Since $c(10)\cdot d(6) = -5^5\cdot 9^9$ we may assume $c(10) = -9^9$, $d(6) = 5^5$. Hence $R(15) = -3^{18}\cdot x(6)^{10}\cdot Q(5)+5^5\cdot x(5)^6\cdot P(9)$. All monomials in the first term are divisible by at least 10 powers of $x(6)$ and those in the second by at most 9, so there is no cancellation of terms. Hence

$$Q(5) = -6^6\cdot a(0)\cdot x(6)^5, \quad P(9) = 10^{10}\cdot a(15)\cdot x(5)^9,$$
$$F = -9^9\cdot x(6)^{10}+10^{10}\cdot a(15)\cdot x(5)^9+P(8)+ \ldots +P(0).$$

Note that $E \bmod (x(5))$ is $3^{15}\cdot a(0)$ times a product of fifteen terms of the form $2^{2/5}\cdot 3^{3/5}\cdot x(6)+\varepsilon_i\cdot 5\cdot a(0)^{3/5}\cdot a(15)^{2/5}$ where the ε_i are various fifth roots of 1. Since $F \bmod (x(5))$ starts with $3^{18}\cdot x(6)^{10}$ it must contain a product, π, of ten of the above factors:

$$\pi = 2^4\cdot 3^6\cdot x(6)^{10}+ \ldots +\varepsilon\cdot 5^{10}\cdot a(0)^6\cdot a(15)^4 \quad \text{with} \quad \varepsilon^5 = 1$$

and

$$F \equiv -3^{18}\cdot x(6)^{10}+ \ldots \equiv -3^{12}\cdot 2^{-4}\cdot \pi \bmod (x(5)).$$

Hence the constant term, $P(0)$, of F is $-3^{12}\cdot 2^{-4}\cdot 5^{10}\cdot a(0)^6\cdot a(15)^4\cdot \varepsilon$. A similar argument gives

$$F \equiv 10^{10}\cdot a(15)\cdot x(5)^9+ \ldots \equiv 10^{10}\cdot 2^{-6}\cdot a(15)\cdot \pi' \bmod (x(6)),$$

where π' is a product of 9 factors of the form

$$2^{2/3}\cdot x(5)+\eta_j\cdot 3\cdot a(0)^{2/3}\cdot a(15)^{1/3} \quad \text{with} \quad \eta_j^3 = 1.$$

Hence $P(0) = 5^{10}\cdot 2^4\cdot 3^9\cdot a(0)^6\cdot a(15)^4\cdot \eta$. If the characteristic of $\bar{F}$ is not 2, 3 or 5, comparison of these two expressions for $P(0)$ gives $\eta\cdot 2^8 = -\varepsilon\cdot 3^3$. Put both sides to the 15th power to get a contradiction if $(\mathrm{Char}(\bar{F}), 2^{120}+ +3^{45}) = 1$. ■

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On sums of sequences of integers, I

by

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1. Throughout this paper, we use the following notation: $c_1, c_2, \ldots, M_0, M_1, \ldots$ denote positive absolute constants. $\theta_1, \theta_2, \ldots$ are real numbers such that $|\theta_i| \leqslant 1$ for all i. We write $e^x = \exp(x)$ and $e^{2\pi i\alpha} = e(\alpha)$. The distance from α to the nearest integer is denoted by $\|\alpha\|$ so that $\|\alpha\| = \min(\alpha-[\alpha], [\alpha]+1-\alpha)$. We put $\min(A, 1/0) = A$. We denote the least prime factor of n by $p(n)$, while the greatest prime factor of n is denoted by $P(n)$. $\nu(n)$ denotes the number of all the prime factors of n:

$$\nu(n) = \sum_{p^\alpha | n, p^{\alpha+1} \nmid n} \alpha.$$

2. In this series, we study the arithmetic nature of the numbers of the form $a+b$ where a, b are taken from "dense" sequences of integers. (See [2] for some related results.) In fact, this paper is devoted to the proof of the following theorem:

THEOREM. *Let $M > M_0$, $\mathscr{A} \subset \{1, 2, \ldots, M\}$ and $\mathscr{B} \subset \{1, 2, \ldots, M\}$. Put*

$$A(n) = \sum_{\substack{a \leqslant n \\ a \in \mathscr{A}}} 1, \quad B(n) = \sum_{\substack{b \leqslant n \\ b \in \mathscr{B}}} 1,$$
$$A = A(M), \quad B = B(M).$$

If

(1) $$AB > M^{5/3}(\log M)^{13},$$

then there exist integers a, b such that $a \in \mathscr{A}$, $b \in \mathscr{B}$ and

(2) $$P(a+b) \leqslant y$$

where

(3) $$y \stackrel{\text{def}}{=} \begin{cases} \exp\{4(\log M \log\log M)^{1/2}\} & \text{for } AB > M^2\exp\{-2(\log M\log\log M)^{1/2}\}, \\ \dfrac{M^2}{AB}\exp\left(4\dfrac{\log M}{\log(M^2/AB)}\log\log M\right) & \text{for } AB \leqslant M^2\exp\{-2(\log M \log\log M)^{1/2}\}. \end{cases}$$

The proof is based on the same method as in [1]. In fact, we need some lemmas here which can be found also in [1] (apart from some trivial modifications of the constants); however, for the sake of completeness, we give all the details also here.

(Note that the term y on the right hand side of (2) can not be replaced by $y^{1/2-\varepsilon}$. This can be shown by the following construction: let p denote the least prime number such that $p > y^{1/2-\varepsilon}$, and let $\mathscr{A} = \mathscr{B} = \{p, 2p, \ldots, [M/p]p\}$.)

3. For $1 \leqslant n \leqslant 2M$, we put

$$T(n) = \sum_{\substack{n/10 < a+b \leqslant n \\ a\in\mathscr{A}, b\in\mathscr{B}}} 1.$$

First we need the following lemma:

LEMMA 1. *Put*

$$\max_{1\leqslant n\leqslant 2M} T(n)\frac{M}{n} = T,$$

and let N be an integer such that

(4) $$T(N)\frac{M}{N} = T.$$

Then we have

(5) $$T(N) > \frac{1}{40}AB\frac{N}{M},$$

(6) $$M^{2/3} < N \leqslant 2M$$

and

(7) $$A(N)B(N) < 13T(N).$$

Proof. Let us define the positive integer k by

$$10^{k-1} < 2M \leqslant 10^k.$$

Then by the definitions of T, N and k, we have

$$AB = \Big(\sum_{a\in\mathscr{A}} 1\Big)\Big(\sum_{b\in\mathscr{B}} 1\Big) = \sum_{\substack{a+b \\ a\in\mathscr{A}, b\in\mathscr{B}}} 1 = \sum_{j=1}^{k} \sum_{\substack{10^{j-1} < a+b \leqslant 10^j \\ a\in\mathscr{A}, b\in\mathscr{B}}} 1 = \sum_{j=1}^{k} T(10^j)$$

$$\leqslant \sum_{j=1}^{k} T\frac{10^j}{M} < 2\cdot 10^k \frac{T}{M} \leqslant 40M\cdot\frac{T}{M} = 40T = 40T(N)\frac{M}{N}$$

which proves (5).

Obviously, for $n \geqslant 2M$ we have $T(n+1) \leqslant T(n)$, so that $T(n)\frac{M}{n}$ is a decreasing function of n in $2M \leqslant n < +\infty$; this implies that

(8) $$N \leqslant 2M.$$

Furthermore, we have

(9) $$T(N) \leqslant \sum_{\substack{a+b\leqslant N \\ a\in\mathscr{A}, b\in\mathscr{B}}} 1 \leqslant \Big(\sum_{\substack{a\leqslant N \\ a\in\mathscr{A}}} 1\Big)\Big(\sum_{\substack{b\leqslant N \\ b\in\mathscr{B}}} 1\Big) \leqslant N\cdot N = N^2.$$

(5) and (9) yield that

$$\frac{1}{40}AB\frac{N}{M} < T(N) \leqslant N^2,$$

thus with respect to (1),

(10) $$N > \frac{1}{40}\frac{AB}{M} > \frac{1}{40}\frac{M^{5/3}(\log M)^{13}}{M} > M^{2/3}.$$

(8) and (10) yield (6).

Finally, define the positive integer r by

$$10^{r-1} < N/10 \leqslant 10^r.$$

Then by the definition of T and with respect to (4) we have

$$A(N)B(N) = \Big(\sum_{\substack{a\leqslant N \\ a\in\mathscr{A}}} 1\Big)\Big(\sum_{\substack{b\leqslant N \\ b\in\mathscr{B}}} 1\Big) = \sum_{\substack{a\leqslant N, b\leqslant N \\ a\in\mathscr{A}, b\in\mathscr{B}}} 1$$

$$\leqslant \sum_{\substack{a+b\leqslant 2N \\ a\in\mathscr{A}, b\in\mathscr{B}}} 1 = \sum_{\substack{a+b\leqslant N/10 \\ a\in\mathscr{A}, b\in\mathscr{B}}} 1 + \sum_{\substack{N/10 < a+b\leqslant N \\ a\in\mathscr{A}, b\in\mathscr{B}}} 1 + \sum_{\substack{N < a+b\leqslant 2N \\ a\in\mathscr{A}, b\in\mathscr{B}}} 1$$

$$\leqslant \sum_{j=1}^{r} T(10^j) + T(N) + T(10N) \leqslant \sum_{j=1}^{r} T\frac{10^j}{M} + T\frac{N}{M} + T\frac{10N}{M}$$

$$< 2T\frac{10^r}{M} + 11\frac{TN}{M} < 2T\frac{N}{M} + 11\frac{TN}{M} = 13\frac{TN}{M} = 13T(N)$$

which completes the proof of Lemma 1.

4. Let N be an integer satisfying the conditions in Lemma 1 and define y by (3). Then by (3) and (6), and with respect to the inequality

$$a + b/a \geqslant 2\sqrt{b}$$

$(a, b \geqslant 0)$, for $AB \leqslant M^2 \exp\{-2(\log M \log\log M)^{1/2}\}$ we have

$$y=\frac{M^2}{AB}\exp\left\{4\frac{\log M}{\log(M^2/AB)}\log\log M\right\}=\exp\left\{\log\frac{M^2}{AB}+\frac{4\log M\log\log M}{\log(M^2/AB)}\right\}$$
$$\geqslant \exp\{2(4\log M\log\log M)^{1/2}\}=\exp\{4(\log M\log\log M)^{1/2}\}$$

so that in both cases in (3) we have

(11) $$y\geqslant \exp\{4(\log M\log\log M)^{1/2}\}$$
$$\geqslant \exp\{3(\log 2M\log\log 2M)^{1/2}\}\geqslant \exp\{3(\log N\log\log N)^{1/2}\}.$$

On the other hand, the function

$$f(x)=x+a/x$$

is increasing for $\sqrt{a}\leqslant x<+\infty$, so that with respect to (1) and (6), for $AB\leqslant M^2\exp\{-2(\log M\log\log M)^{1/2}\}$ we have

$$y=\frac{M^2}{AB}\exp\left\{4\frac{\log M}{\log(M^2/AB)}\log\log M\right\}=\exp\left\{\log\frac{M^2}{AB}+\frac{4\log M\log\log M}{\log(M^2/AB)}\right\}$$
$$\leqslant \exp\left\{\log\frac{M^{1/3}}{(\log M)^{13}}+\frac{4\log M\log\log M}{\log\left(M^{1/3}/(\log M)^{13}\right)}\right\}$$
$$<\exp\{(\tfrac{1}{3}\log M-13\log\log M)+13\log\log M\}=M^{1/3}<N^{2/3},$$

and the inequality

(12) $$y<N^{2/3}$$

holds trivially also for $AB>M^2\exp\{-2(\log M\log\log M)^{1/2}\}$; in fact, with respect to (6), in this case we have

$$y=\exp\{4(\log M\log\log M)^{1/2}\}<\exp\{4(\log N^2\log\log N^2)^{1/2}\}$$
$$<\exp\{8(\log N\log\log N)^{1/2}\}=N^{o(1)}.$$

Put

$$z=\frac{1}{2}y^{1/2},\quad Q=\frac{N}{z}=2\frac{N}{y^{1/2}}\quad\text{and}\quad U=[4N/y]+1.$$

Let $\mathscr{K}$ denote the set of the integers k such that $N/y<k\leqslant 2N/y$ and $z<p(k)$, $P(k)\leqslant y$. We write

$$K=\sum_{k\in\mathscr{K}}1,$$
$$d_n=\sum_{\substack{mk=n\\ m\leqslant y\\ k\in\mathscr{K}}}1,$$
$$S_x(\alpha)=\sum_{n\leqslant x}d_n e(n\alpha)\ (\text{for } 0\leqslant x\leqslant N),$$

$$S(\alpha)=S_N(\alpha)=\sum_{n=1}^{N}d_n e(n\alpha),$$
$$S=S(0)=\sum_{n=1}^{N}d_n,$$
$$U(\alpha)=\sum_{n=0}^{U-1}e(n\alpha),$$
$$S(\alpha)U(\alpha)=\sum_{n=1}^{N+U-1}v_n e(n\alpha)\quad\Big(\text{so that } v_n=\sum_{n-U<j\leqslant n}d_j\Big),$$
$$F(\alpha)=\sum_{\substack{a\leqslant N\\ a\in\mathscr{A}}}e(a\alpha),$$
$$G(\alpha)=\sum_{\substack{b\leqslant N\\ b\in\mathscr{B}}}e(b\alpha)$$

and

$$H(\alpha)=F(\alpha)G(\alpha)=\sum_{\substack{a\leqslant N,b\leqslant N\\ a\in\mathscr{A},b\in\mathscr{B}}}e\big((a+b)\alpha\big)=\sum_{n=1}^{2N}h_n e(n\alpha)$$
$$\Big(\text{so that } h_n=\sum_{\substack{a+b=n\\ a\leqslant N,b\leqslant N\\ a\in\mathscr{A},b\in\mathscr{B}}}1\Big).$$

We start out from the integral

$$J=\int_0^1 F(\alpha)G(\alpha)S(-\alpha)\,d\alpha=\int_0^1 H(\alpha)S(-\alpha)\,d\alpha$$
$$=\int_0^1\sum_{n=1}^{2N}\sum_{m=1}^{N}h_n d_m e\big((n-m)\alpha\big)\,d\alpha=\sum_{n=1}^{N}h_n d_n.$$

Obviously, $d_n>0$ implies that

$$P(n)\leqslant y$$

while $h_n>0$ implies that n can be written in the form

$$a+b=n\quad(a\in\mathscr{A},\ b\in\mathscr{B}).$$

Thus in order to prove the solvability of (2), it is sufficient to show that

(13) $$J=\sum_{n=1}^{N}h_n d_n>0.$$

In order to prove this (by using the Hardy–Littlewood method), we need some lemmas.

5. In this section, we assert some preliminary lemmas.

LEMMA 2. *If V is a positive integer, α a real number then we have*

$$\Big|\sum_{n=0}^{V-1} e(n\alpha)-V\Big| \leqslant 4V^2|\alpha|.$$

Proof. With respect to the well-known inequality

$$|1-e(\beta)| \leqslant 2\pi|\beta| \tag{14}$$

we have

$$\Big|\sum_{n=0}^{V-1} e(n\alpha)-V\Big| \leqslant \sum_{n=0}^{V-1}|e(n\alpha)-1| \leqslant \sum_{n=0}^{V-1} 2\pi n|\alpha| = \pi(V-1)V|\alpha| \leqslant 4V^2|\alpha|.$$

LEMMA 3. *For arbitrary real numbers α, x we have*

$$\Big|\sum_{1\leqslant m\leqslant x} e(m\alpha)\Big| \leqslant \min\Big\{x, \frac{1}{2\|\alpha\|}\Big\}.$$

See e.g. [3], p. 9.

LEMMA 4. *If α, r are real numbers and a, q, f are integers such that $q>0$, $(a, q)=1$ and $|\alpha-a/q| \leqslant 1/q^2$ then we have*

$$\sum_{x=f+1}^{f+q} \min\Big(r, \frac{1}{2\|\alpha x\|}\Big) \leqslant 6r+q\log q.$$

See e.g. [3], p. 23.

LEMMA 5. *If α, r, s are real numbers and a, q are integers such that $s \geqslant 1$, $q>0$, $(a, q)=1$ and $|\alpha-a/q| \leqslant 1/q^2$ then we have*

$$\sum_{x\leqslant s} \min\Big(r, \frac{1}{2\|\alpha x\|}\Big) \leqslant \Big(\frac{s}{q}+1\Big)(6r+q\log q).$$

Proof. With respect to Lemma 4, we have

$$\sum_{x\leqslant s} \min\Big(r, \frac{1}{2\|\alpha x\|}\Big) \leqslant \sum_{k=1}^{[s/q]+1} \sum_{x=(k-1)q+1}^{kq} \min\Big(r, \frac{1}{2\|\alpha x\|}\Big)$$
$$\leqslant \sum_{k=1}^{[s/q]+1} (6r+q\log q) = \Big(\Big[\frac{s}{q}\Big]+1\Big)(6r+q\log q)$$
$$< \Big(\frac{s}{q}+1\Big)(6r+q\log q).$$

6. In this section, we estimate S, $S(\alpha)$, v_n and K.

LEMMA 6. *We have $S \leqslant 2N$.*

Proof.

$$S = \sum_{n=1}^{N} d_n = \sum_{n=1}^{N} \sum_{\substack{mk=n\\ m\leqslant y\\ k\in\mathscr{K}}} 1 = \sum_{m\leqslant y} \sum_{\substack{k\leqslant N/m\\ k\in\mathscr{K}}} 1$$
$$\leqslant \sum_{m\leqslant y} \sum_{k\in\mathscr{K}} 1 \leqslant \sum_{m\leqslant y} \sum_{k\leqslant 2N/y} 1 \leqslant \sum_{m\leqslant y} 2\frac{N}{y} \leqslant y\cdot 2\frac{N}{y} = 2N.$$

LEMMA 7. *If $1 \leqslant u \leqslant N$ and a, q are integers such that $2 \leqslant q \leqslant z$ and $(a, q)=1$ then we have*

$$|S_u(a/q)| \leqslant 2\frac{Nq}{y}.$$

Proof. We have

$$S_u(a/q) = \sum_{n\leqslant u} d_n e(na/q) = \sum_{b=1}^{q} \Big(\sum_{\substack{n\leqslant u\\ na\equiv b \pmod q}} d_n\Big) e(b/q). \tag{15}$$

Here the inner sum can be rewritten in the following form:

$$\sum_{\substack{n\leqslant u\\ na\equiv b \pmod q}} d_n = \sum_{\substack{n\leqslant u\\ na\equiv b \pmod q}} \sum_{\substack{mk=n\\ m\leqslant y\\ k\in\mathscr{K}}} 1 \tag{16}$$
$$= \sum_{\substack{mk\leqslant u\\ mka\equiv b \pmod q\\ m\leqslant y\\ k\in\mathscr{K}}} 1 = \sum_{\substack{k\leqslant u\\ k\in\mathscr{K}}} \sum_{\substack{m\leqslant u/k\\ mka\equiv b \pmod q\\ m\leqslant y}} 1$$
$$= \sum_{\substack{k\leqslant u/y\\ k\in\mathscr{K}}} \sum_{\substack{m\leqslant y\\ mka\equiv b \pmod q}} 1 + \sum_{\substack{u/y<k\leqslant u\\ k\in\mathscr{K}}} \sum_{\substack{m\leqslant u/k\\ mka\equiv b \pmod q}} 1.$$

$p(k) > z \geqslant q$ and $(a, q)=1$ imply that $(ka, q)=1$ hence

$$\sum_{\substack{m\leqslant y\\ mka\equiv b \pmod q}} 1 = y/q+\theta_1$$

and

$$\sum_{\substack{m\leqslant u/k\\ mka\equiv b \pmod q}} 1 = u/kq+\theta_2$$

(where $\theta_1=\theta_1(a, b, k)$, $\theta_2=\theta_2(a, b, k)$). Thus we obtain from (15) and (16) that

$$S_u(a/q) = \sum_{b=1}^{q} \Big(\sum_{\substack{n\leqslant u\\ na\equiv b \pmod q}} d_n\Big) e(b/q)$$

$$= \sum_{b=1}^{q}\Big(\sum_{\substack{k\le u/y\\ k\in\mathcal{K}}}\Big(\frac{y}{q}+\theta_1\Big)+\sum_{\substack{u/y<k\le u\\ k\in\mathcal{K}}}\Big(\frac{u}{kq}+\theta_2\Big)\Big)e(b/q)$$

$$= \sum_{b=1}^{q}\Big\{\Big(\sum_{\substack{k\le u/y\\ k\in\mathcal{K}}}\frac{y}{q}+\sum_{\substack{u/y<k\le u\\ k\in\mathcal{K}}}\frac{u}{kq}\Big)+\theta_3\sum_{\substack{k\le u\\ k\in\mathcal{K}}}1\Big\}e(b/q)$$

$$= \Big(\sum_{\substack{k\le u/y\\ k\in\mathcal{K}}}\frac{y}{q}+\sum_{\substack{u/y<k\le u\\ k\in\mathcal{K}}}\frac{u}{kq}\Big)\sum_{b=1}^{q}e(b/q)+\sum_{b=1}^{q}\Big(\theta_4\sum_{k\in\mathcal{K}}1\Big)$$

$$= \theta_5 q\sum_{k\le 2N/y}1 = 2\theta_6\frac{Nq}{y}\le 2\frac{Nq}{y}$$

(where the numbers θ_i depend on a, b, k) since by $q\ge 2$,

$$\sum_{b=1}^{q}e(b/q)=0.$$

Lemma 8. *If α is a real number and a, q are integers such that $2\le q\le z$, $(a,q)=1$ and $|\alpha-a/q|<1/qQ$ then we have*

$$|S(\alpha)|<8\frac{N}{y^{1/2}}.$$

Proof. We write $\beta=\alpha-a/q$ so that

$$|\beta|=\Big|\alpha-\frac{a}{q}\Big|<\frac{1}{qQ}.$$

Then by using Lemma 7 and (14), we obtain by partial summation that

$$|S(\alpha)|=\Big|\sum_{n=1}^{N}\big(S_n(a/q)-S_{n-1}(a/q)\big)e(n\beta)\Big|$$

$$=\Big|\sum_{n=1}^{N}S_n(a/q)\big(e(n\beta)-e((n+1)\beta)\big)+S_N(u/q)e((N+1)\beta)\Big|$$

$$\le\sum_{n=1}^{N}2\frac{Nq}{y}\cdot 2\pi|\beta|+2\frac{Nq}{y}=2\frac{Nq}{y}(1+2\pi N|\beta|)<2\frac{Nq}{y}\Big(1+7N\frac{1}{qQ}\Big)$$

$$=2\frac{Nq}{y}\Big(\frac{N}{zQ}+7\frac{N}{qQ}\Big)\le 2\frac{Nq}{y}\cdot 8\frac{N}{qQ}=16\frac{N^2}{yQ}=8\frac{N}{y^{1/2}}.$$

Lemma 9. *If α is a real number and a, q are integers such that $z<q\le Q$, $(a,q)=1$ and $|\alpha-a/q|\le 1/q^2$ then for large M (then also N is large) we have*

$$|S(\alpha)|<7\frac{N}{y^{1/2}}\log N.$$

Proof. By using Lemmas 3 and 5, and with respect to (12), we obtain for large N that

$$|S(\alpha)|=\Big|\sum_{n=1}^{N}d_n e(n\alpha)\Big|=\Big|\sum_{\substack{mk\le N\\ m\le y\\ k\in\mathcal{K}}}e(mk\alpha)\Big|$$

$$=\Big|\sum_{k\in\mathcal{K}}\Big(\sum_{m\le\min(N/k,y)}e(mk\alpha)\Big)\Big|\le\sum_{k\in\mathcal{K}}\Big|\sum_{m\le\min(N/k,y)}e(mk\alpha)\Big|$$

$$\le\sum_{k\le 2N/y}\min\Big(\min(N/k,y),\frac{1}{2\|k\alpha\|}\Big)\le\sum_{k\le 2N/y}\min\Big(y,\frac{1}{2\|k\alpha\|}\Big)$$

$$\le\Big(2\frac{N}{qy}+1\Big)(6y+q\log q)=12\frac{N}{q}+6y+2\frac{N}{y}\log q+q\log q$$

$$<12\frac{N}{z}+3\cdot 2\frac{N}{y^{1/2}}\cdot\frac{y^{3/2}}{N}+2\frac{N}{y}\log N+Q\log N$$

$$<12\frac{N}{z}+3\frac{N}{z}+2\frac{N}{z}\log N+\frac{N}{z}\log N=(3+o(1))\frac{N}{z}\log N$$

$$=(6+o(1))\frac{N}{y^{1/2}}\log N<7\frac{N}{y^{1/2}}\log N.$$

Lemma 10. *If*

$$1/Q<\alpha<1-1/Q, \tag{17}$$

then for large M (and N) we have

$$|S(\alpha)|<7\frac{N}{y^{1/2}}\log N. \tag{18}$$

Proof. By Dirichlet's theorem, there exist integers a, q such that $1\le q\le Q$, $(a,q)=1$ and

$$\Big|\alpha-\frac{a}{q}\Big|<\frac{1}{qQ}\ \Big(\le\frac{1}{q^2}\Big).$$

(17) implies that $q>1$. If $2\le q\le z$ then (18) is a consequence of Lemma 8 while if $z<q\le Q$ then (18) holds by Lemma 9.

Lemma 11. *If n is a positive integer satisfying $U\le n\le N$ then we have*

$$v_n\ge K.$$

Proof. For $U\le n\le N$ we have

$$v_n=\sum_{j=n-U+1}^{n}d_j=\sum_{j=n-U+1}^{n}\sum_{\substack{mk=j\\ m\le y\\ k\in\mathcal{K}}}1$$

$$= \sum_{\substack{n-U<mk\leqslant n\\ m\leqslant y\\ k\in\mathcal{K}}} 1 = \sum_{k\in\mathcal{K}} \sum_{\substack{(n-U)/k<m\leqslant n/k\\ m\leqslant y}} 1$$

$$= \sum_{k\in\mathcal{K}}\sum_{(n-U)/k<m\leqslant n/k} 1 \geqslant \sum_{k\in\mathcal{K}}\left(\frac{U}{k}-1\right)$$

$$\geqslant \sum_{k\in\mathcal{K}}\left(\frac{U}{k}-\frac{U}{4N/y}\right) \geqslant \sum_{k\in\mathcal{K}}\left(\frac{U}{k}-\frac{U}{2k}\right)=\frac{U}{2}\sum_{k\in\mathcal{K}}\frac{1}{k}$$

$$\geqslant \frac{U}{2}\sum_{k\in\mathcal{K}}\frac{1}{2N/y}=\frac{Uy}{4N}\sum_{k\in\mathcal{K}}1=\frac{Uy}{4N}K>K$$

since for $k\in\mathcal{K}$ and $n\leqslant N$,

$$\frac{n}{k}<\frac{N}{N/y}=y.$$

LEMMA 12. *For $t>0$ and $j=1,2,\ldots$, let*

$$K_j(t)=\sum_{\substack{t/2<k\leqslant t\\ z<p(k)\leqslant P(k)\leqslant y\\ \nu(k)\leqslant j}} 1.$$

If M (and thus also N) is large and $2\leqslant j$ then for

(19) $$2z<t\leqslant y^j/2^{j-2}$$

we have

(20) $$K_j(t)>\frac{t}{j!(5\log y)^j}.$$

Proof. We prove the assertion by induction (with respect to j). Assume first that $j=2$ so that

$$2z<t\leqslant y^2.$$

If $2z<t\leqslant y$ then for large M (then also N and y are large) we have

(21) $$K_2(t)=\sum_{\substack{t/2<k\leqslant t\\ z<p(k)\leqslant P(k)\leqslant y\\ \nu(k)\leqslant 2}} 1 \geqslant \sum_{\substack{t/2<p\leqslant t\\ z<p\leqslant y}} 1$$

$$=\sum_{t/2<p\leqslant t} 1>\frac{1}{3}\frac{t}{\log t}\geqslant\frac{1}{3}\frac{t}{\log y}>\frac{t}{2!(5\log y)^2},$$

while for $y<t\leqslant y^2$ and large M we have

(22) $$K_2(t)=\sum_{\substack{t/2<k\leqslant t\\ z<p(k)\leqslant P(k)\leqslant y\\ \nu(k)\leqslant 2}} 1\geqslant \sum_{\substack{t/2<pq\leqslant t\\ z<p\leqslant q\leqslant y}} 1$$

$$\geqslant\frac{1}{2}\Big(\sum_{\sqrt{t/2}<p\leqslant\sqrt{t}} 1\Big)^2\geqslant\frac{1}{2}\left(\frac{1}{4}\frac{\sqrt{t}}{\log\sqrt{t}}\right)^2$$

$$>\frac{1}{10}\frac{t}{\log^2 t}\geqslant\frac{1}{40}\frac{t}{\log^2 y}>\frac{t}{2!(5\log y)^2}$$

(since $\sqrt{t/2}>\sqrt{y/2}=\sqrt{(2z)^2/2}>z$ and $\sqrt{t}\geqslant y$).

(21) and (22) yield (20) (with $j=2$) in both cases.

Assume now that (20) holds for all t satisfying (19). We have to show that

$$2z<t\leqslant y^{j+1}/2^{j-1}$$

implies

(23) $$K_{j+1}(t)>\frac{t}{(j+1)!(5\log y)^{j+1}}.$$

If $2z<t\leqslant y^j/2^{j-2}$ then this is a consequence of (20) and the trivial inequality $K_j(t)\leqslant K_{j+1}(t)$. (Note that the right hand side of (20) is a decreasing function of j.) Thus it is sufficient to study the case

(24) $$y^j/2^{j-2}<t\leqslant y^{j+1}/2^{j-1}.$$

Then we have

(25) $$K_{j+1}(t)=\sum_{\substack{t/2<k\leqslant t\\ z<p(k)\leqslant P(k)\leqslant y\\ \nu(k)\leqslant j+1}} 1$$

$$\geqslant\frac{1}{j+1}\sum_{y/2<p\leqslant y}\sum_{\substack{t/2<pl\leqslant t\\ z<p(l)\leqslant P(l)\leqslant y\\ \nu(l)\leqslant j}} 1=\frac{1}{j+1}\sum_{y/2<p\leqslant y}K_j(t/p).$$

If t satisfies (24) and $y/2<p\leqslant y$ then

$$\frac{t}{p}\geqslant\frac{y^j/2^{j-2}}{y}\geqslant\frac{y^2/2^{2-2}}{y}=y>2z\quad\text{and}\quad\frac{t}{p}<\frac{y^{j+1}/2^{j-1}}{y/2}=\frac{y^j}{2^{j-2}}$$

so that (19) holds and thus (20) can be used in order to estimate $K_j(t/p)$. We obtain from (25) that for large M,

$$K_{j+1}(t)\geqslant\frac{1}{j+1}\sum_{y/2<p\leqslant y}K_j(t/p)>\frac{1}{j+1}\sum_{y/2<p\leqslant y}\frac{t/p}{j!(5\log y)^j}$$

$$=\frac{t}{(j+1)!(5\log y)^j}\sum_{y/2<p\leqslant y}\frac{1}{p}\geqslant\frac{t}{(j+1)!(5\log y)^j}\frac{1}{y}\sum_{y/2<p\leqslant y}1$$

$$>\frac{t}{(j+1)!(5\log y)^j}\frac{1}{y}\cdot\frac{1}{3}\frac{y}{\log y}>\frac{t}{(j+1)!(5\log y)^{j+1}}$$

which proves (23) and this completes the proof of Lemma 12.

LEMMA 13. *For large M we have*

$$K > \frac{N}{y}\exp\left(-\frac{6}{5}\frac{\log N}{\log y}\log\log N\right).$$

Proof. Define the positive integer j by

$$\frac{y^{j-1}}{2^{j-3}} < 2\frac{N}{y} \leqslant \frac{y^j}{2^{j-2}}$$

so that

$$\left(\frac{y}{2}\right)^j \leqslant \frac{N}{4} < N, \quad j < \frac{\log N}{\log(y/2)}.$$

Then for large N, Lemma 12 yields that

$$K = \sum_{\substack{N/y < k \leqslant 2N/y \\ z < p(k) \leqslant P(k) \leqslant y}} 1 \geqslant \sum_{\substack{N/y < k \leqslant 2N/y \\ z < p(k) \leqslant P(k) \leqslant y \\ \nu(k) \leqslant j}} 1 = K_j(2N/y)$$

$$> \frac{2N/y}{j!(5\log y)^j} > \frac{N}{y}\frac{1}{(5j\log y)^j} = \frac{N}{y}\exp\{-j\log(5j\log y)\}$$

$$> \frac{N}{y}\exp\left\{-\frac{\log N}{\log(y/2)}\log\left(7\frac{\log N}{\log(y/2)}\log y\right)\right\}$$

$$= \frac{N}{y}\exp\left\{-(1+o(1))\frac{\log N}{\log y}\log\log N\right\} > \frac{N}{y}\exp\left(-\frac{6}{5}\frac{\log N}{\log y}\log\log N\right).$$

7. In this section, we complete the proof of the theorem.

By using Lemmas 2, 6, 10, Cauchy's inequality and Parseval's formula, and with respect to (7), we obtain that

$$(26)\qquad \left|J - \frac{1}{U}\int_0^1 F(\alpha)G(\alpha)U(-\alpha)S(-\alpha)\,d\alpha\right|$$

$$= \left|\int_{-1/Q}^{+1/Q} F(\alpha)G(\alpha)S(-\alpha)\left(1-\frac{U(-\alpha)}{U}\right)d\alpha + \int_{+1/Q}^{1-1/Q} F(\alpha)G(\alpha)S(-\alpha)\left(1-\frac{U(-\alpha)}{U}\right)d\alpha\right|$$

$$\leqslant \int_{-1/Q}^{+1/Q} |F(\alpha)|\,|G(\alpha)|\,|S(-\alpha)|\left|\frac{U-U(-\alpha)}{U}\right|d\alpha + \int_{+1/Q}^{1-1/Q} |F(\alpha)|\,|G(\alpha)|\,|S(-\alpha)|\left(1+\left|\frac{U(-\alpha)}{U}\right|\right)d\alpha$$

$$\leqslant \int_{-1/Q}^{+1/Q} |F(\alpha)|\,|G(\alpha)|\,S\frac{4U^2|\alpha|}{U}\,d\alpha + \int_{+1/Q}^{1-1/Q} |F(\alpha)|\,|G(\alpha)|\left(\max_{+1/Q<\beta<1-1/Q}|S(\beta)|\right)\cdot 2\,d\alpha$$

$$\leqslant \int_{-1/Q}^{1+1/Q} |F(\alpha)|\,|G(\alpha)|\cdot 8NU\frac{1}{Q}\,d\alpha + \int_{+1/Q}^{1-1/Q} |F(\alpha)|\,|G(\alpha)|\cdot 2\cdot 7\frac{N}{y^{1/2}}\log N\,d\alpha$$

$$\leqslant \left(8\frac{NU}{Q} + 14\frac{N}{y^{1/2}}\log N\right)\int_0^1 |F(\alpha)|\,|G(\alpha)|\,d\alpha$$

$$< \left(8\frac{N\cdot 5N/y}{2N/y^{1/2}} + 14\frac{N}{y^{1/2}}\log N\right)\left\{\left(\int_0^1 |F(\alpha)|^2 d\alpha\right)\left(\int_0^1 |G(\alpha)|^2 d\alpha\right)\right\}^{1/2}$$

$$= \left(20\frac{N}{y^{1/2}} + 14\frac{N}{y^{1/2}}\log N\right)(A(N)B(N))^{1/2}$$

$$< 15\frac{N}{y^{1/2}}\log N(A(N)B(N))^{1/2} < 15\frac{N}{y^{1/2}}\log N(13T(N))^{1/2}$$

$$< 60\frac{N}{y^{1/2}}\log N(T(N))^{1/2}.$$

Furthermore, by Lemma 11 and since $h_n \geqslant 0$ and $v_n \geqslant 0$, for large N we have

$$(27)\qquad \int_0^1 F(\alpha)G(\alpha)U(-\alpha)S(-\alpha)\,d\alpha$$

$$= \int_0^1\Big(\sum_{n=1}^{2N} h_n e(n\alpha)\Big)\Big(\sum_{n=1}^{2N+U-1} v_n e(-n\alpha)\Big)d\alpha = \sum_{n=1}^{2N} h_n v_n \geqslant \sum_{U<n\leqslant N} h_n v_n$$

$$\geqslant \sum_{U<n\leqslant N} h_n K = K\sum_{\substack{U<a+b\leqslant N \\ a\in\mathcal{A},\, b\in\mathcal{B}}} 1 \geqslant K\sum_{\substack{5N/y<a+b\leqslant N \\ a\in\mathcal{A},\, b\in\mathcal{B}}} 1 \geqslant K\sum_{\substack{N/10<a+b\leqslant N \\ a\in\mathcal{A},\, b\in\mathcal{B}}} 1$$

$$= KT(N).$$

(26) and (27) yield that

$$(28)\qquad |J| \geq \frac{1}{U}\left|\int_0^1 F(\alpha)G(\alpha)U(-\alpha)S(-\alpha)d\alpha\right| - 60\frac{N}{y^{1/2}}\log N\,(T(N))^{1/2}$$
$$\geq \frac{1}{5N/y}KT(N) - 60\frac{N}{y^{1/2}}\log N\,(T(N))^{1/2}$$
$$\geq \frac{1}{5}(T(N))^{1/2}\frac{N}{y^{1/2}}\log N\left(\frac{K}{N/y}(T(N))^{1/2}\frac{y^{1/2}}{N\log N} - 300\right).$$

By (5), (6) and Lemma 13, here we have

$$(29)\qquad \frac{K}{N/y}(T(N))^{1/2}\frac{y^{1/2}}{N\log N} - 300$$
$$> \exp\left(-\frac{6}{5}\frac{\log N\log\log N}{\log y}\right)\left(\frac{1}{40}AB\frac{N}{M}\right)^{1/2}\frac{y^{1/2}}{N\log N} - 300$$
$$> \frac{1}{10\log M}\left(\frac{ABy}{M^2}\right)^{1/2}\exp\left(-\frac{3}{2}\frac{\log M\log\log M}{\log y}\right) - 300.$$

(3) easily implies that

$$(30)\qquad \frac{ABy}{M^2} > \exp\left(4\frac{\log M\log\log M}{\log y}\right)$$

and finally we obtain by combining (29), (30) and the fact $y < M^{1/3}$ (see the line before (12))

$$(31)\qquad \frac{K}{N/y}(T(N))^{1/2}\frac{y^{1/2}}{N\log N} - 300$$
$$> \frac{1}{10\log M}\exp\left(\frac{1}{2}\frac{\log M\log\log M}{\log y}\right) - 300 > \frac{\log^{1/2} M}{10} - 300 > 0.$$

(28) and (31) yield that $|J| > 0$ which proves (13) and this completes the proof of the theorem.

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