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A note on the Hausdorff–Besicovitch dimension of systems of linear forms

by

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Let $\sum_{i=1}^{m} x_i a_{ij}$, $1 \leqslant j \leqslant n$, be a system of n real linear forms in m variables. Suppose that the function $\psi(\boldsymbol{q})$ is defined for integral vectors $\boldsymbol{q}$ in $\boldsymbol{Z}^m$ and satisfies

$$(1) \qquad \sum_{\boldsymbol{q}\in \boldsymbol{Z}^m} \psi(\boldsymbol{q})^n < \infty .$$

Then a straightforward volume estimate shows that the set of matrices (a_{ij}) (identified with the points $(a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{mn})$ in $\boldsymbol{R}^{mn}$) for which the inequalities

$$(2) \qquad \Big\|\sum_{i=1}^{m} q_i a_{ij}\Big\| < \psi(q_1, \ldots, q_m), \qquad 1 \leqslant j \leqslant n,$$

have infinitely many solutions $(q_1, \ldots, q_m)$ in $\boldsymbol{Z}^m$ has Lebesgue measure 0. (As usual $\|x\| = \inf\{|x-k|\colon k \in \boldsymbol{Z}\}$, the distance of the real number x from the integer nearest to it). This result will be recognised as the easy part of a "Khintchine type" theorem and is a special case of a very general theorem due to Sprindžuk ([8], p. 37, Theorem 13). In this note the problem of determining the Hausdorff–Besicovitch or fractional dimension of this set of measure 0 in $\boldsymbol{R}^{mn}$ is reduced, for a class of error terms ψ, to considering the dimension of a related set in $\boldsymbol{R}^n$, corresponding to the case when $m = 1$. More precisely, when $\psi(\boldsymbol{q})$ is "small on average" compared to the restriction

$$(3) \qquad \psi_1(q) = \psi(q, 0, \ldots, 0),$$

the Hausdorff–Besicovitch dimension of the set of matrices (a_{ij}) is that of the set of points $(x_1, \ldots, x_n)$ in $\boldsymbol{R}^n$ such that

$$\max_{1\leqslant j\leqslant n} \|q x_j\| < \psi_1(q)$$

for infinitely many integers q, augmented by $(m-1)n$, the number of "degrees of freedom". Note that no use is made of the convergence condition (1) but the results are of interest only in this case. As an illustration of the result, the function

$$\psi(q_1, \ldots, q_m) = \prod_{i=1}^{m} (\bar{q}_i)^{-\alpha}, \tag{4}$$

where $\alpha > 1/n$ and $\bar{x} = \max\{|x|, 1\}$, which occurs as an error term in simultaneous Diophantine approximation [9], is "small on average" compared to

$$\psi_1(q) = (\bar{q})^{-\alpha}.$$

Now Jarnik ([7], p. 508, Satz 1) and later Eggleston ([6], p. 60, Theorem 7) proved that when $\alpha > 1/n$, the set of points $(x_1, \ldots, x_n)$ in $\boldsymbol{R}^n$ satisfying

$$\max_{1 \leqslant j \leqslant n} \|qx_j\| < q^{-\alpha} \tag{5}$$

for infinitely many positive integers q has Hausdorff–Besicovitch dimension $\frac{n+1}{\alpha+1}$. It follows that the set of matrices (a_{ij}) which satisfy

$$\max_{1 \leqslant j \leqslant n} \Big\|\sum_{i=1}^{m} q_i a_{ij}\Big\| < \prod_{i=1}^{m} (\bar{q}_i)^{-\alpha}, \quad 1 \leqslant j \leqslant n, \tag{6}$$

where $\alpha > 1/n$, for infinitely many $(q_1, \ldots, q_m)$ in $\boldsymbol{Z}^m$ has dimension

$$\frac{n+1}{\alpha+1} + (m-1)n$$

(see Corollary 1 below).

The proof of this result does not depend on any special properties of the supremum metric associated with the form of the approximation in (2) but on a general boundedness condition that can be expressed simply in terms of distance functions. A distance function F: $\boldsymbol{R}^n \to \boldsymbol{R}$ is continuous, non-negative and satisfies

$$F(ax) = aF(x)$$

for all $\boldsymbol{x}$ in $\boldsymbol{R}^n$ and non-negative numbers a. It allows different forms of rational approximation, including the case

$$F(x_1, \ldots, x_n) = \Big(\prod_{j=1}^{n} |x_j|\Big)^{1/n} \tag{7}$$

studied by Bovey and Dodson [2] and Yu [9], to be bought together and has the additional advantage of making the notation more concise and simple ([4], § V.10.2).

It is convenient to introduce some more definitions and notation. The Hausdorff–Besicovitch dimension of a set X in $\boldsymbol{R}^k$ can be defined as follows. Let $\mathscr{C}$ be any countable or finite cover of X by k-dimensional hypercubes C. For each real s, define the s-volume of a cover $\mathscr{C}$ of X to be

$$L^s(\mathscr{C}) = \sum_{C \in \mathscr{C}} L(C)^s,$$

where $L(C)$ is the length of a side of the hypercube C. For each positive ϱ and real s write

$$\Lambda_\varrho^s(X) = \inf L^s(\mathscr{C}),$$

where the infimum is taken over covers $\mathscr{C}$ of X with $L(C) \leqslant \varrho$ for all C in $\mathscr{C}$. Clearly $\Lambda_\varrho^s(X)$ cannot decrease as ϱ decreases, and if $s' \geqslant s$,

$$\Lambda_\varrho^{s'}(X) \leqslant \varrho^{s'-s} \Lambda_\varrho^s(X).$$

Thus if $\Lambda^s(X) = \sup_{\varrho > 0} \Lambda_\varrho^s(X)$ is finite, $\Lambda^{s'}(X)$ vanishes. The Hausdorff–Besicovitch dimension $h(X)$ of X is the supremum of all real s for which $\Lambda^s(X)$ is positive. It follows from the definition that $0 \leqslant h(X) \leqslant k$ and that if $X = \bigcup_{j=1}^{\infty} X_j$,

$$h(X) = \sup_j h(X_j). \tag{8}$$

Note that the cover $\mathscr{C}$ was chosen to consist of hypercubes C for convenience and, as is well known, a collection $\mathscr{F}$ of sets F, such as k-dimensional discs, with the property that for each F in $\mathscr{F}$, there exists C, C' in $\mathscr{C}$ with

$$C \subseteq F \subseteq C'$$

and

$$1 \ll L(C')/L(C) \ll 1,$$

could be used instead.

Next, let $I = (-\frac{1}{2}, \frac{1}{2}]$. For each vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ in $\boldsymbol{R}^n$, define

$$\langle \boldsymbol{x} \rangle$$

to be the unique vector $\boldsymbol{x} - \boldsymbol{p}$ in I^n, where $\boldsymbol{p} \in \boldsymbol{Z}^n$. Also define

$$|\boldsymbol{x}| = \max_{1 \leqslant j \leqslant n} |x_j|,$$

so that

$$|\langle \boldsymbol{x} \rangle| = \max_{1 \leqslant j \leqslant n} \|x_j\| \in [0, \tfrac{1}{2}].$$

Note that $F(x) = |x|$ is a distance function and that (5) can be written

$$|\langle qx\rangle| < q^{-\alpha}$$

and that the left hand side of (6) can be written

$$|\langle \boldsymbol{q}A\rangle|,$$

where $A = (a_{ij})$ and $\boldsymbol{q} = (q_1, \ldots, q_m) \in \boldsymbol{Z}^m$.

Define

$$W(m, n) = \{A \in \boldsymbol{R}^{mn}\colon F(\langle \boldsymbol{q}A\rangle) < \psi(\boldsymbol{q}) \text{ for infinitely many } \boldsymbol{q} \text{ in } \boldsymbol{Z}^m\};$$

this set is symmetric and

$$W(1, n) = \{\boldsymbol{x} \in \boldsymbol{R}^n\colon F(\langle q\boldsymbol{x}\rangle) < \psi_1(q) \text{ for infinitely many } q \text{ in } \boldsymbol{Z}\},$$

where $\psi_1(q)$ is given by (3).

The Hausdorff–Besicovitch dimension $h(W(1, n))$ of $W(1, n)$ is known for certain F when $\psi_1(q) = |q|^{-\alpha}$, where $\alpha > 1/n$ and $q \neq 0$. In particular when $F(x) = |x|$, $W(1, n)$ is essentially the set of well-approximable points studied by Jarnik [7] and Eggleston [6], and has dimension $h(W(1, n)) = \dfrac{n+1}{\alpha+1}$. When $F(x)$ is given by (7) it follows from [2] that the dimension is $n-1+\dfrac{2}{1+n\alpha}$. However the question of a general formula for $h(W(1, n))$ is not discussed here and the dimension $h(W(m, n))$ of $W(m, n)$ is determined in terms of $h(W(1, n))$. Indeed the first lemma gives a simple and fairly general lower bound of this kind for $h(W(m, n))$; a special case is considered in [9]. Note that for convenience $\psi(\boldsymbol{q})$ will often be written as r.

LEMMA 1. *Let $f\colon \boldsymbol{R}^n \to \boldsymbol{R}$, $\psi\colon \boldsymbol{R}^m \to \boldsymbol{R}$ be functions and let*

$$X_m = \{A \in \boldsymbol{R}^{mn}\colon f(\langle \boldsymbol{q}A\rangle) < \psi(\boldsymbol{q}) \text{ infinitely often}\}$$

and

$$X_1 = \{\boldsymbol{x} \in \boldsymbol{R}^n\colon f(\langle q\boldsymbol{x}\rangle) < \psi_1(q) \text{ infinitely often}\},$$

where

$$\psi_1(q) = \psi(q, 0, \ldots, 0).$$

Then

$$h(X_m) \geqslant h(X_1) + (m-1)n.$$

Proof. Since

$$\boldsymbol{R}^k = \bigcup_{\boldsymbol{j} \in \boldsymbol{Z}^k} (I^k + \boldsymbol{j}),$$

it follows from (8) and the definition of X_1 and X_m that

$$h(X_1) = h(X_1 \cap I^n) \quad \text{and} \quad h(X_m) = h(X_m \cap I^{mn}).$$

For each $\boldsymbol{x}$ in $X_1 \cap I^n$, the $m \times n$ matrix A which has $\boldsymbol{x}$ as its first row and its other entries from I is in $(X_1 \cap I^n) \times I^{(m-1)n}$, and satisfies for infinitely many $\boldsymbol{q} = (q, 0, \ldots, 0)$ in $\boldsymbol{Z}^m$

$$f(\langle \boldsymbol{q}A\rangle) = f(\langle q\boldsymbol{x}\rangle) < \psi_1(q) = \psi(\boldsymbol{q}).$$

Thus

$$(X_1 \cap I^n) \times I^{(m-1)n} \subseteq X_n \cap I^{mn},$$

whence

$$h((X_1 \cap I^n) \times I^{(m-1)n}) \leqslant h(X_n \cap I^{mn}).$$

For any set S in I^k, it is well known that the Hausdorff–Besicovitch dimension $h(S \times I)$ of the cartesian product $S \times I$ in I^{k+1} is given by

$$h(S \times I) = h(S) + 1$$

(a proof is given in [2], p. 216, Lemma 2) and the lemma follows from repeated application.

COROLLARY.

$$h(W(1, n)) + (m-1)n \leqslant h(W(m, n)).$$

It remains to establish the complementary inequality

$$h(W(1, n)) + (m-1)n \geqslant h(W(m, n))$$

and to do this additional restrictions need to be placed on the distance function F and the function $r = \psi(\boldsymbol{q})$. First a cover for $W(m, n) \cap I^{mn}$ is obtained.

For any $\boldsymbol{u}$ in $\boldsymbol{R}^m$, $\boldsymbol{v}$ in $\boldsymbol{R}^n$ and positive real number ϱ, define

$$B(\boldsymbol{u}, \boldsymbol{v}; \varrho) = \{A \in I^{mn}\colon F(\boldsymbol{u}A - \boldsymbol{v}) < \varrho, |\boldsymbol{u}A - \boldsymbol{v}| \leqslant 1/2\}$$

and for any positive number u define

$$B(u, \boldsymbol{v}; \varrho) = \{\boldsymbol{x} \in I^n\colon F(u\boldsymbol{x} - \boldsymbol{v}) < \varrho, |u\boldsymbol{x} - \boldsymbol{v}| \leqslant 1/2\}.$$

Note that for each A in $B(\boldsymbol{u}, \boldsymbol{p}; \varrho)$, where $\boldsymbol{p}$ is in $\boldsymbol{Z}^n$,

$$\boldsymbol{u}A - \boldsymbol{p} = \langle \boldsymbol{u}A\rangle \in I^n$$

and for each $\boldsymbol{x}$ in $B(\boldsymbol{u}, \boldsymbol{p}; \varrho)$

$$u\boldsymbol{x} - \boldsymbol{p} = \langle u\boldsymbol{x}\rangle \in I^n.$$

It follows that for $r = \psi(\boldsymbol{q})$ and each $Q = 1, 2, \ldots$,

$$W(m, n) \cap I^{mn} \subseteq \bigcup_{\boldsymbol{q}} \bigcup_{\boldsymbol{p}} B(\boldsymbol{q}, \boldsymbol{p}; r), \tag{9}$$

where the unions are over those q in Z^m with $|q| \geq Q$ and over those p in Z^n with $|p| \leq |q|$. Thus

$$\{B(q, p; r): |p| \leq |q|, |q| \geq Q\}$$

is a cover for $W(m, n) \cap I^{mn}$, though of course not by hypercubes in general.

LEMMA 2. *Let q be a sufficiently large integer and let $\mathscr{C}(q, p; \varrho)$ be a cover of $B(q, p; \varrho)$ by n-dimensional hypercubes C. Then for each q in Z^m with $|q| = q$, $B(q, p; \varrho)$ has a cover $\mathscr{B}(q, p; \varrho)$ of mn-dimensional hypercubes B with $L(B) = mL(C)$ and*

$$L^t(\mathscr{B}(q, p; \varrho)) \ll L^{t-(m-1)n}(\mathscr{C}(q, p; \varrho))$$

for any real number t.

Proof. For definiteness take $q = |q_1|$. Define the invertible matrix $T = T(q)$ by

$$T = \begin{bmatrix} 1 & -\frac{q_2}{q_1} & \dots & -\frac{q_m}{q_1} \\ 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 1 \end{bmatrix}.$$

It is easy to verify that for any real $m \times n$ matrix A

$$|TA| \leq m|A|$$

where $|A| = \max_{i,j} |a_{ij}|$ (recall that A is regarded as a vector in R^{mn}). Also since $qT = (q_1, 0, \dots, 0)$,

$$qTA = q_1 A_1, \quad qA = q_1 (T^{-1}A)_1,$$

where A_1 is the first row of A and it follows that

$$B(q, p; \varrho) \subseteq T(B(q_1, p; \varrho) \times I^{(m-1)n}).$$

Now $B(q_1, p; \varrho)$ has a cover $\mathscr{C}(q_1, p; \varrho)$ of n-dimensional hypercubes C. Hence $B(q_1, p; \varrho) \times I^{(m-1)n}$ has a cover $\mathscr{K}$ of mn-dimensional hypercubes K of length $L(C)$, where

$$K = C \times J_{21} \times \dots \times J_{m1} \times \dots \times J_{2n} \times \dots \times J_{mn},$$

where

$$J_{ij} = [k_{ij} L(C), (k_{ij}+1) L(C)], \quad 2 \leq i \leq m, \ 1 \leq j \leq n,$$

and where

$$k_{ij} = 0, \pm 1, \dots, \pm [1/2L(C)] \quad \text{since } |TA| \leq m|A|$$

($[\theta]$ is the integer part of the real number θ). Hence by (9), $B(q, p; \varrho)$ has a cover $\mathscr{B}(q, p; \varrho)$ of mn-dimensional hypercubes B of length $L(B) = mL(K) = mL(C)$ such that $B \subseteq TK$ and

$$L^t(\mathscr{B}(q, p; \varrho)) = m^t L^t(\mathscr{K}) = m^t \sum_C L(C)^t (1 + [1/L(C)])^{(m-1)n}$$

where the summation is over hypercubes C in $\mathscr{C}(q_1, p; \varrho)$. Hence for each real number t

$$L^t(\mathscr{B}(q, p; \varrho)) \ll L^{t-(m-1)n}(\mathscr{C}(q_1, p; \varrho)).$$

It is evident that the last estimate holds for $|q| = |q_i|$ for each $i = 1, \dots, m$. Moreover without loss of generality q_i can be taken to be positive since each A can be replaced by $-A$ without altering the s-length of the cover, whence the result.

COROLLARY. *Let Q be a sufficiently large positive integer. Then $W(m, n) \cap I^{mn}$ has a cover $\mathscr{H}_Q$ of mn-dimensional hypercubes such that*

(10) $$L^t(\mathscr{H}_Q) \ll \sum_q \sum_p L^{t-(m-1)n}(\mathscr{C}(|q|, p; r))$$

where the sums are over those p in Z^n with $|p| \leq |q|$ and over those q in Z^m with $|q| \geq Q$.

Proof. By (9)

$$W(m, n) \cap I^{mn} \subseteq \bigcup_q \bigcup_p B(q, p; r) \subseteq \bigcup_q \bigcup_p \bigcup_B B,$$

where the right-hand union is over hypercubes B in $\mathscr{B}(q, p; r)$. Hence by the lemma,

$$L^t(\mathscr{B}(q, p; r)) \ll L^{t-(m-1)n}(\mathscr{C}(|q|, p; r)),$$

whence

$$\mathscr{H}_Q = \{B \in \mathscr{B}(q, p; r): |p| \leq |q|, |q| \geq Q\}$$

is the desired cover for $W(m, n) \cap I^{mn}$.

LEMMA 3. *Let $F: R^n \to R$ be a distance function which satisfies*

(11) $$F^{-1}([0, \varrho)) = \{x \in R^n: F(x) < \varrho\} \subseteq I^n$$

for ϱ sufficiently small. Let $\psi: Z^m \to R$ be non-negative and satisfy

(12) $$\sum_{|q|=q} \psi(q)^s \ll \psi_1(q)^s$$

for each $s > h(W(1, n))$. Then

$$h(W(m, n)) \leq h(W(1, n)) + (m-1)n.$$

Proof. Let $t > h(W(1, n)) + (m-1)n$, so that

$$s = t - (m-1)n > h(W(1, n)).$$

Let δ be a positive number and let Q be a sufficiently large positive integer. For each integer $q \geq Q$ and $\boldsymbol{p}$ in $\boldsymbol{Z}^n$ with $|\boldsymbol{p}| \leq q$, let $\mathscr{C}^*(q, \boldsymbol{p}; \psi_1(q))$ be a cover of $B(q, \boldsymbol{p}; \psi_1(q))$ by n-dimensional hypercubes C^*, so that

$$\{C^* \in \mathscr{C}^*(q, \boldsymbol{p}; \psi_1(q)) \colon |\boldsymbol{p}| \leq q, q \geq Q\}$$

is a cover of $W(1, n) \cap I^n$. Since $s > h(W(1, n))$,

$$\sum_{q \geq Q} \sum_{|\boldsymbol{p}| \leq q} L^s(\mathscr{C}^*(q, \boldsymbol{p}, \psi_1(q))) < \delta \tag{13}$$

providing each $L(C^*)$ is sufficiently small.

Let v and ϱ be positive numbers. For each $\boldsymbol{u}$ in $\boldsymbol{R}^n$

$$B(v, \boldsymbol{u}; \varrho) \subseteq B(v, \boldsymbol{0}; \varrho) + v^{-1}\boldsymbol{u}$$

whence

$$L(B(v, \boldsymbol{u}; \varrho)) \leq L(B(v, \boldsymbol{0}; \varrho)).$$

Also if ϱ is sufficiently small, then for each positive λ,

$$B(v, \boldsymbol{0}; \lambda\varrho) \subseteq \lambda B(v, \boldsymbol{0}; \varrho)$$

since for each $\boldsymbol{x}$ in $B(v, \boldsymbol{0}; \lambda\varrho)$,

$$F(v\lambda^{-1}x) = \lambda^{-1}F(vx) < \varrho,$$

which by (11) implies that $|v\lambda^{-1}\boldsymbol{x}| < 1/2$, so that $\lambda^{-1}\boldsymbol{x}$ is in $B(v, \boldsymbol{0}; \varrho)$. Put

$$\lambda = \psi_1(|\boldsymbol{q}|)^{-1}\psi(\boldsymbol{q}).$$

Then for each $\boldsymbol{q}$ with $|\boldsymbol{q}|$ sufficiently large, $B(|\boldsymbol{q}|, \boldsymbol{p}; \psi(\boldsymbol{q}))$ is contained in the set $B(|\boldsymbol{q}|, \boldsymbol{p}; \psi_1(|\boldsymbol{q}|))$ shrunk by a factor λ. Hence $B(|\boldsymbol{q}|, \boldsymbol{p}; \psi(\boldsymbol{q}))$ has a cover $\mathscr{C}^\dagger(|\boldsymbol{q}|, \boldsymbol{p}; \psi(\boldsymbol{q}))$ of n-dimensional hypercubes of length $\lambda L(C^*)$, where C^* is a member of the cover $\mathscr{C}^*(|\boldsymbol{q}|, \boldsymbol{p}; \psi_1(|\boldsymbol{q}|))$ for $B(|\boldsymbol{q}|, \boldsymbol{p}; \psi_1(|\boldsymbol{q}|))$.

Choose C^* so that $L(C^*)$ is sufficiently small to ensure that the δ in (13) is sufficiently small. Then by (10)

$$\begin{aligned} L^t(\mathscr{H}_Q) &\leq \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} L^{t-(m-1)n}(\mathscr{C}^\dagger(|\boldsymbol{q}|, \boldsymbol{p}; \psi(\boldsymbol{q}))) \\ &\leq \sum_{\boldsymbol{q}} \sum_{\boldsymbol{p}} \lambda^s L^s(\mathscr{C}^*(|\boldsymbol{q}|, \boldsymbol{p}; \psi_1(|\boldsymbol{q}|))) \\ &\leq \sum_{q \geq Q} \psi_1(q)^{-s}\Big(\sum_{|\boldsymbol{q}|=q} \psi(\boldsymbol{q})^s\Big) \cdot \sum_{|\boldsymbol{p}| \leq q} L^s(\mathscr{C}^*(q, \boldsymbol{p}; \psi_1(q))) \\ &\leq \sum_{q \geq Q} \sum_{|\boldsymbol{p}| \leq q} L^s(\mathscr{C}^*(q, \boldsymbol{p}; \psi_1(q))) \\ &\leq \delta \end{aligned}$$

by (12) and since $s > h(W(1, n))$. It follows that $L^t(\mathscr{H}_Q)$ can be made arbitrarily small, whence

$$h(W(m, n) \cap I^{mn}) = h(W(m, n)) \leq h(W(1, n)) + (m-1)n.$$

Note that if for a distance function F, $F^{-1}(0) = \boldsymbol{0}$, then the continuity of F implies that (11) holds or that the inverse image of a neighbourhood of the origin is a bounded set. Evidently Lemma 1 and Lemma 3 are complementary and combining their hypotheses gives the

THEOREM. *Let $F\colon \boldsymbol{R}^n \to \boldsymbol{R}$ be a distance function which satisfies*

$$F^{-1}([0, \varrho)) \subseteq I^n \tag{14}$$

for ϱ sufficiently small. Let $\psi(\boldsymbol{q})$ be a non-negative function of $\boldsymbol{q}$ in $\boldsymbol{Z}^m$ with restriction $\psi_1(q) = \psi(q, 0, \ldots, 0)$, and let the Hausdorff–Besicovitch dimension of the set $W(1, n)$ be $h(W(1, n))$. Then if

$$\sum_{|\boldsymbol{q}|=q} \psi(\boldsymbol{q})^s \ll \psi_1(q)^s \tag{15}$$

for each $s > h(W(1, n))$, the Hausdorff–Besicovitch dimension $h(W(m, n))$ of $W(m, n)$ is given by

$$h(W(m, n)) = h(W(1, n)) + (m-1)n.$$

COROLLARY 1. *The set of real $m \times n$ matrices A which satisfy* (6), *i.e.*

$$|\langle \boldsymbol{q}A \rangle| = \prod_{i=1}^{m} (\bar{q}_i)^{-\alpha},$$

for infinitely many $\boldsymbol{q} = (q_1, \ldots, q_m)$ in $\boldsymbol{Z}^m$ has Hausdorff–Besicovitch dimension h given by

$$h = \begin{cases} (m-1)n + (n+1)/(\alpha+1), & \alpha > 1/n, \\ mn, & \alpha \leq 1/n. \end{cases}$$

Proof. Since

$$\{\boldsymbol{x} \in \boldsymbol{R}^n \colon |\boldsymbol{x}| < \varrho\} = (-\varrho, \varrho)^n \subset I^n$$

for $0 < \varrho \leq 1/2$, the distance function $F(\boldsymbol{x}) = |\boldsymbol{x}|$ satisfies (14). The restriction $\psi_1(q)$ of

$$\psi(\boldsymbol{q}) = \prod_{i=1}^{m} (\bar{q}_i)^{-\alpha}$$

is given by

$$\psi_1(q) = \psi(q, 0, \ldots, 0) = (\bar{q})^{-\alpha}.$$

By [6] or [7], when $\alpha > 1/n$, the dimension of the set W_1 say of points x in R^n satisfying
$$|\langle q x\rangle| < \psi_1(q)$$
for infinitely many integers q is
$$\frac{n+1}{\alpha+1} > \frac{1}{\alpha},$$
so that $s > (n+1)/(\alpha+1)$ implies $\alpha s > 1$. Now
$$\sum_{|q|=q} \psi(q)^s = \sum_{|q|=q} \prod_{i=1}^{m} (\bar{q}_i)^{-\alpha s} = q^{-\alpha s} \sum_{i=1}^{m} \prod_{\substack{j=1\\ j\neq i}}^{m} \sum_{q_j=-q}^{q} (\bar{q}_j)^{-\alpha s}$$
$$\ll q^{-\alpha s} \Big(\sum_{k=1}^{\infty} k^{-\alpha s}\Big)^{m-1} \ll \psi_1(q)^s$$
when $\alpha s > 1$. Hence (15) is satisfied when $\alpha > 1/n$ and
$$h = (m-1)n + \frac{n+1}{\alpha+1}.$$
When $\alpha \leqslant 1/n$, the dimension of W_1 is n and it follows from the Corollary to Lemma 1 that
$$h = mn.$$

Other forms of approximation can also be considered. For instance:

COROLLARY 2. *Let v, w be positive real numbers and let $\nu(q)$ be the number of non-zero components of $q = (q_1, \ldots, q_m)$. Then the set of real $m \times n$ matrices (a_{ij}) which satisfy*
$$\sum_{j=1}^{n} \Big\|\sum_{i=1}^{m} q_i a_{ij}\Big\|^v < \Big(\sum_{i=1}^{m} |q_i|^w\Big)^{-\nu(q)\alpha}$$
for infinitely many $q = (q_1, \ldots, q_m)$ in Z^m has Hausdorff–Besicovitch dimension h given by
$$h = \begin{cases} (m-1)n + v(n+1)/(w\alpha+v), & \alpha > v/wn,\\ mn, & \alpha \leqslant v/wn. \end{cases}$$

Proof. Since $\{(x_1, \ldots, x_n)\colon \sum_{j=1}^{n} |x_j|^v < \varrho^v\} \subset (-\varrho, \varrho)^n \subset I^n$ for $0 < \varrho \leqslant 1/2$, the distance function
$$F(x_1, \ldots, x_m) = \Big(\sum_{j=1}^{n} |x_j|^v\Big)^{1/v}$$
satisfies (14). Moreover, for $q \neq 0$,
$$\psi(q, 0, \ldots, 0) = \psi_1(q) = |q|^{-\alpha w/v}.$$

The set
$$F(c, \varrho) = \{x \in R^n\colon \sum_{j=1}^{n} |x_j - c_j|^v < \varrho^v\}$$
satisfies
$$C(c, \varrho n^{-1/v}) \subset F(c, \varrho) \subset C(c, \varrho)$$
where
$$C(c, \varrho) = \{x \in R^n\colon |x - c| < \varrho\}$$
and
$$1 \ll L(C(c, \varrho))/L(C(c, \varrho n^{-1/v})) = n^{1/v} \ll 1.$$
It follows that the Hausdorff–Besicovitch dimension of the set
$$\{x \in R^n\colon \Big(\sum_{j=1}^{n} \|q x_j\|^v\Big)^{1/v} < |q|^{-\alpha w/v} \text{ infinitely often}\}$$
is $v(n+1)/(w\alpha+v) > v/w\alpha$ when $\alpha > v/wn$ and n otherwise. Now
$$\sum_{|q|=q} \psi(q)^s \ll \sum_{k=1}^{m} \sum_{\substack{\nu(q)=k\\ |q|=q}} q^{-\nu(q)\alpha w s/v} \ll \sum_{k=1}^{m} q^{-k\alpha w s/v} q^{k-1}$$
$$\ll q^{-1} \sum_{k=1}^{m} q^{k(1-\alpha w s/v)} \ll q^{-\alpha w s/v} = \psi_1(q)^s$$
when $s > v/w\alpha$ and *a fortiori* when $s > v(n+1)/(w\alpha+v)$. Hence (15) is satisfied and the result follows.

Recently Yu has shown that when $F(x)$ is given by (7) and $\psi(q)$ by (4) the set $W(m, n)$ has dimension $mn - 1 + 2/(1+n\alpha)$ ([9]; the present notation differs from Yu's). The distance function (7) does not satisfy (11) and so the arguments of this paper, although derived in part from [9], do not apply. Nevertheless for $\alpha > 1/n$ and the distance function (7), it follows from [2] that the dimension of the set
$$\{(x_1, \ldots, x_n)\colon \prod_{j=1}^{n} \|q x_j\|^{1/n} < |q|^{-\alpha} \text{ infinitely often}\}$$
is
$$n - 1 + \frac{2}{1+n\alpha} > \frac{1}{\alpha}$$
so that by Corollary 1 to the theorem, (15) holds when $\psi(q)$ is given by (4). Moreover
$$mn - 1 + \frac{2}{1+n\alpha} = (m-1)n + n - 1 + \frac{2}{1+n\alpha},$$

so that the conclusion of the theorem holds, suggesting that a similar result might hold for a large class of distance functions and might be compared with the additivity of the Hausdorff–Besicovitch dimension for the cartesian product of certain sets. However, the results of this paper and [9] rely on the dimension being small enough to coincide with a relatively simple lower bound. The methods of this paper will not work for functions which do not satisfy (15) and in particular will not work for the function

$$\psi(q) = |q|^{-\alpha} \tag{16}$$

where $q \neq 0$, which is a natural generalization of the right-hand side of (5). Indeed with $\psi(q)$ given by (16) and $F(x) = |x|$, the dimension $h(W(m, n))$ of $W(m, n)$ is given by

$$h(W(m, n)) = (m-1)n + \frac{m+n}{\alpha+1}$$

when $\alpha > m/n$ [3].

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On some generalizations of the diophantine equation $1^k+2^k+\ldots+x^k = y^z$

by

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1. Introduction. In [5] J. J. Schäffer proved that for fixed integers $k > 0$ and $z > 1$ the equation

$$1^k+2^k+\ldots+x^k = y^z \tag{1}$$

has an infinite number of solutions in positive integers x and y only in the cases

(I) $k = 1, z = 2$; (II) $k = 3, z \in \{2, 4\}$; (III) $k = 5, z = 2$.

In all other cases the number of solutions was shown to be bounded by a constant depending only on k. In [2] and [7] K. Győry, R. Tijdeman and M. Voorhoeve have extended Schäffer's result by proving that for fixed $R(x) \in Z[x]$, $0 \neq b \in Z$ and fixed $k > 0$, $k \notin \{1, 3, 5\}$ the equation

$$1^k+2^k+\ldots+x^k+R(x) = by^z \tag{2}$$

has only finitely many solutions in integers $x, y > 1$, $z > 1$. If $R(x) \equiv r \in Z$, then their result is effective (cf. [2]).

The purpose of the present paper is to give some effective generalizations and extensions of the results of [2] and [7]. As a special case we get an effective version of the above-quoted finiteness theorem concerning the equation (2).

For brevity let us set $S_k(x) = 1^k+2^k+\ldots+x^k$, $A = Z[x]$ and $\varkappa = (k+1)\prod_{(p-1)|(k+1)!} p$ (p prime). Let

$$F(y) = Q_n y^n+\ldots+Q_1 y+Q_0 \in A[y].$$

Consider the solutions of the equation

$$F(S_k(x)) = by^z \tag{3}$$

in integers $x, y, z > 1$.