Also ist wegen (5)

$$(19)\qquad N_j N_{j+1}^{Y-1} \ll |u_j|.$$

Man wählt $\delta_j := c_8 [1 + N_{j+1}^{Z} N_j^{-1}]$, $c_8 \in N$, und setzt $\mathfrak{X}_j := \delta_j \mathfrak{G}_j$. Dann zeigen (19) und (12), dass für genügend grosses c_8 gilt

$$(20)\qquad \mathfrak{x}_j \in \Phi(\varrho, 0).$$

Wäre $N_{j+1}^{Z} < N_j$ für unendlich viele j, so $\delta_j = c_8$ und $c_8 \mathfrak{g}_j \in \Phi(\varrho, 0)$, sowie nach dem Lemma $|L(c_8 \mathfrak{G}_j)| \ll N_{j+1}^{-Y} \ll N_j^{-2} \ll \langle c_8 \mathfrak{g}_j \rangle^{-r}$ unendlich oft. Dies widerspricht (7), so dass man

$$N_{j+1}^{Z} \geqslant N_j$$

annehmen darf für alle genügend grossen j. Dann ist $\delta_j \ll N_{j+1}^{Z} N_j^{-1}$ und mit (10) bis (13), sowie (4)

$$\langle \mathfrak{x}_j \rangle \ll N_{j+1}^{Z},$$

$$|L(\mathfrak{X}_j)| \ll N_{j+1}^{Z-Y} N_j^{-1} \ll N_{j+1}^{Z-Y-X} \ll N_{j+1}^{-rZ} \ll \langle \mathfrak{x}_j \rangle^{-r}.$$

Für grosse j widerspricht dies, zusammen mit (20), erneut (7). Damit ist der Satz bewiesen.

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Dedekind type sums and Hecke operators

by

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1. Introduction. For $\alpha, \gamma \in \boldsymbol{Z}$ with $\gamma > 0$, the Dedekind sum $s(\alpha, \gamma)$ is defined by

$$s(\alpha, \gamma) = \sum_{\mu (\mathrm{mod}\, \gamma)} \left(\left(\frac{\mu}{\gamma}\right)\right)\left(\left(\frac{\alpha\mu}{\gamma}\right)\right),$$

where $((x)) = x - [x] - 1/2$ is $x \notin \boldsymbol{Z}$ and $((x)) = 0$ if $x \in \boldsymbol{Z}$. We regard $s(\alpha, \gamma)$ as a function on $\boldsymbol{Z} \times \boldsymbol{N}$ and call it the *homogeneous Dedekind sum*. In fact, it is well known [10] that

$$(1)\qquad s(\alpha + \gamma, \gamma) = s(\alpha, \gamma),$$

$$(2)\qquad s(n\alpha, n\gamma) = s(\alpha, \gamma), \qquad n \in \boldsymbol{N}.$$

For this function Dedekind [3] discovered the following relation: for a prime number p

$$(3)\qquad s(p\alpha, \gamma) + \sum_{b=0}^{p-1} s(\alpha + b\gamma, p\gamma) = (p+1)\, s(\alpha, \gamma).$$

Recently Knopp [5] obtained an extension of (3): for every $n \in \boldsymbol{N}$

$$(4)\qquad \sum_{\substack{ad = n \\ d > 0}} \sum_{b=0}^{d-1} s(a\alpha + b\gamma, d\gamma) = \sigma(n)\, s(\alpha, \gamma),$$

where $\sigma(n)$ means the sum of all positive divisors of n. However, before Knopp, Subrahmanyam [11] had given another extension of (3): for every $n \in N$

$$(5)\qquad \sum_{b=0}^{n-1} s(\alpha + b\gamma, n\gamma) = \sum_{d \mid n} \mu(d)\, \sigma\left(\frac{n}{d}\right) s(d\alpha, \gamma),$$

where $\mu(n)$ means Möbius' function. We shall see later that (4) is equivalent to (5). Goldberg [4] has already derived (4) from (5).

Following Dedekind we define

$$D\left(\frac{\alpha}{\gamma}\right) = s(\alpha, \gamma).$$

Then the function $D(x)$ on $\boldsymbol{Q}$ is well-defined because of (2) and it is periodic with period 1 from (1). We call $D(x)$ the *inhomogeneous* Dedekind sum. When the relations (3), (4), and (5) are expressed in terms of $D(x)$, the meaning appears more clearly. Namely,

$$D(px)+\sum_{b=0}^{p-1} D\left(\frac{x+b}{p}\right)=\sigma(p)\,D(x); \tag{3'}$$

$$T_0(n)\,D(x)=\frac{1}{n}\sigma(n)\,D(x); \tag{4'}$$

$$\sum_{b=0}^{n-1} D\left(\frac{x+b}{n}\right)=\sum_{d|n}\mu(d)\,\sigma\left(\frac{n}{d}\right)D(dx), \tag{5'}$$

where $T_0(n)$ is the ordinary Hecke operator of weight 0. Parson [8] really pointed out (4′).

In this paper we extend the above relations to certain Dedekind type sums. Our results also contain those of Carlitz [2], Theorems 3 and 6, Parson [8], Parson and Rosen [9], Apostol and Vu [1], and the author [7], Section 5.

2. Homogeneous Dedekind type sums. Let K be a field of characteristic 0 and V an algebra over K. For each $i=1, 2$ let $Q_i(x)$ be a periodic V-valued function of period 1 on $\boldsymbol{Q}$ and satisfy the relation: for any $n\in\boldsymbol{N}$

$$n^{k_i}\sum_{b=0}^{n-1} Q_i\left(x+\frac{b}{n}\right)=Q_i(nx). \tag{6}$$

Let both χ_1 and χ_2 be Dirichlet characters modulo N. For the system

$$\mathscr{Q}=(Q_1(x),\,Q_2(x);\,\chi_1,\,\chi_2)$$

we define the function $S_{\mathscr{Q}}(\alpha, \gamma)$ on $\boldsymbol{Z}\times\boldsymbol{N}$ by

$$S_{\mathscr{Q}}(\alpha,\gamma)=\sum_{\mu(\mathrm{mod}\,\gamma N)}\ \sum_{\nu(\mathrm{mod}\,N)}\chi_1(\mu)\,\chi_2(\nu)\,Q_1\left(\frac{\mu}{\gamma N}\right)Q_2\left(\frac{\alpha\mu+\gamma\nu}{\gamma N}\right).$$

We call this the *homogeneous Dedekind type sum associated with* $\mathscr{Q}$. In fact, $S_{\mathscr{Q}}(\alpha, \gamma)$ has the following properties.

LEMMA 1. *Let* $n\in\boldsymbol{N}$. *Then*

$$S_{\mathscr{Q}}(\alpha+N\gamma,\gamma)=S_{\mathscr{Q}}(\alpha,\gamma), \tag{7}$$

$$S_{\mathscr{Q}}(n\alpha,n\gamma)=n^{-k_1}S_{\mathscr{Q}}(\alpha,\gamma). \tag{8}$$

Proof. The property (7) is immediate from the definition. We prove (8). By (6), $S_{\mathscr{Q}}(n\alpha, n\gamma)$ equals

$$\sum_{\mu(\mathrm{mod}\,n\gamma N)}\ \sum_{\nu(\mathrm{mod}\,N)}\chi_1(\mu)\,\chi_2(\nu)\,Q_1\left(\frac{\mu}{n\gamma N}\right)Q_2\left(\frac{\alpha\mu+\gamma\nu}{\gamma N}\right)$$

$$=\sum_{\substack{\varkappa(\mathrm{mod}\,n)\\ \lambda(\mathrm{mod}\,\gamma N)\\ \nu(\mathrm{mod}\,N)}}\chi_1(\lambda)\,\chi_2(\nu)\,Q_1\left(\frac{\varkappa\gamma N+\lambda}{n\gamma N}\right)Q_2\left(\frac{\alpha(\varkappa\gamma N+\lambda)+\gamma\nu}{\gamma N}\right)$$

$$=\sum_{\substack{\lambda(\mathrm{mod}\,\gamma N)\\ \nu(\mathrm{mod}\,N)}}\chi_1(\lambda)\,\chi_2(\nu)\left(\sum_{\varkappa(\mathrm{mod}\,n)}Q_1\left(\frac{\lambda}{n\gamma N}+\frac{\varkappa}{n}\right)\right)Q_2\left(\frac{\alpha\lambda+\gamma\nu}{\gamma N}\right)$$

$$=\sum_{\lambda(\mathrm{mod}\,\gamma N)}\ \sum_{\nu(\mathrm{mod}\,N)}\chi_1(\lambda)\,\chi_2(\nu)\,n^{-k_1}Q_1\left(\frac{\lambda}{\gamma N}\right)Q_2\left(\frac{\alpha\lambda+\gamma\nu}{\gamma N}\right),$$

which is exactly the right-hand side of (8).

EXAMPLES. (a) For the system $\mathscr{Q}=(((x)),\,((x));\,\varepsilon,\,\varepsilon)$ with ε the Dirichlet character modulo 1, $S_{\mathscr{Q}}(\alpha, \gamma)=s(\alpha, \gamma)$.

(b) Let $\mathscr{Q}=(P_m(x),\,P_{2k-m}(x);\,\varepsilon,\,\varepsilon)$, where $k\in\boldsymbol{N}$, $0\leqslant m\leqslant 2k$, $m\in\boldsymbol{Z}$, $P_m(x)$ are Bernoulli functions, $P_1(x)=((x))$. Then $S_{\mathscr{Q}}(\alpha, \gamma)=S_{2k}^{(m)}(\alpha, \gamma)$ are investigated by Carlitz [2], Parson and Rosen [9], and Parson [8].

(c) Provided that $\mathscr{Q}=(P_m(x),\,P_{2k-m}(x)/\tau(\chi);\,\chi,\,\chi)$ where χ is primitive and $\tau(\chi)$ is the normalized Gauss sum attached to χ, $S_{\mathscr{Q}}(\alpha, \gamma)=S_{2k,\chi}^{(m)}(\alpha, \gamma)$ with some restriction are studied in [7].

(d) Apostol and Vu [1] made a discussion on $S_{\mathscr{Q}}(\alpha, \gamma)$ with $\chi_1=\chi_2=\varepsilon$.

We say that a function $f(x)$ is *of parity* δ ($\delta=0$ or 1) if $f(-x)=(-1)^{\delta}f(x)$ for any x in its domain.

PROPOSITION 1. *Let i be either 1 or 2. If χ_i and $Q_i(x)$ are of parity δ_i and of parity δ_i' respectively, then*

$$S_{\mathscr{Q}}(-\alpha,\gamma)=(-1)^{\delta_i+\delta_i'}S_{\mathscr{Q}}(\alpha,\gamma).$$

Proof. When $i=1$, $S_{\mathscr{Q}}(-\alpha, \gamma)$ equals

$$\sum_{\mu(\mathrm{mod}\,\gamma N)}\ \sum_{\nu(\mathrm{mod}\,N)}\chi_1(\mu)\,\chi_2(\nu)\,Q_1\left(\frac{\mu}{\gamma N}\right)Q_2\left(\frac{-\alpha\mu+\gamma\nu}{\gamma N}\right)$$

$$=\sum_{\substack{\mu(\mathrm{mod}\,\gamma N)\\ \nu(\mathrm{mod}\,N)}}(-1)^{\delta_1}\chi_1(-\mu)\,\chi_2(\nu)(-1)^{\delta_1'}Q_1\left(\frac{-\mu}{\gamma N}\right)Q_2\left(\frac{\alpha(-\mu)+\gamma\nu}{\gamma N}\right)$$

$$=(-1)^{\delta_1+\delta_1'}S_{\mathscr{Q}}(\alpha,\gamma).$$

Similarly we can prove the case $i=2$.

COROLLARY. *Let $Q_1(x)$ and $Q_2(x)$ be of parity δ_1' and of parity δ_2' respectively. Then $S_{\mathscr{Q}}(\alpha, \gamma)$ always vanishes unless*

$$\delta_1+\delta_1'\equiv\delta_2+\delta_2'\ (\mathrm{mod}\,2).$$

3. Extension of Dedekind's relation and inhomogeneous Dedekind type sums. We can extend the Dedekind relation (3) to the case of our Dedekind type sums. Namely, we have

LEMMA 2. *For a prime number p not dividing N*

$$\sum_{b=0}^{p-1} S_{\mathscr{Q}}(\alpha+Nb\gamma,\, p\gamma) \tag{9}$$
$$= \big(\chi_1(p)\,p+\bar{\chi}_2(p)\,p^{-k_1-k_2}\big) S_{\mathscr{Q}}(\alpha,\gamma)-\chi_1(p)\,\bar{\chi}_2(p)\,p^{-k_2} S_{\mathscr{Q}}(p\alpha,\gamma),$$

where $\bar{\chi}_2$ means the inverse of χ_2.

Proof. By (6) the left-hand side of (9) equals

$$\sum_{b=0}^{p-1}\sum_{\substack{\mu \pmod{p\gamma N}\\ \nu \pmod N}} \chi_1(\mu)\chi_2(\nu)\,Q_1\Big(\frac{\mu}{p\gamma N}\Big)Q_2\Big(\frac{(\alpha+Nb\gamma)\mu+p\gamma\nu}{p\gamma N}\Big)$$
$$=\sum_{\substack{\mu \pmod{p\gamma N}\\ \nu \pmod N}} \chi_1(\mu)\chi_2(\nu)\,Q_1\Big(\frac{\mu}{p\gamma N}\Big)\sum_{b=0}^{p-1} Q_2\Big(\frac{\alpha\mu+p\gamma\nu}{p\gamma N}+\frac{b\mu}{p}\Big)$$
$$=\sum_{\substack{\mu \pmod{p\gamma N}\\ \nu \pmod N}} \chi_1(\mu)\chi_2(\nu)\,Q_1\Big(\frac{\mu}{p\gamma N}\Big)p^{-k_2}Q_2\Big(\frac{\alpha\mu+p\gamma\nu}{\gamma N}\Big)+$$
$$+\sum_{\substack{\lambda \pmod{\gamma N}\\ \nu \pmod N}} \chi_1(p\lambda)\chi_2(\nu)\,Q_1\Big(\frac{\lambda}{\gamma N}\Big)\Big\{pQ_2\Big(\frac{\alpha\lambda+\gamma\nu}{\gamma N}\Big)-p^{-k_2}Q_2\Big(\frac{\alpha p\lambda+p\gamma\nu}{\gamma N}\Big)\Big\}$$
$$=\bar{\chi}_2(p)\,p^{-k_2}\sum_{\substack{\mu \pmod{p\gamma N}\\ \nu \pmod N}} \chi_1(\mu)\chi_2(p\nu)\,Q_1\Big(\frac{\mu}{p\gamma N}\Big)Q_2\Big(\frac{\alpha\mu+\gamma p\nu}{\gamma N}\Big)+$$
$$+\chi_1(p)\,p\sum_{\substack{\lambda \pmod{\gamma N}\\ \nu \pmod N}} \chi_1(\lambda)\chi_2(\nu)\,Q_1\Big(\frac{\lambda}{\gamma N}\Big)Q_2\Big(\frac{\alpha\lambda+\gamma\nu}{\gamma N}\Big)-$$
$$-\chi_1(p)\,\bar{\chi}_2(p)\,p^{-k_2}\sum_{\substack{\lambda \pmod{\gamma N}\\ \nu \pmod N}} \chi_1(\lambda)\chi_2(p\nu)\,Q_1\Big(\frac{\lambda}{\gamma N}\Big)Q_2\Big(\frac{p\alpha\lambda+\gamma p\nu}{\gamma N}\Big)$$
$$=\bar{\chi}_2(p)\,p^{-k_2}S_{\mathscr{Q}}(p\alpha,p\gamma)+\chi_1(p)\,pS_{\mathscr{Q}}(\alpha,\gamma)-\chi_1(p)\,\bar{\chi}_2(p)\,p^{-k_2}S_{\mathscr{Q}}(p\alpha,\gamma),$$

which is equal to the right-hand side of (9) by (8), [2], [10].

Now we define

$$D_{\mathscr{Q}}\Big(\frac{\alpha}{\gamma}\Big)=\gamma^{k_1}S_{\mathscr{Q}}(\alpha,\gamma).$$

Then the function $D_{\mathscr{Q}}(x)$ on $\boldsymbol{Q}$ is well-defined by (8) and it is a periodic function of period N from (7). We call $D_{\mathscr{Q}}(x)$ the *inhomogeneous* Dedekind type sum. Through the modification

$$D_{\mathscr{Q}}^*(x)=D_{\mathscr{Q}}(Nx),$$

$D_{\mathscr{Q}}^*(x)$ becomes periodic with period 1. We write

$$\psi=\chi_1\bar{\chi}_2$$

and for $n\in N$

$$\sigma_{\mathscr{Q}}(n)=\sum_{d\mid n}\chi_1(d)\,\bar{\chi}_2\Big(\frac{n}{d}\Big)d^{k_1+k_2+1}.$$

Then Lemma 2 reads in terms of $D_{\mathscr{Q}}^*(x)$ as follows:

LEMMA 2'. *For a prime number p not dividing N*

$$\sum_{b=0}^{p-1}D_{\mathscr{Q}}^*\Big(\frac{x+b}{p}\Big)=p^{-k_2}\sigma_{\mathscr{Q}}(p)\,D_{\mathscr{Q}}^*(x)-p^{k_1-k_2}\psi(p)\,D_{\mathscr{Q}}^*(px).$$

Remark. Suppose that K contains all the values of χ_1 and χ_2. We set for each $n\in N$

$$\varphi_n(x)=\varrho^{-n}\big(\varrho D_{\mathscr{Q}}^*(x)-p^{k_1-k_2}\psi(p)\,D_{\mathscr{Q}}^*(px)\big),$$

where $\varrho=\chi_1(p)\,p^{k_1+1}$ or $\bar{\chi}_2(p)\,p^{-k_2}$. Then the family $\{\varphi_n\}$ defines a V-valued distribution on the projective system $\{(1/m_0p^n)\boldsymbol{Z}/\boldsymbol{Z}\}$, where m_0 is a fixed rational integer [6], XII, Theorem 2.1.

4. Action of Hecke operators on $D_{\mathscr{Q}}^*(x)$. For a function $f(z)$ the Hecke operators $T_{k,\psi}(n)$ $(n\in N)$ are defined by

$$T_{k,\psi}(n)f(z)=n^{k-1}\sum_{\substack{ad=n\\ d>0}}\sum_{b=0}^{d-1}\psi(a)\,d^{-k}f\Big(\frac{az+b}{d}\Big).$$

We can easily verify that on periodic functions of period 1

$$T_{k,\psi}(m)\,T_{k,\psi}(n)=\sum_{d\mid(m,n)}\psi(d)\,d^{k-1}\,T_{k,\psi}\Big(\frac{mn}{d^2}\Big).$$

THEOREM 1. *Let n be a positive integer prime to N. Then*

$$T_{k_1-k_2,\psi}(n)\,D_{\mathscr{Q}}^*(x)=n^{-k_2-1}\sigma_{\mathscr{Q}}(n)\,D_{\mathscr{Q}}^*(x). \tag{10}$$

In fact, for prime $p\nmid N$, (10) holds from Lemma 2'. Therefore, making use of the above composition formula for Hecke operators and the readily verified relation

$$\sigma_{\mathscr{Q}}(m)\,\sigma_{\mathscr{Q}}(n)=\sum_{d\mid(m,n)}\psi(d)\,d^{k_1+k_2+1}\,\sigma_{\mathscr{Q}}\Big(\frac{mn}{d^2}\Big),$$

we have our assertion by the usual induction argument [8]. Proof of Theorem 4.1.

5. Action of averaging operators on $D_{\mathcal{D}}^*(x)$. For a periodic function $g(x)$ of period 1, the averaging operators A_n $(n \in N)$ are defined by

$$A_n g(x) = \sum_{b=0}^{n-1} g\left(\frac{x+b}{n}\right).$$

It is obvious [6], XII, § 2 that

$$A_m A_n = A_{mn},$$
$$A_n(g \circ d) = d A_{n/d}\, g \qquad \text{if } d|n,$$
$$= (A_n g) \circ d \qquad \text{if } (d, n) = 1,$$

where $g \circ d(x) = g(dx)$.

THEOREM 2. *For every positive integer n prime to N*

$$A_n D_{\mathcal{D}}^*(x) = n^{-k_2} \sum_{d|n} \mu(d)\psi(d) d^{k_1} \sigma_{\mathcal{D}}\left(\frac{n}{d}\right) D_{\mathcal{D}}^*(dx). \tag{11}$$

Proof. When $n = p$ with p prime and $(p, N) = 1$, (11) is valid from Lemma 2′. We assume that (11) holds for p^r, $r \geqslant 1$. Then

$$\begin{aligned} A_{p^{r+1}} D_{\mathcal{D}}^*(x) &= A_p A_{p^r} D_{\mathcal{D}}^*(x) \\ &= p^{-rk_2} A_p\left(\sigma_{\mathcal{D}}(p^r) D_{\mathcal{D}}^*(x) - \psi(p) p^{k_1} \sigma_{\mathcal{D}}(p^{r-1}) D_{\mathcal{D}}^*(px)\right) \\ &= p^{-rk_2} \sigma_{\mathcal{D}}(p^r)\left\{p^{-k_2}\left(\sigma_{\mathcal{D}}(p) D_{\mathcal{D}}^*(x) - \psi(p) p^{k_1} D_{\mathcal{D}}^*(px)\right)\right\} - \\ &\qquad - \psi(p) p^{-rk_2+k_1+1} \sigma_{\mathcal{D}}(p^{r-1}) D_{\mathcal{D}}^*(x) \\ &= p^{-(r+1)k_2}\left\{\left(\sigma_{\mathcal{D}}(p^r)\sigma_{\mathcal{D}}(p) - \psi(p) p^{k_1+k_2+1} \sigma_{\mathcal{D}}(p^{r-1})\right) D_{\mathcal{D}}^*(x) - \right. \\ &\qquad \left. - \psi(p) p^{k_1} \sigma_{\mathcal{D}}(p^r) D_{\mathcal{D}}^*(px)\right\} \\ &= p^{-(r+1)k_2}\left(\sigma_{\mathcal{D}}(p^{r+1}) D_{\mathcal{D}}^*(x) - \psi(p) p^{k_1} \sigma_{\mathcal{D}}(p^r) D_{\mathcal{D}}^*(px)\right). \end{aligned}$$

Hence (11) is valid for any prime power prime to N. Let $n = n'n''$ with both divisors $n', n'' > 1$ and $(n, N) = (n', n'') = 1$. Assume that (11) holds for n' and n''. Then

$$\begin{aligned} A_n D_{\mathcal{D}}^*(x) &= A_{n'} A_{n''} D_{\mathcal{D}}^*(x) \\ &= A_{n'}\left(n''^{-k_2} \sum_{d''|n''} \mu(d'')\psi(d'') d''^{k_1} \sigma_{\mathcal{D}}\left(\frac{n''}{d''}\right) D_{\mathcal{D}}^*(d''x)\right) \\ &= n''^{-k_2} \sum_{d''|n''} \mu(d'')\psi(d'') d''^{k_1} \sigma_{\mathcal{D}}\left(\frac{n''}{d''}\right) \times \\ &\qquad \times n'^{-k_2} \sum_{d'|n'} \mu(d')\psi(d') d'^{k_1} \sigma_{\mathcal{D}}\left(\frac{n'}{d'}\right) D_{\mathcal{D}}^*(d'd''x) \\ &= n^{-k_2} \sum_{d|n} \mu(d)\psi(d) d^{k_1} \sigma_{\mathcal{D}}\left(\frac{n}{d}\right) D_{\mathcal{D}}^*(dx). \end{aligned}$$

Therefore we complete the proof.

Remark. We are also able to prove our formula (11) after the manner employed in [1] and [11] which makes no use of induction.

6. Equivalence between Theorems 1 and 2. We prove that Theorems 1 and 2 are equivalent. Let us fix $x \in Q$ arbitrarily. Let n be a positive integer prime to N. Then Theorem 1 asserts that

$$n^{k_1} \sum_{\substack{ad=n \\ d>0}} \sum_{b=0}^{d-1} \psi(a) d^{k_2-k_1} D_{\mathcal{D}}^*\left(\frac{ax+b}{d}\right) = \sigma_{\mathcal{D}}(n) D_{\mathcal{D}}^*(x).$$

Since $\psi(a) = \psi(n)\bar{\psi}(d)$, we replace x by x/n and multiply both sides by $\bar{\psi}(n) n^{-k_1}$ to obtain

$$\sum_{d|n} \sum_{b=0}^{d-1} \bar{\psi}(d) d^{k_2-k_1} D_{\mathcal{D}}^*\left(\frac{x/d+b}{d}\right) = \bar{\psi}(n) n^{-k_1} \sigma_{\mathcal{D}}(n) D_{\mathcal{D}}^*\left(\frac{x}{n}\right).$$

By the Möbius inversion formula we get

$$\bar{\psi}(n) n^{k_2-k_1} \sum_{b=0}^{n-1} D_{\mathcal{D}}^*\left(\frac{x/n+b}{n}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right)\bar{\psi}(d) d^{-k_1} \sigma_{\mathcal{D}}(d) D_{\mathcal{D}}^*\left(\frac{x}{d}\right).$$

Altering x/n into x and multiplying both sides by $\psi(n) n^{k_1-k_2}$, we have

$$\begin{aligned} \sum_{b=0}^{n-1} D_{\mathcal{D}}^*\left(\frac{x+b}{n}\right) &= \psi(n) n^{k_1-k_2} \sum_{d|n} \mu(d)\bar{\psi}\left(\frac{n}{d}\right)\left(\frac{n}{d}\right)^{-k_1} \sigma_{\mathcal{D}}\left(\frac{n}{d}\right) D_{\mathcal{D}}^*(dx) \\ &= n^{-k_2} \sum_{d|n} \mu(d)\psi(d) d^{k_1} \sigma_{\mathcal{D}}\left(\frac{n}{d}\right) D_{\mathcal{D}}^*(dx). \end{aligned}$$

This is precisely the assertion of Theorem 2. Through the reverse process we deduce Theorem 1 from Theorem 2, and so conclude the proof. In particular, considering the case of $\mathcal{D}$ in Example (a), we find the equivalence of (4′) and (5′), namely, that of (4) and (5) as mentioned before.

7. Quasi-inhomogeneous Dedekind type sums. After Parson [8] we define the function $E_{\mathcal{D}}(x)$ on Q by

$$E_{\mathcal{D}}\left(\frac{\alpha}{\gamma}\right) = (\alpha, \gamma)^{k_1} S_{\mathcal{D}}(\alpha, \gamma).$$

We call $E_{\mathcal{D}}(x)$ the *quasi-inhomogeneous* Dedekind type sum. We can extend Parson's result [8], (5.2) to $E_{\mathcal{D}}(x)$.

PROPOSITION 2. *Let n be a positive integer prime to N. Assume that $\mathrm{ord}_p x < 0$ for every prime divisor p of n. Then, on such rationals x*

$$\sum_{\substack{ad=n \\ d>0}} \sum_{b=0}^{d-1} \psi(a) d^{k_1+k_2} E_{\mathcal{D}}\left(\frac{ax+Nb}{d}\right) = \sigma_{\mathcal{D}}(n) E_{\mathcal{D}}(x).$$

The proof is quite similar to those of Theorem 1 and Parson, so we may omit it.

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Über die Verteilung ganzer Zahlen mit ausgezeichneten Eigenschaften der Faktorzerlegung in algebraischen Zahlkörpern*

von

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0. Einleitung und Ergebnisse. Ist K ein algebraischer Zahlkörper mit einer Klassenzahl $h > 1$, so weist die multiplikative Struktur des Ringes O_K der ganzen Zahlen von K wesentliche Unterschiede zum rationalen Fall auf. Beispielsweise ist die Tatsache wohlbekannt, daß für Zahlen aus O_K im allgemeinen keine eindeutige Faktorzerlegung in unzerlegbare Zahlen existiert. Insbesondere sind für $h > 1$ unzerlegbare und prime Zahlen wohl zu unterscheiden. Daher stellt sich unmittelbar die Frage nach dem Anteil und der Verteilung von Primzahlen, unzerlegbaren Zahlen und Zahlen mit eindeutiger Faktorzerlegung in unzerlegbare Zahlen in O_K. In Arbeiten von Rémond [10] (Theorem 2.4.5) und Narkiewicz [5] (Theorem 2) ist das asymptotische Verhalten von Funktionen der Form

$$F(x) = \sum_{N(\omega) < x} 1 \tag{a}$$

wobei über Hauptideale (ω) von O_K mit unzerlegbaren bzw. eindeutig zerlegbaren Zahlen ω als erzeugenden Elementen zu summieren ist, angegeben worden.

Die Untersuchungen von Narkiewicz und Rémond stützen sich dabei wesentlich auf den Taubersatz von Delange–Ikehara und liefern als Ergebnis für unzerlegbare Zahlen:

$$F(x) = (C_0 + o(1))\frac{x}{\log x}(\log\log x)^{D-1}, \tag{b}$$

sowie für Zahlen mit k-deutiger Faktorzerlegung in unzerlegbare ganze Zahlen:

$$F_k(x) = (C_k + o(1))\frac{x}{(\log x)^{1-1/h}}(\log\log x)^{a_k}. \tag{c}$$

Die in (b) und (c) auftretenden Konstanten D und a_k sind lediglich von der Klassengruppe $H(K)$ des Körpers abhängig. Die Konstante C_0 ist von

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