+1}A_{j+1}$. But this is equivalent to $-1+q_jA_j+q_{j-1}A_{j-1}<0$.

(3) $b_j=a_{j+1}$, $j-1=i_k$ and $A_{j-1}>0$. Let $\mathfrak{G}'=\mathfrak{G}\cup\{j-1\}$, $\mathfrak{F}^{(2)\prime}=\mathfrak{F}^{(2)}\setminus\{j-1\}$. Then $\mathfrak{B}'=(\mathfrak{B}\setminus\{j-2\})\cup\{j\}$, $\mathfrak{D}'=\mathfrak{D}\setminus\{j\}$, $\mathfrak{C}'=\mathfrak{C}\cup\{j-2\}$. Let $c_{j-1}'=c_j'=0$ and $c_{j+1}'=a_{j+2}$. Then $i_{k'}=i_k+2$ and we get

$$0<-q_{j-1}A_{j-1}+(a_{j+1}-1)(1-q_jA_j)+q_{j+1}A_{j+1}=-1+q_jA_j<0.$$

Lemma 1. *Let j be odd and $s\leqslant j\leqslant t$. Then $A_{j-1}>0$.*

Proof. Assume the contrary. Then there is a largest j, $s\leqslant j\leqslant t$, denoted by u, such that $A_{j-1}<0$. Note that $u\in\mathfrak{F}$, $b_u\neq 0$. Let $\mathfrak{G}'=\mathfrak{G}\cup[u,t]$, $\mathfrak{F}^{(1)\prime}=\mathfrak{F}^{(1)}\setminus[u,t]$ and $\mathfrak{F}^{(2)\prime}=\mathfrak{F}^{(2)}\setminus[u,t]$. Let $c_j'=0$ for $j\in[u,t]$ and $j\neq i_k$. Note that for even j, $u<j<t$ we have $A_j>0$. Let $c_j'=b_j$ if j is even and $u<j<t$. Let $c_{t+1}'=c_{t+1}+1$ (Remark 2), except in the case $t+2\in\mathfrak{G}$, $b_{t+1}=0$ and $A_{t+1}>0$. Furthermore let $c_{u-1}'=c_{u-1}$.

First we prove that the contribution of the difference $\sigma^{-1}(k)-N\{k\alpha\}-(\sigma^{-1}(k')-N\{k'\alpha\})$ furnished by $t+1$ is $\leqslant q_{t+1}A_{t+1}$.

If $t+2\in\mathfrak{G}$ and $b_{t+1}\neq 0$, then $t+1\in\mathfrak{A}$, $t+1\in\mathfrak{C}'$, $c_{t+1}=b_{t+1}-1$. We get

$$c_{t+1}(1-q_{t+1}A_{t+1})+1-c_{t+1}'(1-q_{t+1}A_{t+1})=q_{t+1}A_{t+1}.$$

If $t+2\in\mathfrak{G}$, $A_{t+1}>0$ and $b_{t+1}=0$, let $c_{t+1}'=0$. Then $c_{t+1}=0$ and we get

$$c_{t+1}(1-q_{t+1}A_{t+1})-c_{t+1}'(1-q_{t+1}A_{t+1})=0<q_{t+1}A_{t+1}.$$

If $t+2\in\mathfrak{G}$ and $A_{t+1}<0$, then $t+1\in\mathfrak{E}^{(1)}$ and $t+1\in\mathfrak{E}^{(2)\prime}$. We get

$$-(a_{t+2}-1)q_{t+1}A_{t+1}+a_{t+2}q_{t+1}A_{t+1}=q_{t+1}A_{t+1}.$$

If $t+2\in\mathfrak{F}^{(1)}$, then $t+1\in\mathfrak{E}^{(1)}$ and $t+1\in\mathfrak{E}^{(2)\prime}$. We get $c_{t+1}=a_{t+2}-1$ if $A_{t+1}<0$ (Remark 2) and $c_{t+1}=b_{t+1}+1$ if $A_{t+1}>0$. In the first case this leads to $b_{t+1}-(a_{t+2}-1)q_{t+1}A_{t+1}-b_{t+1}+a_{t+2}q_{t+1}A_{t+1}$ and in the second case we get $b_{t+1}(1-q_{t+1}A_{t+1})-b_{t+1}+(b_{t+1}+1)q_{t+1}A_{t+1}$ and this is $q_{t+1}A_{t+1}$ in any case.

Let us consider two cases:

(1) $u-2\in\mathfrak{G}$. Then $u=s$, $A_{s-1}<0$ and $b_s\neq 0$.

(1.1) $u=1$ and $i_k=0$. This would imply $A_0>0$ immediately.

(1.2) $u=i_k$. Then $u\in\mathfrak{F}^{(2)}$. If $A_{u+1}<0$ we would get $u=s=t$, $u+1\in\mathfrak{E}^{(2)\prime}$; if we choose $c_u'=0$ and $c_{u+1}'=a_{u+2}$ we have $i_{k'}=i_k+1$ and $0<-1-b_u-a_{u+1}q_uA_u+1+q_{u+1}A_{u+1}=q_{u-1}A_{u-1}<0$. Therefore $A_{u+1}>0$. If $c_{u+1}'=a_{u+2}$ we would get $u=s=t$, $b_{u+1}=a_{u+2}-1$, $u+1\in\mathfrak{E}^{(1)}$, $u+1\in\mathfrak{E}^{(2)\prime}$ and $c_{u+1}=b_{u+1}$, $i_{k'}=i_k+1$ and $0<-1-b_u-a_{u+1}q_uA_u+1+q_{u+1}A_{u+1}=q_{u-1}A_{u-1}<0$ (note that $c_{u+2}=0$). Therefore $A_{u+1}>0$ and $c_{u+1}'<a_{u+2}$. Let $c_u'=1$. Using Remark 4 we get

$$0<-1-\sum_{\substack{j=u\\ j\in\mathfrak{F}^{(2)}}}^{t}(b_j+a_{j+1}q_jA_j)-\sum_{\substack{j=u\\ j\in\mathfrak{F}^{(1)}}}^{t}(b_j+c_jq_jA_j)+\sum_{\substack{u<j<t\\ j+1\in\mathfrak{F}^{(1)}}}b_j(1-q_jA_j)-\sum_{\substack{u<j<t\\ 2|j}}b_j(1-q_jA_j)+1+q_uA_u+q_{t+1}A_{t+1}$$

$$\leqslant-\sum_{\substack{j=u\\ 2|j}}^{t}(b_j-a_{j+1}q_jA_j)-\sum_{\substack{u<j<t\\ j+1\in\mathfrak{F}^{(1)}}}b_j(1-q_jA_j)+q_uA_u+q_{t+1}A_{t+1}$$

$$=\sum_{\substack{j=u\\ 2|j}}^{t}(q_{t+1}A_{t+1}-q_{t-1}A_{t-1})+q_uA_u+q_{t+1}A_{t+1}=q_{u-1}A_{u-1}+q_uA_u.$$

(1.3) $u-2\geqslant i_k$. Lemma 6 of § 4 implies $u-2>i_k$. Note that $u-1\in\mathfrak{E}^{(2)}$, $u-1\in\mathfrak{E}^{(2)\prime}$, $c_{u-1}=c_{u-1}'=a_u$. We get

$$0<-\sum_{\substack{j=u\\ j\in\mathfrak{F}^{(2)}}}^{t}(b_j+a_{j+1}q_jA_j)-\sum_{\substack{j=u\\ j\in\mathfrak{F}^{(1)}}}^{t}(b_j+c_jq_jA_j)+\sum_{\substack{u<j<t\\ j+1\in\mathfrak{F}^{(1)}}}b_j(1-q_jA_j)-\sum_{\substack{u<j<t\\ 2|j}}b_j(1-q_jA_j)+q_{t+1}A_{t+1}\leqslant q_{u-1}A_{u-1}<0,$$

like above.

(2) $u-2\in\mathfrak{F}$. Note that $u-1\in\mathfrak{E}^{(1)}$, $u-1\in\mathfrak{E}^{(1)\prime}$, $c_{u-1}=c_{u-1}'$. Like above we get $0\leqslant q_{u-1}A_{u-1}<0$.

Lemma 2. *$s\neq i_k$.*

Proof. Assume that $s=i_k$. In a first step we prove that $b_{s-1}=a_s$ (resp. $b_0=a_1-1$ if $s=1$). We consider two cases:

(1) $s=1$. Let $\mathfrak{F}^{(1)\prime}=\mathfrak{F}^{(1)}\cup\{1\}$, $\mathfrak{F}^{(2)\prime}=\mathfrak{F}^{(2)}\setminus\{1\}$. Note that $A_0>0$ (Lemma 1) and that $A_1<0$. Therefore $b_0\neq 0$. If $b_0<a_1-1$ let $c_0'=b_0+1$

and $c_1' = a_2 - 1$. Then

$$0 < -b_0 + (b_0+1)q_0A_0 - 1 - b_1 - a_2q_1A_1 + 1 + b_1 + (a_2-1)q_1A_1 = -(a_1-b_0-1)q_0A_0 < 0.$$

We get $b_0 = a_1 - 1$.

(2) $s \geqslant 3$. We assume that $b_{s-1} < a_s$ and again we consider two cases:

(2.1) $b_{s-2} \neq 0$. Let $\mathfrak{G}' = \mathfrak{G} \cup [s,t]$, $\mathfrak{F}^{(1)\prime} = \mathfrak{F}^{(1)} \setminus [s,t]$, $\mathfrak{F}^{(2)\prime} = \mathfrak{F}^{(2)} \setminus [s,t]$, $c'_{s-2} = 1$ and $c'_{s-1} = b_{s-1}$. Note that, like in Lemma 1, the contribution of the difference $\sigma^{-1}(k) - N\{k\alpha\} - (\sigma^{-1}(k') - N\{k'\alpha\})$ furnished by $t+1$ is $\leqslant q_{t+1}A_{t+1}$. Therefore we get

$$\begin{aligned} 0 < -1 &- \sum_{\substack{j=s\\ j\in\mathfrak{F}^{(2)}}}^{t} (b_j + a_{j+1}q_jA_j) - \sum_{\substack{j=s\\ j\in\mathfrak{F}^{(1)}}}^{t} (b_j + c_jq_jA_j) + \\ &+ \sum_{\substack{j=s\\ j+1\in\mathfrak{F}^{(1)}}}^{t-1} b_j(1-q_jA_j) - \sum_{\substack{j=s\\ 2|j}}^{t} b_j(1-q_jA_j) + 1 + q_{s-2}A_{s-2} - \\ &- b_{s-1}(1-q_{s-1}A_{s-1}) + q_{t+1}A_{t+1} \\ \leqslant q_{s-2}A_{s-2} &+ q_{s-1}A_{s-1} - b_{s-1} - (a_s - b_{s-1})q_{s-1}A_{s-1} + a_sq_{s-1}A_{s-1} \\ = q_sA_s &- (a_s - b_{s-1} - 1)q_{s-1}A_{s-1} < 0. \end{aligned}$$

(2.2) $b_{s-2} = 0$. Let $\mathfrak{G}' = (\mathfrak{G} \cup [s,t]) \setminus \{s-2\}$, $\mathfrak{F}^{(1)\prime} = \mathfrak{F}^{(1)} \setminus [s,t]$, $\mathfrak{F}^{(2)\prime} = (\mathfrak{F}^{(2)} \setminus [s,t]) \cup \{s-2\}$, $c'_{s-2} = 1$ and let $c'_{s-1} = b_{s-1} - 1$ if $b_{s-1} \neq 0$, $c'_{s-1} = b_{s-1}$ if $b_{s-1} = 0$. Note that, like in Lemma 1, the contribution of the difference $\sigma^{-1}(k) - N\{k\alpha\} - (\sigma^{-1}(k') - N\{k'\alpha\})$ furnished by $t+1$ is $\leqslant q_{t+1}A_{t+1}$.

Note that $b_{s-1} \neq 0$ implies $s-1 \in \mathfrak{A}'$. Therefore we get

$$\begin{aligned} 0 < -1 &- \sum_{\substack{j=s\\ j\in\mathfrak{F}^{(2)}}}^{t} (b_j + a_{j+1}q_jA_j) - \sum_{\substack{j=s\\ j\in\mathfrak{F}^{(1)}}}^{t} (b_j + c_jq_jA_j) + \\ &+ \sum_{\substack{j=s\\ j+1\in\mathfrak{F}^{(1)}}}^{t-1} b_j(1-q_jA_j) - \sum_{\substack{j=s\\ 2|j}}^{t} b_j(1-q_jA_j) + \\ &+ 1 + q_{s-2}A_{s-2} - c'_{s-1}(1-q_{s-1}A_{s-1}) - (1-\delta_{0,b_{s-1}}) + q_{t+1}A_{t+1} \\ \leqslant q_{s-2}A_{s-2} &+ q_{s-1}A_{s-1} - c'_{s-1}(1-q_{s-1}A_{s-1}) - (1-\delta_{0,b_{s-1}}). \end{aligned}$$

If $b_{s-1} \neq 0$ we get

$$0 < q_{s-2}A_{s-2} - (b_{s-1}-1)(1-q_{s-1}A_{s-1}) - 1 + q_{s-1}A_{s-1} = q_sA_s - (a_s - b_{s-1})q_{s-1}A_{s-1} < 0.$$

If $b_{s-1} = 0$ we get

$$0 < q_{s-2}A_{s-2} + q_{s-1}A_{s-1} = (q_{s-2} + q_{s-1})A_s - \frac{q_s - q_{s-1} - q_{s-2}}{q_sq_{s-1}} N_{s-2} < 0.$$

Therefore $b_{s-1} = a_s$.

Now having proved this conclusion, in the case $s \geqslant 3$ we may conclude like in (2) to get $0 \leqslant q_sA_s < 0$.

If $s = 1$ we note that $b_0 \neq 0$. Let $c'_0 = b_0$, $\mathfrak{G}' = \mathfrak{G} \cup [s,t]$, $\mathfrak{F}^{(1)\prime} = \mathfrak{F}^{(1)} \setminus [s,t]$ and $\mathfrak{F}^{(2)\prime} = \mathfrak{F}^{(2)} \setminus [s,t]$. Then $i_{k'} = 0$ and we get

$$\begin{aligned} 0 < -1 &- \sum_{\substack{j=1\\ j\in\mathfrak{F}^{(2)}}}^{t} (b_j + a_{j+1}q_jA_j) - \sum_{\substack{j=1\\ j\in\mathfrak{F}^{(1)}}}^{t} (b_j + c_jq_jA_j) + \sum_{\substack{j=1\\ j+1\in\mathfrak{F}^{(1)}}}^{t-1} b_j(1-q_jA_j) - \\ &- \sum_{\substack{j=1\\ 2|j}}^{t} b_j(1-q_jA_j) + q_{t+1}A_{t+1} + 1 - (a_1-1)(1-q_0A_0) \\ \leqslant q_0A_0 &- (a_1-1)(1-q_0A_0) = q_1A_1 < 0. \end{aligned}$$

PROPOSITION 1. *$\mathfrak{F} = \{j \mid i_k \leqslant j \leqslant m,\ j$ odd, $A_j > 0$ or $(b_j = 0$ and $A_{j-1} < 0)\}$.*

Proof. Let $\mathfrak{G}' = \mathfrak{G} \cup [s,t]$, $\mathfrak{F}^{(1)\prime} = \mathfrak{F}^{(1)} \setminus [s,t]$, $\mathfrak{F}^{(2)\prime} = \mathfrak{F}^{(2)} \setminus [s,t]$. Like in Lemma 1 the contribution of the difference $\sigma^{-1}(k) - N\{k\alpha\} - (\sigma^{-1}(k') - N\{k'\alpha\})$ furnished by $t+1$ is $\leqslant q_{t+1}A_{t+1}$. If j is even, $s < j < t$, then $A_j > 0$ by Lemma 1. Let $c'_j = b_j$ for these j and let $c'_j = 0$ for $j \in [s,t]$.

We consider two cases:

(1) $s-2 \in \mathfrak{G}$. Here we have to consider three subcases:

(1.1) $s = 1$ and $i_k = 0$. Note that $A_0 > 0$ and that $A_1 < 0$. Therefore $b_0 \neq 0$ and $1 \in \mathfrak{F}^{(1)}$. We get $c_0 = b_0 + 1$. Let $c'_0 = b_0$ and note that $c_1 < a_2$. We get

$$\begin{aligned} 0 < b_0 &- (b_0+1)q_0A_0 - b_0(1-q_0A_0) - \sum_{\substack{j=1\\ j\in\mathfrak{F}^{(2)}}}^{t} (b_j + a_{j+1}q_jA_j) - \\ &- \sum_{\substack{j=1\\ j\in\mathfrak{F}^{(1)}}}^{t} (b_j + c_jq_jA_j) + \sum_{\substack{j=1\\ j+1\in\mathfrak{F}^{(1)}}}^{t-1} b_j(1-q_jA_j) - \sum_{\substack{j=1\\ 2|j}}^{t} b_j(1-q_jA_j) + q_{t+1}A_{t+1} < 0 \end{aligned}$$

like in Lemma 1.

(1.2) $s = i_k$. This case is impossible because of Lemma 2.

(1.3) $s-2 \geqslant i_k$. Note that $A_{s-1} > 0$, $s-1 \in \mathfrak{B} \cup \mathfrak{E}^{(2)}$ and that $c_{s-1} = 0$ or $= b_{s-1}+1$. Let $c'_{s-1} = b_{s-1}$. We get

$$0 < c_{\mathfrak{E}^{(2)}}(s-1)\big(b_{s-1}-(b_{s-1}+1)q_{s-1}A_{s-1}\big)-b_{s-1}(1-q_{s-1}A_{s-1})-c_{\mathfrak{B}}(s-1)-$$
$$-\sum_{\substack{j=s\\ j\in\mathfrak{F}^{(2)}}}^{t}(b_j+a_{j+1}q_jA_j)-\sum_{\substack{j=s\\ j\in\mathfrak{F}^{(1)}}}^{t}(b_j+c_jq_jA_j)+$$
$$+\sum_{\substack{j=s\\ j+1\in\mathfrak{F}^{(1)}}}^{t}b_j(1-q_jA_j)-\sum_{\substack{j=s\\ 2|j}}^{t}b_j(1-q_jA_j)+q_{t+1}A_{t+1}$$
$$\leqslant c_{\mathfrak{E}^{(2)}}(s-1)\big(b_{s-1}-(b_{s-1}+1)q_{s-1}A_{s-1}\big)-$$
$$-b_{s-1}(1-q_{s-1}A_{s-1})-c_{\mathfrak{B}}(s-1)+q_{s-1}A_{s-1}.$$

If $s-1 \in \mathfrak{E}^{(2)}$ note that $c_s < a_{s+1}$; we get a contradiction. If $s-1 \in \mathfrak{B}$ note that $s-2 > i_k$; we get $0 < -a_s(1-q_{s-1}A_{s-1})-1+q_{s-1}A_{s-1} < 0$, a contradiction too.

(2) $s-2 \in \mathfrak{F}$. Here we note that $A_{s-1} > 0$ and that $\Omega(s-2)$. This implies $A_{s-2} > 0$. $A_s < 0$ implies $b_{s-1} \neq 0$. We have $s-1 \in \mathfrak{D} \cup \mathfrak{E}^{(1)}$, $c_{s-1} = 0$ or $= b_{s-1}$. Let $c'_{s-1} = b_{s-1}-1$ and note that $s-1 \in \mathfrak{A}'$. We get $0 < c_{\mathfrak{E}^{(1)}}(s-1)b_{s-1}(1-q_{s-1}A_{s-1})-(b_{s-1}-1)(1-q_{s-1}A_{s-1})-1+q_{s-1}A_{s-1}$ like in (1.3). Therefore $s-1 \in \mathfrak{E}^{(1)}$. But then $c_s < a_{s+1}$ and we get a contradiction.

Remark 5. If j is odd, $j \neq i_k$, then $c_j = 0$. This is proved in the very same way as Remark 1 if one uses Proposition 1.

Remark 6. It follows from Proposition 1 that $\mathfrak{F}^{(2)} \subseteq \{i_k\}$. Therefore $\mathfrak{D} = \emptyset$ and $\mathfrak{B} \subseteq \{i_k-1\}$. This implies that for $j \in \mathfrak{B}$ we get $c_j = 0$.

7. Determination of i_k.

LEMMA 1. *Let $j = i_k \in \mathfrak{F}^{(2)}$. Then $A_j > 0$.*

Proof. Assume the contrary. Proposition 1 of § 6 implies $A_{j-1} < 0$ and $b_j = 0$. We get $c_j = a_{j+1}$ and $A_{j+1} < 0$ and $c_{j+1} < a_{j+2}$. Let $c'_j = 0$ and $c'_{j+1} = a_{j+2}$. Then $i_{k'} = i_k+1$ and

$$0 < -1-a_{j+1}q_jA_j+1+(a_{j+2}-c_{j+1})q_{j+1}A_{j+1}$$
$$= q_{j-1}A_{j-1}+(a_{j+2}-c_{j+1}-1)q_{j+1}A_{j+1} < 0.$$

Notation. Let

$$s = \min\{j \mid 1 \leqslant j \leqslant m,\ j \text{ odd},\ A_j > 0,\ A_{j+2} > 0 \Rightarrow b_{j+1} < a_{j+2}\}$$

and

$$t = \min\{j \mid 1 \leqslant j \leqslant m,\ j \text{ odd},\ A_{j-1} < 0 < A_{j+1},\ A_{j+2} > 0 \Rightarrow b_{j+1} < a_{j+2}-1\},$$

where we put $\min\emptyset = \infty$.

LEMMA 2. $i_k \leqslant \min(s, t)$.

Proof. We assume that $u := \min(s, t) < i_k$. Then i_k is odd. Let

$$\mathfrak{G}' = \mathfrak{G} \setminus \{j \mid u \leqslant j \leqslant i_k,\ j \text{ odd},\ A_j > 0 \text{ or } (b_j = 0 \text{ and } A_{j-1} < 0)\},$$
$$\mathfrak{F}^{(1)'} = \mathfrak{F}^{(1)} \cup \{j \mid u < j \leqslant i_k,\ j \text{ odd},\ A_j > 0 \text{ or } (b_j = 0 \text{ and } A_{j-1} < 0)\}.$$

Then $u = t$ implies $u \in \mathfrak{G}'$; if $u = s$ let $\mathfrak{F}^{(2)'} = \{u\}$. For simplicity we write i for i_k. Let $c'_u = 1$ and $c'_j = 0$ if j is odd, $j > u$. Note that $c_i = 1$ (Lemma 1). Then we get

$$0 \leqslant -1-q_iA_i+1+q_uA_u-\sum_{\substack{j=u+1\\ j\in\mathfrak{G}'\cup\mathfrak{E}^{(1)'},A_j>0}}^{i-1}b_j(1-q_jA_j)-$$
$$-\sum_{\substack{j=u+1\\ j\in\mathfrak{E}^{(2)'},A_j>0}}^{i-1}\big(b_j-(b_j+1)q_jA_j\big)-$$
$$-\sum_{\substack{j=u+1\\ j\in\mathfrak{A}'}}^{i-1}\big(b_j(1-q_jA_j)+q_jA_j\big)+\sum_{\substack{j=u+1\\ A_j<0,2|j}}^{i-1}a_{j+1}q_jA_j$$
$$= -q_iA_i+q_uA_u-\sum_{\substack{j=u+1\\ j\in\mathfrak{G}'\cup\mathfrak{E}^{(1)'},A_j>0}}^{i-1}(b_j-a_{j+1}q_jA_j)-$$
$$-\sum_{\substack{j=u+1\\ j\in\mathfrak{G}'\cup\mathfrak{E}^{(1)'},A_j>0}}^{i-1}(a_{j+1}-b_j)q_jA_j-\sum_{\substack{j=u+1\\ j\in\mathfrak{E}^{(2)'},A_j>0}}^{i-1}(b_j-a_{j+1}q_jA_j)-$$
$$-\sum_{\substack{j=u+1\\ j\in\mathfrak{E}^{(2)'},A_j>0}}^{i-1}(a_{j+1}-1-b_j)q_jA_j-\sum_{\substack{j=u+1\\ j\in\mathfrak{A}'}}^{i-1}(b_j-a_{j+1}q_jA_j)-$$
$$-\sum_{\substack{j=u+1\\ j\in\mathfrak{A}'}}^{i-1}(a_{j+1}+1-b_j)q_jA_j-\sum_{\substack{j=u+1\\ A_j<0,2|j}}^{i-1}(b_j-a_{j+1}q_jA_j)$$
$$\leqslant -\sum_{\substack{j=u+1\\ j\in\mathfrak{G}'\cup\mathfrak{E}^{(1)'},A_j>0}}^{i-1}(a_{j+1}-b_j)q_jA_j-\sum_{\substack{j=u+1\\ j\in\mathfrak{E}^{(2)'},A_j>0}}^{i-1}(a_{j+1}-1-b_j)q_jA_j-$$
$$-\sum_{\substack{j=u+1\\ j\in\mathfrak{A}'}}^{i-1}(a_{j+1}+1-b_j)q_jA_j.$$

All these sums are not negative.

Assume first that $u = s$. If $s+1 \in \mathfrak{A}'$, then the third sum is positive. If $s+1 \in \mathfrak{E}^{(1)'}$, then $b_{s+1} < a_{s+2}$ and therefore the first sum is positive. We get a contradiction.

If $u=t$, then $t+1\in\mathfrak{C}'\cup\mathfrak{E}^{(2)\prime}$. If $t+1\in\mathfrak{C}'$, then $b_t\neq 0$, $b_{t+1}<a_{t+2}$ and therefore the first sum is positive. If $t+1\in\mathfrak{E}^{(2)\prime}$, then $A_{t+2}>0$ and therefore $b_{t+1}<a_{t+2}-1$. This implies that the second sum is positive. Again we get a contradiction.

PROPOSITION 1. (1) *Let i_N be even. Then*

(1.1) *If $b_0=a_1-1$ and $A_1>0$ we get $i_k=s$;*

(1.2) *If $b_0<a_1-1$ or $A_1<0$ we get $i_k=0$.*

(2) *Let i_N be odd. Then $i_k=\min(s,t)$.*

Proof. (1.1) Proposition 4 (5) of § 4 implies $1\in\mathfrak{F}^{(1)}$; therefore $i_k\neq 0$. We get $k-1=(a_1-1)q_0+a_3q_2+\ldots+a_{i_k}q_{i_k-1}+(c_{i_k}-1)q_{i_k}+\ldots+c_mq_m$. $\{(k-1)\alpha\}<\{N\alpha\}$ implies $b_0=a_1-1$, $b_1=0,\ldots,b_{i_k-1}=a_{i_k}$, $b_{i_k}\leqslant c_{i_k}-1<c_{i_k}$. Therefore $i_k\in\mathfrak{F}^{(2)}$. Lemma 1 implies $A_{i_k}>0$. If $A_{i_k+2}>0$, then $i_k+1\in\mathfrak{E}^{(1)}$ and $i_k+2\in\mathfrak{F}^{(2)}$. Therefore $a_{i_k+2}>c_{i_k+1}\geqslant b_{i_k+1}$. We get $s\leqslant i_k$. Lemma 2 implies $i_k\leqslant s$.

(1.2) Assume that i_k is odd. Again $k-1=(a_1-1)q_0+a_3q_2+\ldots+a_{i_k}q_{i_k-1}+(c_{i_k}-1)q_{i_k}+\ldots+c_mq_m$ and $b_0=a_1-1$, $b_1=0,\ldots,b_{i_k-1}=a_{i_k}$, $b_{i_k}<c_{i_k}$. We get $i_k\in\mathfrak{F}^{(2)}$ and $A_{i_k}>0$. This implies $A_1>0$.

(2) Note that i_k is odd. If $i_k\in\mathfrak{F}^{(2)}$, then $A_{i_k}>0$ and $A_{i_k+2}>0\Rightarrow b_{i_k+1}<a_{i_k+2}$ like in (1.1). Therefore $s\leqslant i_k\leqslant\min(s,t)$ and $i_k=\min(s,t)$. If $i_k\in\mathfrak{G}$, then $A_{i_k-1}<0<A_{i_k+1}$ by Proposition 10 and Lemma 6 of § 4. If $A_{i_k+2}>0$, then $i_k+1\in\mathfrak{E}^{(2)}$ and therefore $1+b_{i_k+1}\leqslant c_{i_k+1}<a_{i_k+2}$. This implies $t\leqslant i_k\leqslant\min(s,t)$ and therefore $i_k=\min(s,t)$.

Remark 1. In the case (1.1) we have $s\leqslant t$ and therefore $i_k=\min(s,t)$ again.

8. The explicit formula for the discrepancy.

PROPOSITION 1. *Let α be irrational and $N\in\mathbf{N}$. Then*

$$ND_N^*(\alpha)=\max_{1\leqslant k\leqslant N}\left(\sigma^{-1}(k)-N\{k\alpha\}\right)+\max\Big(0,\,2\sum_{n=1}^{N}\{n\alpha\}-N\Big)$$
$$=\max_{1\leqslant k\leqslant N}\left(\sigma^{-1}(k)-N\{k\alpha\}\right)+\max\Big(0,\,A_0-\sum_{j=0}^{m}b_j\big((-1)^j-q_jA_j\big)\Big).$$

Proof. By Proposition 1 of § 1 we get

$$ND_N^*(\alpha)=\tfrac{1}{2}+\max_{1\leqslant k\leqslant N}\left|\sigma^{-1}(k)-N\{k\alpha\}-\tfrac{1}{2}\right|$$
$$=\max\Big(\max_{1\leqslant k\leqslant N}\left(\sigma^{-1}(k)-N\{k\alpha\}\right),\,1+\max_{1\leqslant k\leqslant N}\left(N\{k\alpha\}-\sigma^{-1}(k)\right)\Big)$$
$$=\text{(Proposition 1 of § 4)}$$
$$=\max\Big(\max_{1\leqslant k\leqslant N}\left(\sigma^{-1}(k)-N\{k\alpha\}\right),\,1+\max_{1\leqslant k\leqslant N}\left(\sigma^{-1}(k)-N\{k\alpha\}\right)-N-1+2\sum_{n=1}^{N}\{n\alpha\}\Big)$$
$$=\max_{1\leqslant k\leqslant N}\left(\sigma^{-1}(k)-N\{k\alpha\}\right)+\max\Big(0,\,2\sum_{n=1}^{N}\{n\alpha\}-N\Big).$$

The second formula follows from Corollary 1 of § 4.

Now let us summarize our notations:

Notation. Let $\alpha\in(0,1/2)$ be irrational with continued fraction expansion $[0;a_1,a_2,\ldots]$ and convergents p_n/q_n.

Let N be a positive integer.

Let $m\geqslant 0$ be chosen such that $q_m\leqslant N<q_{m+1}$.

Determine non-negative integers $b_0,\ldots,b_m,N_0,\ldots,N_{m-1}$ by the following algorithm: $N=b_mq_m+N_{m-1}$, $0\leqslant N_{m-1}<q_m$; $N_{m-1}=b_{m-1}q_{m-1}+N_{m-2}$, $0\leqslant N_{m-2}<q_{m-1}$; …; $N_0=b_0q_0$. Let $N_{m+1}=N_m=N$, $N_{-1}=0$.

For $0\leqslant j\leqslant m+2$ let $A_j=N_{j-1}(\alpha-p_j/q_j)+\sum_{t=j}^{m}b_t(q_t\alpha-p_t)$ and let $A_{-1}=0$.

Let i_N be the smallest j such that $b_j\neq 0$.

Let

$$s=\min\{j\mid 1\leqslant j\leqslant m,\ A_j>0,\ A_{j+2}>0\Rightarrow b_{j+1}<a_{j+2},\ j\text{ odd}\},$$
$$t=\min\{j\mid 1\leqslant j\leqslant m,\ A_{j-1}<0<A_{j+1},\ A_{j+2}>0\Rightarrow b_{j+1}<a_{j+2}-1,\ j\text{ odd}\},$$

where $\min\emptyset=\infty$.

Let

$$u=\begin{cases}0 & \text{if } i_N \text{ is even and } (b_0<a_1-1 \text{ or } A_1<0),\\ \min(s,t) & \text{else.}\end{cases}$$

With these notations the following explicit formula holds:

THEOREM 1.

$$ND_N^*(\alpha)=\sum_{\substack{j=u\\2|j}}^{m}b_j(1-q_jA_j)+\sum_{\substack{j=u\\A_{j+1}<0<A_{j-1}\\2|j}}^{m}q_jA_j-\sum_{\substack{j=u\\A_{j-1}\leqslant 0<A_{j+1}\\2|j}}^{m}q_jA_j-$$
$$-\sum_{\substack{j=u\\A_j<0,\,2|j}}^{m}a_{j+1}q_jA_j+(\delta_{u,0}-1)q_uA_u+\max\Big(0,\,A_0-\sum_{j=0}^{m}b_j\big((-1)^j-q_jA_j\big)\Big).$$

Proof. If $j\geqslant u$, $A_j<0$ and if j is even, then $c_j=a_{j+1}$. Therefore Remark 1 of § 4 and Proposition 1 imply

$$ND_N^*(\alpha)=\sum_{j\in\mathfrak{A}}\big(b_j(1-q_jA_j)+q_jA_j\big)+$$
$$+\sum_{\substack{j=u\\j\in\mathfrak{C}\cup\mathfrak{F}^{(1)},A_j>0}}^{m}b_j(1-q_jA_j)+\sum_{j\in\mathfrak{E}^{(2)},A_j>0}\big(b_j(1-q_jA_j)-q_jA_j\big)+$$
$$+\sum_{\substack{j=u\\A_j<0,\,2|j}}^{m}a_{j+1}q_jA_j+$$
$$+(\delta_{u,0}-1)q_uA_u+\max\Big(0,\,A_0-\sum_{j=0}^{m}b_j\big((-1)^j-q_jA_j\big)\Big).$$

Now it is easily seen that $j \in \mathfrak{A}$ if and only if $2|j$, $u \leqslant j$ and $A_{j+1} < 0 < A_{j-1}$. Furthermore $j \in \mathfrak{E}^{(2)}$ and $A_j > 0$ if and only if $2|j$, $j \geqslant u$ and $A_{j-1} \leqslant 0 < A_{j+1}$.

9. Corollaries and examples. Let us compute $ND_N^*(\pi)$ for $N = 1000000$. The continued fraction expansion is the following (see [8]): $\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, \ldots]$.

For $\pi - 3$ we get the following table:

$p_0 = 0$	$q_0 = 1$	$1000000 = 2 \cdot 364913 + 270174$
$p_1 = 1$	$q_1 = 7$	$270174 = 1 \cdot 265381 + 4793$
$p_2 = 15$	$q_2 = 106$	$4793 = 0 \cdot 99532 + 4793$
$p_3 = 16$	$q_3 = 113$	$4793 = 0 \cdot 66317 + 4793$
$p_4 = 4687$	$q_4 = 33102$	$4793 = 0 \cdot 33215 + 4793$
$p_5 = 4703$	$q_5 = 33215$	$4793 = 0 \cdot 33102 + 4793$
$p_6 = 9390$	$q_6 = 66317$	$4793 = 42 \cdot 113 + 47$
$p_7 = 14093$	$q_7 = 99532$	$47 = 0 \cdot 106 + 47$
$p_8 = 37576$	$q_8 = 265381$	$47 = 6 \cdot 7 + 5$
$p_9 = 51669$	$q_9 = 364913$	$5 = 5 \cdot 1 + 0$
$p_{10} = 192583$	$q_{10} = 1360120$	

Therefore $b_0 = 5$, $b_1 = 6$, $b_2 = 0$, $b_3 = 42$, $b_4 = b_5 = b_6 = b_7 = 0$, $b_8 = 1$, $b_9 = 2$, $m = 9$, $N_0 = 5$, $N_1 = N_2 = 47$, $N_3 = N_4 = N_5 = N_6 = N_7 = 4793$, $N_8 = 270174$, $N_9 = 1000000$. Because of $i_N = 0$ and $b_0 = 5 < a_1 - 1 = 6$ we get $u = 0$. Now let us determine the numbers $q_j A_j$:

$$
\begin{aligned}
q_0 A_0 &= 10^6(\pi-3) - 142592 = 0.653589793 \ldots \\
q_1 A_1 &= 7 \cdot 10^6(\pi-3) - 991149 = -0.424871447 \ldots \\
q_2 A_2 &= 106 \cdot 10^6(\pi-3) - 15008821 = 0.280518083 \ldots \\
q_3 A_3 &= 113 \cdot 10^6(\pi-3) - 15999970 = -0.144353364 \ldots \\
q_4 A_4 &= 33102 \cdot 10^6(\pi-3) - 4687000019 = 0.129335779 \ldots \\
q_5 A_5 &= 33215 \cdot 10^6(\pi-3) - 4702999989 = -0.015017584 \ldots \\
q_6 A_6 &= 66317 \cdot 10^6(\pi-3) - 9390000008 = 0.114318195 \ldots \\
q_7 A_7 &= 99532 \cdot 10^6(\pi-3) - 14092999997 = 0.099300610 \ldots \\
q_8 A_8 &= 265381 \cdot 10^6(\pi-3) - 37576000002 = 0.312919416 \ldots \\
q_9 A_9 &= 364913 \cdot 10^6(\pi-3) - 51669000000 = -0.587779972 \ldots
\end{aligned}
$$

From this we deduce (§ 4, Corollary 1) that

$$2 \sum_{n=1}^{10^6} \{n\pi\} - 10^6 = (10^6+1)\,10^6(\pi-3) - 141592795144 = 38.4468282558 \ldots$$

Theorem 1 implies

$$
\begin{aligned}
10^6 D_{10^6}^*(\pi) &= b_0(1 - q_0 A_0) + b_8(1 - q_8 A_8) + q_8 A_8 - q_6 A_6 + 2 \sum_{n=1}^{10^6} \{n\pi\} - 10^6 \\
&= 933679 \cdot 10^6(\pi-3) - 132202087170 = 41.064561094 \ldots
\end{aligned}
$$

Note that $\mathfrak{F}^{(1)} = \{7\}$, $\mathfrak{F}^{(2)} = \emptyset$, $\mathfrak{G} = \{-1, 1, 3, 5, 9\}$, $\mathfrak{A} = \{8\}$, $\mathfrak{B} = \mathfrak{D} = \emptyset$, $\mathfrak{C} = \{0, 2, 4\}$, $\mathfrak{E}^{(1)} = \emptyset$, $\mathfrak{E}^{(2)} = \{6\}$.

$\max_{1 \leqslant k \leqslant 10^6} (\sigma^{-1}(k) - 10^6 \{k\pi\})$ is attained at $k = 933679$ and

$\max_{1 \leqslant k \leqslant 10^6} |\sigma^{-1}(k) - 10^6 \{k\pi\} - \frac{1}{2}|$ is attained at $k = 10^6 - 933679 + 1 = 66322$.

Now we deduce some corollaries from Theorem 1 of § 8:

COROLLARY 1. *Let α be irrational, $m \in N$ and $1 \leqslant b \leqslant a_{m+1}$, $b \in N$. Then*

$$bq_m D_{bq_m}^*(\alpha) = b - (b-1)\, bq_m |q_m \alpha - p_m| - b |q_m \alpha - p_m|.$$

Proof. If m is even, then $u = 0$. If m is odd, then $s = \infty$, $t = m$ and therefore $u = m$. Furthermore

$$2 \sum_{n=1}^{bq_m} \{n\alpha\} - bq_m = A_0 - b((-1)^m - q_m A_m) = A_m - b((-1)^m - q_m A_m),$$

where $A_m = b(q_m \alpha - p_m)$. It is easily seen that $A_m - b((-1)^m - q_m A_m) > 0$ if and only if m is odd. From now on we assume that m is even. Then

$$
\begin{aligned}
bq_m D_{bq_m}^*(\alpha) &= b(1 - q_m A_m) + q_m A_m - q_0 A_0 \\
&= b - (b-1)\, bq_m (q_m \alpha - p) - b(q_m \alpha - p_m).
\end{aligned}
$$

COROLLARY 2. *Let α be irrational. Then $\lim_{N \to \infty} ND_N^*(\alpha) = 1$.*

Proof. Take $b = 1$ in Corollary 1 to get $\underline{\lim}_{N \to \infty} ND_N^*(\alpha) \leqslant 1$. To prove the converse inequality let $\varepsilon > 0$ and $N > 1/\varepsilon$. There is an n, $1 \leqslant n \leqslant N$, and a $p \in Z$ such that $|nN\alpha - p| < 1/N < \varepsilon$. We get

$$
\begin{aligned}
|\sigma^{-1}(n) - N\{n\alpha\} - \tfrac{1}{2}| &= |\sigma^{-1}(n) + N[n\alpha] - p - Nn\alpha - \tfrac{1}{2} + p| \\
&\geqslant |\sigma^{-1}(n) + N[n\alpha] - p - \tfrac{1}{2}| - |Nn\alpha - p| \geqslant \tfrac{1}{2} - \varepsilon.
\end{aligned}
$$

Therefore

$$ND_N^*(\alpha) \geqslant \tfrac{1}{2} + |\sigma^{-1}(n) - N\{n\alpha\} - \tfrac{1}{2}| \geqslant 1 - \varepsilon$$

if N is large enough.

Now we give a formula for $ND_N^*(\alpha)$ which is good enough for almost all α.

COROLLARY 3.

$$ND_N^*(\alpha) = \max\left(\sum_{\substack{j=0 \\ 2|j}}^{m} b_j\left(1 - \frac{b_j}{a_{j+1}}\right), \sum_{\substack{j=0 \\ 2\nmid j}}^{m} b_j\left(1 - \frac{b_j}{a_{j+1}}\right)\right) + O(m),$$

where the O-constant is an absolute one.

Proof.

$$q_j A_j = N_j(q_j\alpha-p_j)+q_j\sum_{t=j+1}^{m} b_t(q_t\alpha-p_t)$$
$$= b_j q_j(q_j\alpha-p_j)+O(N_{j-1}|q_j\alpha-p_j|+q_j|q_j\alpha-p_j|)$$
$$= b_j q_j(-1)^j|q_j\alpha-p_j|+O(q_j|q_j\alpha-p_j|)$$
$$= b_j q_j(-1)^j\frac{1}{q_{j+1}}+O\left(b_j q_j\left(\frac{1}{q_{j+1}}-\frac{1}{q_j\alpha_{j+1}+q_{j-1}}\right)+\frac{q_j}{q_{j+1}}\right)$$
$$= b_j\frac{(-1)^j}{a_{j+1}}+O\left(\frac{1}{a_{j+1}}-\frac{q_j}{q_{j+1}}+\frac{b_j q_j^2}{\alpha_{j+2}q_{j+1}^2}+\frac{1}{a_{j+1}}\right)$$
$$= b_j\frac{(-1)^j}{a_{j+1}}+O\left(\frac{b_j}{a_{j+1}^2}+\frac{1}{a_{j+1}}\right)= b_j\frac{(-1)^j}{a_{j+1}}+O\left(\frac{1}{a_{j+1}}\right),$$

where we put $\alpha_0=\alpha$ and $\alpha_{j+1}=\dfrac{1}{\alpha_j-a_j}$. Note that the O-constant is absolute. Now we prove that we may assume that $u=0$. If i_N is even, $b_0=a_1-1$ and $A_1>0$, then take $N-1$ instead of N: i_{N-1} is even, $b_0'=b_0-1<a_1-1$ and $|ND_N^*(\alpha)-(N-1)D_{N-1}^*(\alpha)|\leqslant 1$. If i_N is odd, take $N+1$ instead of N and use that $i_{N+1}=0$. Theorem 1 of § 8 implies

$$ND_N^*(\alpha)=\sum_{\substack{j=0\\ 2|j}}^{m} b_j\left(1-\frac{b_j}{a_{j+1}}\right)+\max\left(0,\ -\sum_{j=0}^{m} b_j(-1)^j\left(1-\frac{b_j}{a_{j+1}}\right)\right)+O(m).$$

Note that for almost all α $m=o(\log N)$, while $ND_N^*(\alpha)\neq O(\log N)$ for almost all α (see Corollary 5 below). For example:

Corollary 4.

$$\overline{\lim_{N\to\infty}}\left(\frac{\log\log N}{\log N}\right)^2 ND_N^*(e)=\frac{1}{8}.$$

Proof. Note that $e-2=[0;1,2,1,1,4,1,1,6,1,1,8,1,\ldots]$, that is $a_{3t+1}=a_{3t+3}=1$ and $a_{3t+2}=2(t+1)$ for $t\geqslant 0$. Therefore

$ND_N^*(e)$

$$=\max\left(\sum_{0\leqslant t\leqslant(m-4)/6} b_{6t+4}\left(1-\frac{b_{6t+4}}{4t+4}\right),\ \sum_{0\leqslant t\leqslant(m-1)/6} b_{6t+1}\left(1-\frac{b_{6t+1}}{4t+2}\right)\right)+O(m).$$

Of course, to get this maximum as large as possible we have to choose b_{6t+1}

$=2t+1$, $b_{6t+4}=2t+2$ and $b_t=0$ if $t\equiv 0,\ 2,\ 3,\ 5 \pmod 6$. Then we get $ND_N^*(e)=m^2/72+O(m)$. Note that

$$\log q_m=\sum_{j=1}^{m}\log a_j+O\left(\sum_{j=1}^{m}\log\left(1+\frac{1}{a_j}\right)\right)$$
$$=\sum_{0\leqslant j\leqslant(m-2)/3}\log 2(j+1)+O(m)=\sum_{0\leqslant j\leqslant(m-2)/3}\log(j+1)+O(m)$$
$$=\frac{m}{3}\log\frac{m}{3}+O(m)=\frac{m}{3}\log m+O(m).$$

Therefore

$$\log q_{m+1}=\frac{m}{3}\log m+O(m).$$

We get $3\log N=m\log m+O(m)$. This implies

$$m=\frac{3\log N}{\log(3\log N)}+O\left(\frac{\log N}{\log^2\log N}\right).$$

Therefore

$$ND_N^*(e)=\frac{1}{8}\frac{\log^2 N}{\log^2\log N}+O\left(\frac{\log N}{\log\log N}\right)$$

for these N.

Now we determine all α with best possible discrepancy:

Corollary 5. *Let* $\alpha=[0;a_1,a_2,\ldots]$ *be irrational. The following conditions are equivalent*:

(1) $ND_N^*(\alpha)=O(\log N)$.

(2) $\left(\frac{1}{m}\sum_{j=1}^{m}a_j\right)_{m\in\mathbb{N}}$ *is bounded.*

Proof. (1) $\Rightarrow$ (2). Let m be even, $b_i=\left[\dfrac{a_{i+1}}{2}\right]+1$ if i is even, $b_i=0$ if i is odd, $0\leqslant i\leqslant m$. Then

$$ND_N^*(\alpha)=\sum_{\substack{i=0\\ 2|i}}^{m}\left[\frac{a_{i+1}}{2}\right]\left(1-\frac{1}{a_{i+1}}\left[\frac{a_{i+1}}{2}\right]\right)+O(m)\geqslant\frac{1}{4}\sum_{\substack{i=0\\ 2|i}}^{m}a_{i+1}+O(m).$$

Therefore $\sum_{\substack{i=0\\ 2|i}}^{m}a_{i+1}=O(\log N)=O(\log q_{m+1})$. Similarly $\sum_{\substack{i=0\\ 2\nmid i}}^{m}a_{i+1}=O(\log q_{m+1})$. We get $\sum_{i=0}^{m}a_{i+1}=O(\log q_{m+1})$. On the other hand

$$\log q_{m+1}=O\Big(\sum_{i=0}^{m+1}\log(a_i+1)\Big)=O\Big(\sum_{i=1}^{m+1}\sqrt{a_i}\Big)=O\Big(\Big(\sum_{i=1}^{m+1}a_i\Big)^{1/2}\sqrt{m}\Big).$$

This implies

$$\sum_{i=1}^{m+1} a_i = O\left(\sqrt{m}\sqrt{\sum_{i=1}^{m+1} a_i}\right).$$

(2) ⇒ (1). Note that for $0 \leq x \leq a$ $x\left(1-\frac{x}{a}\right) \leq \frac{a}{4}$. Therefore Corollary 3 implies

$$ND_N^*(\alpha) = \max\left(\sum_{\substack{i=0\\2|i}}^{m} b_i\left(1-\frac{b_i}{a_{i+1}}\right), \sum_{\substack{i=0\\2\nmid i}}^{m} b_i\left(1-\frac{b_i}{a_{i+1}}\right)\right)+O(m)$$

$$= O\left(\sum_{i=0}^{m} \frac{a_{i+1}}{4}+m\right) = O(m) = O(\log N).$$

10. Notes. The representation of N as $\sum_{i=0}^{m} b_i q_i$ was often used (see [2], [7], [10]).

Theorem 1 of § 2 was first proved in [6]. See also [3]. Our analytic method enables us to give a very short proof for it.

The explicit formula for σ^{-1} (§ 3) is easily obtained from the main result in [11] (and is in fact equivalent to it). V. T. Sós proved her theorem by the "art of counting". Our more analytic method is completely different and gives us this shorter proof.

Let k be chosen such that $\max_{1\leq n\leq N} (\sigma^{-1}(n)-N\{n\alpha\})$ is attained at k. We used that $\min_{1\leq n\leq N} (\sigma^{-1}(n)-N\{n\alpha\})$ is attained at $N-k+1$, as it follows from the formula of Proposition 1 of § 4, immediately. This fact is in close connection with Theorem 2 of [9].

Corollary 3 of § 9 should be compared with the formula given in [4].

Note that one can use the proof in [7] to get the result of (2) ⇒ (1) in Corollary 5 of § 9. This corollary should also be compared with [1].

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