so that the conclusion of the theorem holds, suggesting that a similar result might hold for a large class of distance functions and might be compared with the additivity of the Hausdorff–Besicovitch dimension for the cartesian product of certain sets. However, the results of this paper and [9] rely on the dimension being small enough to coincide with a relatively simple lower bound. The methods of this paper will not work for functions which do not satisfy (15) and in particular will not work for the function

$$\psi(q) = |q|^{-\alpha} \tag{16}$$

where $q \neq 0$, which is a natural generalization of the right-hand side of (5). Indeed with $\psi(q)$ given by (16) and $F(x) = |x|$, the dimension $h(W(m, n))$ of $W(m, n)$ is given by

$$h(W(m, n)) = (m-1)n + \frac{m+n}{\alpha+1}$$

when $\alpha > m/n$ [3].

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On some generalizations of the diophantine equation $1^k + 2^k + \ldots + x^k = y^z$

by

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1. Introduction. In [5] J. J. Schäffer proved that for fixed integers $k > 0$ and $z > 1$ the equation

$$1^k + 2^k + \ldots + x^k = y^z \tag{1}$$

has an infinite number of solutions in positive integers x and y only in the cases

(I) $k = 1, z = 2$; (II) $k = 3, z \in \{2, 4\}$; (III) $k = 5, z = 2$.

In all other cases the number of solutions was shown to be bounded by a constant depending only on k. In [2] and [7] K. Győry, R. Tijdeman and M. Voorhoeve have extended Schäffer's result by proving that for fixed $R(x) \in \mathbf{Z}[x]$, $0 \neq b \in \mathbf{Z}$ and fixed $k > 0$, $k \notin \{1, 3, 5\}$ the equation

$$1^k + 2^k + \ldots + x^k + R(x) = by^z \tag{2}$$

has only finitely many solutions in integers $x, y > 1$, $z > 1$. If $R(x) \equiv r \in \mathbf{Z}$, then their result is effective (cf. [2]).

The purpose of the present paper is to give some effective generalizations and extensions of the results of [2] and [7]. As a special case we get an effective version of the above-quoted finiteness theorem concerning the equation (2).

For brevity let us set $S_k(x) = 1^k + 2^k + \ldots + x^k$, $A = \mathbf{Z}[x]$ and $\varkappa = (k+1) \prod_{(p-1)|(k+1)!} p$ (p prime). Let

$$F(y) = Q_n y^n + \ldots + Q_1 y + Q_0 \in A[y].$$

Consider the solutions of the equation

$$F(S_k(x)) = by^z \tag{3}$$

in integers $x, y, z > 1$.

THEOREM 1. *If $Q_1(-1)$ is odd and $k>1$, then all solutions of the equation* (3) *in integers $x, y, z>1$ satisfy $z<C_1$, where $C_1=C_1(F, k, b)$ is an effectively computable constant.*

THEOREM 2. *If $k>1$, $Q_i(x)\equiv q_i\in\mathbf{Z}$ $(i=0, 1, \ldots, n)$ and the polynomial F has simple zeros, then all solutions of the equation* (3) *in integers x, y, z with $x, y>1$, $z\notin\{1, 2, 3, 4, 6\}$ satisfy $\max\{x, y, z\}<C_2$, where $C_2=C_2(F, k, b)$ is an effectively computable constant.*

Our Theorem 2 is a generalization of the main result of [2]. Indeed let b, r, s and $k>1$ be fixed rational integers, and consider the special case

$$r+s(1^k+2^k+\ldots+x^k)=by^z$$

of the equation (3). It was proved in [2] that if s is square-free, then this equation has only finitely many solutions in integers x, y, z with $x, y>1$ and $z\notin\{1, 2, 3, 4, 6\}$ and all these can be effectively determined. Our above theorem implies this result without any assumption concerning s.

THEOREM 3. *If $Q_i(x)\equiv 0 \pmod{x^i}$ for $i=2, \ldots, n$; $Q_1(x)\equiv\pm 1 \pmod 4$ and $k\notin\{1, 2, 3, 5\}$, then all solutions of the equation* (3) *in integers $x, y, z>1$ satisfy $\max\{x, y, z\}<C_3$, where C_3 is an effectively computable constant depending only on F, k and b.*

Let in particular $Q_n(x)=\ldots=Q_2(x)\equiv 0$, and let $Q_1(x)\equiv s$, where s is an odd integer. If $k\notin\{1, 2, 3, 5\}$, then by Theorem 3 all solutions of the equation

$$s(1^k+2^k+\ldots+x^k)+Q_0(x)=by^z$$

in integers $x, y, z>1$ satisfy $\max\{x, y, z\}<C_4$, where C_4 is an effectively computable constant depending only on s, $Q_0(x)$, k and b. This is an effective version of the above-cited main result of [7].

The ineffective character of the results of [7] is due to the application of an ineffective theorem of LeVeque [3] concerning the hyperelliptic equation. All other arguments of [7] are effective. In proving our results we shall use a recent effective version (cf. [1]) of LeVeque's theorem and some arguments of [7].

2. Auxiliary results. For $q=0, 1, 2, \ldots$, the Bernoulli polynomials $B_q(x)$ are defined by

$$\frac{ze^{zx}}{e^z-1}=\sum_{q=0}^{\infty}\frac{B_q(x)z^q}{q!}, \quad |z|<2\pi.$$

Their expansion around the origin is given by

$$B_q(x)=\sum_{i=0}^{q}\binom{q}{i}B_i x^{q-i},$$

where $B_n=B_n(0)$ $(n=0, 1, 2, \ldots)$ are the Bernoulli numbers. For the following properties of Bernoulli polynomials we refer to Rademacher [4]. We have

(4) $$B_{2k+1}=0, \quad k=1, 2, \ldots,$$

(5) $$B_k(x+1)=B_k(x)+kx^{k-1},$$

(6) $$B_k'(x)=kB_{k-1}(x),$$

(7) $$1^k+2^k+\ldots+x^k=\frac{1}{k+1}\big(B_{k+1}(x+1)-B_{k+1}\big).$$

Further, by Staudt–Clausen theorem

$$B_{2k}=G_{2k}-\sum_{(p-1)|2k}\frac{1}{p} \quad (p \text{ prime})$$

where $G_{2k}\in\mathbf{Z}$.

LEMMA 1 (A. Schinzel and R. Tijdeman). *Let $0\neq b\in\mathbf{Z}$, and let $P(x)\in\mathbf{Z}[x]$ be a polynomial with at least two distinct zeros. Then the equation*

$$P(x)=by^z$$

in integers x, $y>1$, z implies that $z<C$, where $C=C(P, b)$ is an effectively computable constant.

Proof. See A. Schinzel and R. Tijdeman [6].

LEMMA 2. *Let $F(y)=\sum_{i=0}^{n}Q_i y^i$ be a non-zero element of $A[y]$. Suppose that $Q_1(0)$ is odd and $q>2$. Then the polynomial*

$$G(x)=F\big((1/q)\,(B_q(x)-B_q)\big)$$

has at least two distinct zeros.

Proof. Supposing the contrary, we can write

$$q^n G(x)=\alpha(x-x_0)^t,$$

with some non-zero $\alpha\in\mathbf{Z}$ and some $x_0\in\mathbf{Q}$. Put $x_0=a/b$, where $a, b\in\mathbf{Z}$ and $(a, b)=1$. Then

$$b^t q^n G(0)=b^t q^n Q_0(0)=\alpha(-a)^t$$

and

$$b^t q^n G(1)=b^t q^n Q_0(1)=\alpha(b-a)^t.$$

It is easy to see that

$$b^t q^n \mid \big(\alpha a^t, \alpha(b-a)^t\big)=\alpha.$$

By using this relation we get

$$G(x)-Q_0(x)=\frac{\alpha}{b^t q^n}(bx-a)^t-Q_0(x)\in \mathbf{Z}[x].$$

We can write

$$G(x)-Q_0(x)=l_1 x+l_2 x^2+\ldots,$$

where

$$l_1=B_{q-1}Q_1(0) \quad \text{and} \quad l_2=\frac{q-1}{2}B_{q-2}Q_1(0)+B_{q-1}Q_1'(0)+Q_2(0)B_{q-1}^2.$$

By the Staudt–Clausen theorem the denominators of B_1, B_{2j} $(j=1,2,\ldots)$ are even. If q is odd then $l_1\notin \mathbf{Z}$, and if q is even then $l_2\notin \mathbf{Z}$. This is a contradiction.

The following lemma is an effective version of a well-known theorem of LeVeque [3].

LEMMA 3. *Let* $G(x)\in \mathbf{Q}[x]$,

$$G(x)=a_0x^N+\ldots+a_N=a_0\prod_{i=1}^{n}(x-\alpha_i)^{r_i},$$

with $a_0\neq 0$ *and* $\alpha_i\neq\alpha_j$ *for* $i\neq j$. *Let* $0\neq b\in \mathbf{Z}$, $2\leqslant m\in \mathbf{Z}$ *and define* $t_i=m/(m, r_i)$. *Suppose that* $\{t_1,\ldots,t_n\}$ *is not a permutation of the n-tuples*

(a) $\{t, 1,\ldots,1\}$, $t\geqslant 1$; (b) $\{2, 2, 1,\ldots,1\}$.

Then all solutions $(x, y)\in \mathbf{Z}^2$ *of the equation*

$$G(x)=by^m$$

satisfy $\max\{|x|, |y|\}<C'$, *where* C' *is an effectively computable constant depending only on G, b and m.*

Proof. See B. Brindza [1].

LEMMA 4 (M. Voorhoeve, K. Győry and R. Tijdeman). *Let* $q\geqslant 2$, $R^*(x)\in \mathbf{Z}[x]$ *and set*

$$P(x)=B_q(x)-B_q+qR^*(x).$$

Then

(i) $P(x)$ *has at least three zeros of odd multiplicity, unless* $q\in\{1, 2, 4, 6\}$.

(ii) *For any odd prime p, at least two zeros of* $P(x)$ *have multiplicities relatively prime to p.*

Proof. See M. Voorhoeve, K. Győry and R. Tijdeman [7].

LEMMA 5. *Let* $q\geqslant 2$ *and* $Q^*(x)$, $R^*(x)\in \mathbf{Z}[x]$, *and set*

$$V(x)=(B_q(x)-B_q)Q^*(x)+qR^*(x).$$

Suppose that $Q^*(x)\equiv\pm 1 \pmod 4$. *Then*

(i) $V(x)$ *has at least three zeros of odd multiplicity, unless* $q\in\{1, 2, 3, 4, 6\}$.

(ii) *For any odd prime p, at least two zeros of* $V(x)$ *have multiplicities relatively prime to p.*

Proof. We shall follow the proof of Lemma 4 of [7]. Choose $d\in \mathbf{N}$ such that $d(B_q(x)-B_q)$ is a primitive polynomial in $\mathbf{Z}[x]$. It is easy to see (see the proof of Lemma 4 in [7]) that d is odd if and only if $q=2^\lambda$ for some $\lambda\geqslant 1$. Further, if $q\neq 2^\lambda$ for any $\lambda\geqslant 1$, then $d\equiv 2 \pmod 4$.

We distinguish three cases:

A. Let $q>3$ be odd. Then $d\equiv 2 \pmod 4$ and

$$dV(x)=d(B_q(x)-B_q)Q^*(x)+dqR^*(x)\equiv d(B_q(x)-B_q)$$
$$\equiv x^{q-1}+\sum_{i=1}^{(q-1)/2}\binom{q}{2i}x^{q-2i} \pmod 2.$$

Hence,

$$d(V(x)+xV'(x))\equiv qx^{q-1}+\sum_{i=1}^{(q-1)/2}\binom{q}{2i}(1+q-2i)x^{q-2i}\equiv x^{q-1} \pmod 2.$$

Any common factor of $dV(x)$ and $dV'(x)$ must therefore be congruent to a power of x (mod 2). Since

$$dV'(0)=dqB_{q-1}Q^*(0)+dqR^{*\prime}(0)\equiv 1 \pmod 2$$

we find that $dV(x)$ and $dV'(x)$ are relatively prime (mod 2). So any common divisor of $dV(x)$ and $dV'(x)$ in $\mathbf{Z}[x]$ is of the shape $2S(x)+1$. Write $dV(x)=T(x)Q(x)$, where $T(x)=\prod_i T_i^{k_i}(x)\in \mathbf{Z}[x]$ contains the multiple factors of dV and $Q\in \mathbf{Z}[x]$ contains its simple factors. Then $T(x)$ is of the shape $2R(x)+1$ with $R(x)\in \mathbf{Z}[x]$, so

$$Q(x)\equiv dV(x)\equiv x^{q-1}+\ldots \pmod 2.$$

Thus the degree of $Q(x)$ is at least $q-1$, proving case A.

In cases B and C we suppose that $Q^*(x)\equiv 1 \pmod 4$. If $Q^*(x)\equiv -1 \pmod 4$ then we write from first to last $-dV(x)$ instead of $dV(x)$ and however the proof is completely similar.

B. Suppose $q=2^\lambda$ for some $\lambda\geqslant 1$, so d is odd. We may assume that $\lambda\geqslant 3$. Now $\binom{q}{2k}$ is divisible by 4 unless $2k=\frac{1}{2}q=2^{\lambda-1}$. Similarly, $\binom{q}{2k}$ is divisible by 8 unless $2k$ is divisible by $2^{\lambda-2}$. We have therefore for some odd d', writing $v=q/4$

$$dV(x)\equiv d(B_q(x)-B_q)\equiv dx^{4v}+2x^{3v}+d'x^{2v}+2x^{v} \pmod 4. \tag{8}$$

Write $dV(x) = T^2(x)Q(x)$, where $T(x)$, $Q(x) \in \mathbf{Z}[x]$ and $Q(x)$ contains each factor of odd multiplicity of $V(x)$ in $\mathbf{Z}[x]$ exactly once. Assume that $\deg Q(x) \leqslant 2$. Since

$$T^2(x)Q(x) \equiv x^{4\nu}+x^{2\nu} = x^{2\nu}(x^{2\nu}+1) \pmod 2,$$

$T^2(x)$ must be divisible by $x^{2\nu-2} \pmod 2$. So

$$T(x) = x^{\nu-1}T_1(x)+2T_2(x),$$
$$T^2(x) = x^{2\nu-2}T_1^2(x)+4T_3(x),$$

for certain T_1, T_2, $T_3 \in \mathbf{Z}[x]$. If $q > 8$, then $\nu > 2$ so the last identity is incompatible with (8) because of the term $2x^\nu$. Hence $\deg Q(x) \geqslant 3$, which proves (i). If $q = 8$, then $d = 3$ and

$$dV(x) \equiv 3x^8+2x^6+x^4+2x^2 \equiv -x^2(x+1)(x-1)(x^2+1)(x^2+2) \pmod 4.$$

All these factors – except x^2 – are simple, so $\deg Q \geqslant 6 > 3$ if $q = 8$, proving (i) in case B.

To prove (ii), let p be an odd prime and write $dV(x) = T^p(x)Q(x)$, where Q, $T \in \mathbf{Z}[x]$ and all the roots of multiplicity divisible by p are incorporated in $T^p(x)$. We have, writing $\mu = 2^{\lambda-1}$,

$$dV(x) = T^p(x)Q(x) \equiv x^\mu(x^\mu+1) = x^\mu(x+1)^\mu \pmod 2.$$

Since μ is relatively prime to p, Q has at least two different zeros, proving (ii) in case B.

C. Suppose q is even and $q \neq 2^\lambda$ for any $\lambda \geqslant 1$. Then $d \equiv 2 \pmod 4$ and hence

$$dV(x) \equiv d(B_q(x)-B_q) \tag{9}$$
$$\equiv 2x^q - qx^{q-1}+\tfrac{1}{6}d\binom{q}{2}x^{q-2}+\ldots+dB_{q-2}\binom{q}{2}x^2 \pmod 4.$$

In order to prove part (i) we may assume that $q \geqslant 10$, because $q = 2, 4, 6$ are the exceptional cases and $q = 8$ is treated in Section B. Write $dV(x) = T^2(x)Q(x)$, where T, $Q \in \mathbf{Z}[x]$ and $Q(x)$ contains each factor of odd multiplicity of V exactly once. Then $\deg Q(x) \geqslant 3$. The assertion easily follows by repeating the corresponding part of the proof of Lemma 4. This proves part (i) of the lemma.

Consider the case (ii). Write $q = 2^\lambda r$, where $r > 1$ is odd. Then

$$dV(x) \equiv d(B_q(x)-B_q) \equiv \sum_{k=1}^{(q-2)/2}\binom{q}{2k}x^{2k} = \sum_{l=1}^{q-1}\binom{q}{l}x^l$$
$$\equiv (x+1)^q - x^q - 1 \equiv ((x+1)^r - x^r - 1)^{2^\lambda} \pmod 2.$$

Since $r > 1$ is odd, $(x+1)^r - x^r - 1$ has x and $x+1$ as simple factors (mod 2).

Thus

$$dV(x) \equiv x^{2^\lambda}(x+1)^{2^\lambda}H(x) \pmod 2,$$

where H is neither divisible by x nor by $x+1$ (mod 2). As in the preceding case, $V(x)$ must have two roots of multiplicity prime to p. The proof of Lemma 5 is thus complete.

LEMMA 6 (K. Győry, R. Tijdeman and M. Voorhoeve). *Each zero of* $T(x) = B_q(x) - B_q$ *is of multiplicity less than* 3.

Proof. If $q = 3$ then

$$2T(x) \equiv 2x^3+x \equiv 2x(x+1)(x-1) \pmod 3,$$

showing that $T(x)$ has three simple roots. First suppose $q > 3$ is odd. Then, following the proof of Theorem 2 in [2], we have

$$d(T(x)+xT'(x)) \equiv x^{q-1} \pmod 2.$$

Since $dT'(0) = qdB_{q-1} \equiv 1 \pmod 2$ we find that $dT(x)$ and $dT'(x)$ are relatively prime (mod 2). So any irreducible common divisor of $dT(x)$ and $dT'(x)$ in $\mathbf{Z}[x]$ must be of shape $2S(x)+1$. Then $dT(x)$ is divisible by $(2S(x)+1)^2$ and the leading coefficient d of $dT(x)$ is divisible by the leading coefficient of $(2S(x)+1)^2$. Since $4 \nmid d$, this is impossible unless $S(x)$ is a constant. All the zeros of $T(x)$ are therefore simple. Next suppose that q is even. Since then $T'(x) = qB_{q-1}(x)$ hence each zero of $T(x)$ is multiplicity less than 3.

3. Proof of the theorems.

Proof of Theorem 1. Putting $Q_i^*(x) = Q_i(x-1)$ $(i = 0, 1, \ldots, n)$, we have $Q_1^*(0) = Q_1(-1)$. Let x, y, $z > 1$ be an arbitrary solution of (3) in rational integers. Then

$$F(S_k(x)) = \sum_{i=0}^{n} Q_i^*(x+1)\left[(B_{k+1}(x+1)-B_{k+1})\frac{1}{k+1}\right]^i,$$

and we get an effective bound for z by applying Lemmas 1 and 2.

Proof of Theorem 2. Let y_0 be a simple zero of $F(y)$, and write

$$F(y) = q_n y^n+\ldots+q_1 y+q_0 = (y-y_0)H(y),$$

where $H(y_0) \neq 0$. Put

$$M(x) = \frac{1}{k+1}(B_{k+1}(x+1)-B_{k+1}) - y_0 = a_0\prod_{i=1}^{u}(x-\alpha_i)^{\gamma_i}$$
$$(\gamma_i > 0,\ \alpha_i \neq \alpha_j \text{ if } i \neq j)$$

and

$$L(x) = H\left(\frac{1}{k+1}(B_{k+1}(x+1)-B_{k+1})\right) = b_0\prod_{i=1}^{v}(x-\beta_i)^{\delta_i}$$
$$(\delta_i > 0,\ \beta_i \neq \beta_j \text{ if } i \neq j).$$

Clearly $\alpha_i \neq \beta_j$ for $i \in \{1, \ldots, u\}$, $j \in \{1, \ldots, v\}$. From (6) we get $M'(x) = B_k(x+1)$. So by Lemma 6 we can write $\gamma_i < 4$ for $i = 1, \ldots, u$. If $k+1 > 3$ then $u \geqslant 2$; if $k = 2$ then

$$M'(x) = B_2(x+1) = (x+1)^2 - (x+1) + \tfrac{1}{6} = (x-x_1)(x-x_2) \quad (x_1 \neq x_2)$$

and again $u \geqslant 2$.

Let now x, y, z be an arbitrary integer solution of (3) under the assumptions of the theorem, with $x, y > 1$, $z \notin \{1, 2, 3, 4, 6\}$. Then Lemma 1 gives $z < C_0$ with an effective C_0. Since $\gamma_i < 4$ and $z \notin \{1, 2, 3, 4, 6\}$ so $z/(z, \gamma_i) \geqslant 3$ for $i = 1, \ldots, u$. Using Lemma 3 we have $\max\{x, y\} < C$, where C is an effectively computable constant depending only on F, k and b. This completes the proof of Theorem 2.

Proof of Theorem 3. Write $Q_i(x) = \varkappa^i K_i(x)$, where $K_i(x) \in \boldsymbol{Z}[x]$ for $i = 2, \ldots, n$. By using the Staudt–Clausen theorem we have

$$(10) \qquad (B_{k+1}(x+1) - B_{k+1}) \prod_{s=1}^{[(k+1)/2]} \prod_{\substack{(p-1)|2s, (p-1)\nmid 2j \text{ for } j<s \\ p \text{ prime}}} p \in \boldsymbol{Z}[x],$$

and

$$\prod_{s=1}^{[(k+1)/2]} \prod_{\substack{(p-1)|2s, (p-1)\nmid 2j \text{ for } j<s \\ p \text{ prime}}} p \,\Big|\, \varkappa/(k+1).$$

From (10) we get

$$Q_i(x)\left\{\frac{1}{k+1}(B_{k+1}(x+1) - B_{k+1})\right\}^i = K_i(x)\left\{\frac{\varkappa}{k+1}(B_{k+1}(x+1) - B_{k+1})\right\}^i \in \boldsymbol{Z}[x]$$

for $i = 2, \ldots, n$. Putting

$$U(x+1) = (k+1)F\left(\frac{1}{k+1}(B_{k+1}(x+1) - B_{k+1})\right)$$

we have

$$U(x+1) = Q_1^*(x+1)(B_{k+1}(x+1) - B_{k+1}) + (k+1)W(x+1),$$

where $W \in \boldsymbol{Z}[x]$ and $Q_1^*(x) = Q_1(x-1)$, $Q_1^*(x+1) \equiv \pm 1 \pmod 4$. By applying Lemmas 5 and 1 we see that z is bounded, i.e. $z < C_1$ with an effectively computable C_1. Write

$$U(x) = c \prod_{i=1}^{N} (x - x_i)^{r_i}$$

where $c \neq 0$, $x_i \neq x_j$ if $i \neq j$ and, for a fixed z let $t_i = z/(z, r_i)$. If z is even, then by Lemma 5 at least three zeros have odd multiplicity, say r_1, r_2, r_3 are odd. Consequently, t_1, t_2 and t_3 are even. If z is odd and $p|z$ for an odd

prime p, then by Lemma 5 at least two zeros of $U(x)$ have multiplicities prime to p. We may assume that $(r_1, p) = (r_2, p) = 1$, so $p|t_1$ and $p|t_2$. Using Lemma 3 we have $\max\{x, y\} < C_2(z)$ with an effectively computable $C_2(z)$ which depend on z. Finally $z < C_1$ implies the required assertion.

Remark. In Theorem 3 it is necessary to assume that $Q_1(x) \equiv \pm 1 \pmod 4$. Indeed, let $Q_n(x) = \ldots = Q_2(x) \equiv 0$, and if $Q_1(x) \not\equiv \pm 1 \pmod 4$, choose d, $k \in N$ such that $d(B_{k+1}(x+1) - B_{k+1})$ is a primitive polynomial in $\boldsymbol{Z}[x]$ and $Q_1(x) = d(k+1) \not\equiv \pm 1 \pmod 4$. If this is the case, there are infinitely many choice for $Q_0(x)$ and b such that (3) has an infinite number of solutions. We may take for example

$$Q_0(x) = x - d(B_{k+1}(x+1) - B_{k+1})$$

when the number of solutions of the corresponding equation

$$F(S_k(x)) = x = by^z$$

is obviously infinite.

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