(XI.19) $|R_{n+1}^{*}(M)-K_{n+1}^{*}(M)\mathfrak{S}^{*}(M)| \ll K_{n+1}^{*}(M)n^{3+(1/4\tau)-(k\tau h/2)}.$

Démonstration. On applique la proposition précédente. Avec (XI.5) et (V.13), il vient

$$|R_i(M)-K_i(M)\mathfrak{S}(M)|$$
$$\ll K_i(M)q^{-ks/2}+i^{1/4\varrho}q^{i-ks+m}m^{-1}+\frac{s}{m}q^{i+3ks+(m/2)}\sum_{\substack{H\in\mathcal{U}\\ d^0H\leqslant ks}}\Phi(H),$$

d'où, avec (XI.7) et (XI.13),

$$|R_i(M)-K_i(M)\mathfrak{S}(M)| \ll K_i(M)\left[q^{-ks/2}+mi^{1/4\varrho}q^{-ks}+\frac{s}{m}q^{5ks-(m/2)}\right],$$

(XI.16) et (XI.18) se déduisent alors de (IV.1) et (IV.7).

On procède de même pour (XI.17) et (XI.19).

Cette démonstration achève la démonstration du théorème. Il suffit de choisir maintenant h suffisamment grand pour que $3+(1/4\tau)-(k\tau h/2)$ soit >0, c'est à dire $h>(3+2^{2k-2})2^{2k+1}/k$. Les nombres $R_i(M)$ et $R_i^{*}(M)$ sont alors asymptotiquement équivalents aux nombres $K_i(M)\mathfrak{S}(M)$ et $K_i^{*}(M)\mathfrak{S}^{*}(M)$ qui sont strictement positifs.

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On elliptic units and class number of a certain dihedral extension of degree $2l$

by

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0. Introduction. G. Gras and M.-N. Gras have introduced an effective method to compute the class number of the real abelian number field, utilizing cyclotomic units ([2]). K. Nakamula has introduced an effective method to compute the class number of a certain non-galois number field, utilizing its elliptic units ([6]). Nakamula also considered more general problems concerning the class number formulas related to elliptic units, and pointed out some essential issues ([7]). Our purpose is to establish the similar algorithms to those in [2] for any abelian extension over an imaginary quadratic number field, utilizing elliptic units instead of cyclotomic units. In the present note, as the first step of our purpose, we shall treat the following special case.

Let L be an abelian extension of degree l with an odd prime number l over an imaginary quadratic number field K such that L is a dihedral extension of degree $2l$ over rational number field Q. For each number field $\_$, we denote by $h_{\_}$, $E_{\_}$ and $\mu_{\_}$ the class number of $\_$, the unit group in $\_$ and the torsion part of $E_{\_}$ respectively. Then the index formula of the following form is well known (cf. [9]):

$$[E_L\colon \mu_L\eta^{Z[G]}] = M\frac{h_L}{h_K},$$

where G denotes the Galois group of L over K, $Z[G]$ the group ring of G over the ring Z of rational integers, η an element of E_L explicitly given by using the values of the Dedekind η-function and M a constant explicitly given depending on the choice of η (§ 1). Our problem treated here is how to give an effective algorithm for the numerical determination of $[E_L\colon \mu_L\eta^{Z[G]}]$. In this note we shall show a procedure to solve this problem. Especially we shall treat more precisely the case where $l=5$. The reason why we treat the case where $l=5$ is only that the case where $l=3$ has been partially treated by Nakamula (I of [6]), though his treatment is slightly different from ours.

It is also possible and interesting to establish the similar arguments in the treatment of the arithmetic of the maximal real subfield Ω of L. This subject will be treated in another paper. Nakamula's treatment for the case where $l=3$ is surely one in Ω.

1. Preliminary results. Let L, Ω and K be the same as in Section 0. Then the Galois group of L over Q is generated by two elements S and T such that $S^l = T^2 = 1$ and $ST = TS^{-1}$. Here we may assume that $G = \langle S \rangle$ and T is the complex conjugation, i.e. $\langle T \rangle$ is the fixed group of Ω. By a result of Martinet ([5]),

$$D_\Omega = f^{l-1} D_K^{(l-1)/2}$$

with a natural number f, where D_Ω (resp. D_K) denotes the discriminant of Ω (resp. K). Moreover L is contained in the ring class field N_f modulo f over K. Let O_f the order in K with a rational conductor f and $\mathfrak{R}(f)$ the group of the equivalent classes of the proper O_f-ideals. We denote by $\sigma\colon \mathfrak{R}(f) \to \mathrm{Gal}(N_f/K)$ the isomorphism from $\mathfrak{R}(f)$ to the Galois group of N_f over K via Artin's reciprocity law and by $\mathfrak{U}$ the subgroup of index l in $\mathfrak{R}(f)$ which corresponds to the extension L/K. Now as a representative O_f-ideal of each class $\mathfrak{k}$ in $\mathfrak{R}(f)$ we may choose $\mathfrak{p}_f = \mathfrak{p} \cap O_f$, where $\mathfrak{p}$ is a prime O_1-ideal (prime ideal in the usual sense) of degree 1 in K such that $(\mathfrak{p}, fD_K) = 1$. Put $(p) = \mathfrak{p} \cap Z$ and $p \sim \mathfrak{p}\bar{\mathfrak{p}}$ in K. Here $\bar{\mathfrak{p}}^{h_K} = (\alpha)$ for some integer α in K. Then, as is well known,

$$\varepsilon_f(\mathfrak{k}) := \left\{ p^{12} \frac{\Delta(\mathfrak{p}_f)}{\Delta(O_f)} \right\}^{h_K} \alpha^{-12}$$

gives a unit in N_f, which is uniquely determined by $\mathfrak{k}$ independently of the choice of $\mathfrak{p}_f$. Here $\Delta(\ )$ means the lattice function which is expressed by means of the Dedekind eta-function as follows:

$$\Delta(\mathfrak{a}) = \left(\frac{2\pi}{w_2}\right)^{12} \eta\left(\frac{w_1}{w_2}\right)^{24},$$

where $\mathfrak{a} = [w_1, w_2]$ is a 2-dimensional complex lattice with Z-basis $\{w_1, w_2\}$, $\mathrm{Im}(w_1/w_2) > 0$. For any class $\mathfrak{k}$ in $\mathfrak{R}(f)$ we define a unit $\varepsilon_L(\mathfrak{k})$ in E_L by

$$\varepsilon_L(\mathfrak{k}) = N_{N_f/L}(\varepsilon_f(\mathfrak{k})) = \alpha^{-12\#\mathfrak{U}} \prod_{\{\mathfrak{a}_f\}} \left(p^{12} \frac{\Delta(\mathfrak{p}_f \mathfrak{a}_f)}{\Delta(\mathfrak{a}_f)} \right)^{h_K}.$$

Herein $\{\mathfrak{a}_f\}$ is a system of the complete representative O_f-ideals of the classes in $\mathfrak{U}$. Then the following lemma is well known.

LEMMA 1 ([9]). (1) $\varepsilon_L(\mathfrak{k}_1) = \varepsilon_L(\mathfrak{k}_2)$ *if* $\mathfrak{k}_1 \equiv \mathfrak{k}_2 \bmod \mathfrak{U}$ *and* $\varepsilon_L(\mathfrak{k}) = 1$ *if* $\mathfrak{k}$ *is in* $\mathfrak{U}$.

(2) *For any* $\mathfrak{k}$, $\mathfrak{h}$ *in* $\mathfrak{R}(f)$,

$$\varepsilon_L(\mathfrak{k})^{\sigma(\mathfrak{h})} = \varepsilon_L(\mathfrak{k}\mathfrak{h}^{-1})/\varepsilon_L(\mathfrak{h}^{-1}).$$

Remark 1. For any $\mathfrak{k}$ in $\mathfrak{R}(f)$, the absolute values $|\varepsilon_L(\mathfrak{k})|^2$ is the h_K-th power of a real positive unit in Ω. Moreover if $\mathfrak{k} \equiv \mathfrak{k}_0^2 \bmod \mathfrak{U}$ for some $\mathfrak{k}_0$ in $\mathfrak{R}(f)$, $|\varepsilon_L(\mathfrak{k})|^2$ is the $24h_K$-th power in E_Ω ([8]).

Now let $\mathfrak{k}_1$ be a fixed class in $\mathfrak{R}(f)\backslash\mathfrak{U}$. Then following the similar arguments to those of Leopoldt ([4]), we have

$$[E_L\colon \mu_L \varepsilon_L(\mathfrak{k}_1)^{Z[G]}] = (24h_K)^{l-1} \frac{h_L}{h_K}. \tag{1.1}$$

This formula (1.1) can be found in [9] and elsewhere. From Remark 1, $|\varepsilon_L(\mathfrak{k}_1)|^2 = \eta(\mathfrak{k}_1)^{24h_K}$ for some real positive unit $\eta(\mathfrak{k}_1)$ in E_Ω. Hereafter as a generator S of G, we take $S = \mathrm{Res}_L \sigma(\mathfrak{k}_1^{-1})$. Then by Lemma 1 we have

$$\left(\eta(\mathfrak{k}_1)^{S^i}\right)^{24h_K} = \frac{\varepsilon_L(\mathfrak{k}_1^{i+1})\varepsilon_L(\mathfrak{k}_1^{i-1})}{\varepsilon_L(\mathfrak{k}_1^i)^2} \quad (i = 0, 1, \ldots, l-1), \tag{1.2}$$

and by a short calculation we have

$$\begin{aligned} R_L\left(\mu_L \eta(\mathfrak{k}_1)^{24h_K Z[G]}\right) &= l R_L(\ll \varepsilon_L(\mathfrak{k}_1), \ldots, \varepsilon_L(\mathfrak{k}_1^{l-1}) \gg) \\ &= l R_L(\mu_L \varepsilon_L(\mathfrak{k}_1)^{Z[G]}), \end{aligned}$$

where $R_L(\_)$ means the regulator of each subgroup $\_$ in E_L. (Especially we denote $R_L(E_L)$ by R_L.) Therefore from the formula (1.1) we have

$$[E_L\colon \mu_L \eta(\mathfrak{k}_1)^{Z[G]}] = l \frac{h_L}{h_K}. \tag{1.3}$$

Hereafter our arguments are based on the formula (1.3).

Remark 2. By using the q-expansion formula of the Dedekind η-function and the relation (1.2), the approximate values of $\eta(\mathfrak{k}_1)$ and its algebraic conjugates $\eta(\mathfrak{k}_1)^{S^i}$ $(i = 1, 2, \ldots, l-1)$ are computable as exactly as desired (see Appendix). Hence the approximate value of any unit in $\mu_L \eta(\mathfrak{k}_1)^{Z[G]}$ is also computable as exactly as desired.

Remark 3. By a short calculation we see that $E_L/\mu_L E_\Omega E_\Omega^S$ is an elementary l-group and $[E_L\colon \mu_L E_\Omega E_\Omega^S] = l^t$ with $0 \leq t \leq (l-1)/2$. Furthermore in the case where L/K is an unramified extension, we have $t = 0$. (The same assertion for $l = 3$ has been proved by Callahan ([1]).) On the other hand, by making use of the Kuroda–Brauer formula (cf. [3]), we have

$$\frac{h_L}{h_K} = \frac{1}{l} [E_L\colon \mu_L E_\Omega E_\Omega^S] h_\Omega^2.$$

Hence $[E_L\colon \mu_L \eta(\mathfrak{k}_1)^{Z[G]}] = l^t h_\Omega^2$, and the l-primary part of $[E_L\colon \mu_L \eta(\mathfrak{k}_1)^{Z[G]}]$ must be equal to a square of rational integer.

2. Gras' theory. In this section we develope briefly the similar arguments to those in [2], which G. Gras and M.-N. Gras used for the effective determination of the class number and the fundamental units of real abelian number field.

Let $\mathscr{L}$: $E_L \to \boldsymbol{R}^l$ be the usual homomorphism from E_L to the l-dimensional Euclidean space $\boldsymbol{R}^l$, given by

$$\mathscr{L}(\varepsilon) = (\ \ldots,\ \log|\varepsilon^{S^i}|^2,\ \ldots\ )_{i=0,1,\ldots,l-1}.$$

Then the kernel of $\mathscr{L}$ is μ_L and the image $\mathscr{L}(E_L)$ is a lattice on the hyper plane $\boldsymbol{P} = \{(x_i) \in \boldsymbol{R}^l;\ \sum x_i = 0\}$. $Z[G]$ acts on E_L and also on $\mathscr{L}(E_L)$. Via this action $Z[G]/(\sum S^i)Z[G]$ acts transitively on $\mathscr{L}(E_L)$. As can be easily seen, $Z[G]/(\sum S^i)Z[G]$ is isomorphic to the maximal order $Z[\zeta_l]$ of the prime cyclotomic field $Q(\zeta_l)$, where $\zeta_l = e^{2\pi i/l}$. Hence $\mathscr{L}(E_L)$ can be seen as $Z[\zeta_l]$-module with Z-free rank $l-1$.

Let η be a unit in E_L such that $\mu_L \eta^{Z[G]}$ forms a subgroup of a finite index in E_L. Then $[E_L: \mu_L \eta^{Z[G]}] = [\mathscr{L}(E_L): \mathscr{L}(\eta^{Z[G]})]$. Moreover since $\mathscr{L}(E_L)$ and $\mathscr{L}(\eta^{Z[G]})$ are isomorphic to some fractional ideals in $Q(\zeta_l)$, $[E_L: \mu_L \eta^{Z[G]}]$ may be expressed as the absolute norm of some integral ideal in $Z[\zeta_l]$.

Proposition 1 ([2]). *Assume that $Z[\zeta_l]$ is a principal ideal ring, then for a prime number p the following* (1), (2) *and* (3) *are equivalent.*

(1) $[E_L: \mu_L \eta^{Z[G]}]$ *is divisible by p.*

(2) $[E_L: \mu_L \eta^{Z[G]}]$ *is divisible by $p^{r(p)}$. Herein $r(p)$ means the order of p modulo l for $p \neq l$ and $r(p) = 1$ for $p = l$.*

(3) *There exists a prime ideal (π) of $Z[\zeta_l]$ with the absolute norm $p^{r(p)}$ such that η^{Π} is in $\mu_L E_L^{p^{r(p)}}$. Herein $\Pi = p^{r(p)}/\pi$, and η^{Π} is due to the action via isomorphism $Z[\zeta_l] \simeq Z[G]/(\sum S^i)Z[G]$.*

Remark 4. Also in the case where $Z[\zeta_l]$ is not principal ideal ring, the similar assertion to one in Proposition 1 holds. However, it is very complicated.

When we use Proposition 1 for our purpose, we need an explicit criterion to judge for any ξ in $Z[G]$ and any natural number n whether η^{ξ} belongs to $\mu_L E_L^n$ or not. For example, for real abelian field G. Gras and M.-N. Gras gave a criterion of this kind by deciding the minimal polynomial of cyclotomic unit over Q from its approximate value. Also in our situation we are able to give such criterion by the similar method to that of Gras.

Namely for any ξ in $Z[G]$, the approximate values of $\eta(\mathfrak{k}_1)^{\xi}$ and $\eta(\mathfrak{k}_1)^{\xi S^i}$ $(i = 1, 2, \ldots, l-1)$ are computable as exactly as desired (Remark 2). Then all the coefficients of the minimal polynomial of $\eta(\mathfrak{k}_1)^{\xi}$ over K are completely decided, because they are expressed by means of the fundamental symmetric polynomials of l-variables. Now let $\eta = \eta_0$ be any unit in $\mu_L \eta(\mathfrak{k}_1)^{Z[G]}$, and $\eta_i = \eta^{S^i}$ $(i = 1, 2, \ldots, l-1)$. For each set of the complex number $(\eta'_0, \eta'_1, \ldots, \eta'_{l-1})$ such that $\eta'^n_i = \eta_i$ $(i = 0, 1, \ldots, l-1)$, compute the approximate values of

$$F_j = \hat{F}_j(\eta'_0, \eta'_1, \ldots, \eta'_{l-1}) \qquad (j = 1, 2, \ldots, l),$$

where $\hat{F}_j(, \ldots,)$ means the jth fundamental symmetric polynomial, and examine whether they simultaneously take the very near values to some integers in K or not. If η_0 is in E_L^n, there exists at least a set $(F_1, F_2, \ldots, F_l)$ of integers in K such that $\hat{F}_j(\eta'_0, \eta'_1, \ldots, \eta'_{l-1}) = F_j$ $(j = 1, 2, \ldots, l)$. Conversely if there exists such a set $(F_1, F_2, \ldots, F_l)$ of integers in K, we may conclude that η_0 is in E_L^n (see Example 1).

Remark 5. With the exception of the cases where $K = Q(\sqrt{-1})$ or $K = Q(\sqrt{-3})$, $\mu_L = \{\pm 1\}$. In the case of $K = Q(\sqrt{-1})$, $\mu_L = \{\pm 1, \pm i;\ i^2+1 = 0\}$, and in the case of $K = Q(\sqrt{-3})$, $\mu_L = \{\pm 1, \pm\varrho, \pm\varrho^2;\ \varrho^2+\varrho+1 = 0\}$. We must take care of the treatment in the case where $\mu_L \neq \{\pm 1\}$.

In the rest of this section, we shall state about a method to seek for an upper bound $\mathrm{Bd}(\eta)$ of $[E_L: \mu_L \eta^{Z[G]}]$. Now $\mathscr{L}(E_L)$ is a lattice on the hyper plane $\boldsymbol{P}$. We denote by $\mathfrak{M}_L$ the volume of the fundamental lattice of $\mathscr{L}(E_L)$. Put

$$\mathfrak{D} = \{(x_i) \in \boldsymbol{R}^l\colon |x_i| \leqslant 1 \text{ for } i = 1, 2, \ldots, l-1\},$$

and $\boldsymbol{D} = \mathfrak{D} \cap \boldsymbol{P}$. Then $\boldsymbol{D}$ is a compact, convex domain with a computable finite volume $\mathfrak{m}$ and symmetric w.r.t. the origin O.

Proposition 2 (Minkowski, cf. [2]). *There exists a unit $\tilde{\varepsilon}$ in E_L such that*

$$\log|\tilde{\varepsilon}|^2 < 0 \quad \textit{and} \quad \big|\log|\tilde{\varepsilon}^{S^i}|^2\big| \leqslant 2\left(\frac{\mathfrak{M}_L}{\mathfrak{m}}\right)^{1/(l-1)} \qquad (i = 1, 2, \ldots, l-1).$$

Now by a short calculation, it is easily confirmed that

$$C_0 := 2\left(\frac{\mathfrak{M}_L}{\mathfrak{m}}\right)^{1/(l-1)} = R_L^{1/(l-1)}.$$

Since $[E_L: \mu_L \eta^{Z[G]}] = R_L(\eta)/R_L$, we have

$$[E_L: \mu_L \eta^{Z[G]}] = C_0^{-(l-1)} R_L(\eta). \tag{2.1}$$

Herein $R_L(\eta)$ means the regulator $R_L(\mu_L \eta^{Z[G]})$. On the other hand, for the unit $\tilde{\varepsilon}$ which satisfies the conditions of Proposition 2, we have

$$|(|\tilde{\varepsilon}|^2)^{S^i}|^2 = |\tilde{\varepsilon}^{S^i}|^2\, |\tilde{\varepsilon}^{S^{-i}}|^2 \leqslant e^{2C_0} \qquad (i = 1, 2, \ldots, l-1),$$

and therefore

$$|D(|\tilde{\varepsilon}|^2)| \leqslant (2e^{C_0})^{l(l-1)},$$

where $D(|\tilde{\varepsilon}|^2)$ means the discriminant of $|\tilde{\varepsilon}|^2$. Moreover since for any integer α in Ω,

$$|D(\alpha)| \geqslant |D_{\Omega}| = f^{l-1}\, |D_{\Sigma}|^{(l-1)/2},$$

we have an inequality

$$C > \frac{1}{l}\log(f\sqrt{|D_K|}/2^l),$$

and equivalently

$$(2.2)\qquad [E_L\colon \mu_L\eta^{Z[G]}] < \left\{\frac{l}{\log(f\sqrt{|D_K|}/2^l)}\right\}^{l-1} R_L(\eta).$$

Herein the finite number of cases that $f\sqrt{|D_K|} \leqslant 2^l$ must be excepted.

Remark 6. The above estimation (2.2) is so rough that it is hoped that more precise one will be realized. At any rate, we must have some supplementary estimation for $[E_L\colon \mu_L\eta^{Z[G]}]$ which is available also for the exceptional case where $f\sqrt{|D_K|} \leqslant 2^l$ (see § 3).

3. The case where $l = 5$. In this section we shall consider more precisely the case where $l = 5$.

From the result of Section 2, we have an inequality

$$[E_L\colon \mu_L\eta^{Z[G]}] < \left\{\frac{5}{\log(f\sqrt{|D_K|}/32)}\right\}^4 R_L(\eta),$$

for $f\sqrt{|D_K|} > 32$. Now we are going to consider the exceptional case. Let $\tilde{\varepsilon}$ be a unit in E_L which satisfies the conditions in Proposition 2. We denote the minimal polynomial of $|\tilde{\varepsilon}|^2$ over Q by

$$X^5 - Q_1X^4 + Q_2X^3 - Q_3X^2 + Q_4X - 1,$$

and by z_1 and z_2 the two algebraic conjugate roots of $|\tilde{\varepsilon}|^2$ which are not complex conjugate to each other. Then

$$e^{-4C_0} \leqslant |\tilde{\varepsilon}|^2 < 1 \quad \text{and} \quad e^{-2C_0} \leqslant |z_i|^2 \leqslant e^{2C_0} \quad (i = 1, 2),$$

where $C_0 = R_L^{1/4}$. Hence the four coefficients Q_1, Q_2, Q_3 and Q_4 are restricted in some finite intervals. Moreover the discriminant $D(|\tilde{\varepsilon}|^2)$ of $|\tilde{\varepsilon}|^2$ is a square of rational integer, because it must be of the form $D(|\tilde{\varepsilon}|^2) = A^2D_\Omega = A^2f^4D_K^2$ with a rational integer A. For example, if $C_0 < 50^{-0.25}$ (with the exception of these possibilities, inequality $[E_L\colon \mu_L\eta^{Z[G]}] < 50R_L(\eta)$ always holds), it is necessary that $|Q_1| \leqslant 6$, $|Q_2| \leqslant 18$, $|Q_3| \leqslant 25$ and $|Q_4| \leqslant 10$. The integer systems (Q_1, Q_2, Q_3, Q_4) in Table I are all of those for which $F(X) := X^5 - Q_1X^4 + Q_2X^3 - Q_3X^2 + Q_4X - 1$ satisfy the following two conditions:

(1) The discriminant of $F(X)$ is a square of rational integer.

(2) Equation $F(X) = 0$ has only one real root θ and two pairs of the complex roots $(z_i, \bar{z}_i)$ $(i = 1, 2)$ such that $0.2221 \leqslant \theta < 1$ and $0.4714 \leqslant |z_i|^2 \leqslant 2.1214$ $(i = 1, 2)$.

For each of six cases in Table I C_0 must be greater than

$\max\{\frac{1}{4}|\log\theta|, \frac{1}{2}|\log|z_1|^2|, \frac{1}{2}|\log|z_2|^2|\}$, and by the equality (2.1) we have an upper bound $\mathrm{Bd}(\eta)$ of $[E_L\colon \mu_L\eta^{Z[G]}]$, which is noted in the last column of table. In the same manner we attempted to seek for all the possibilities that $C_0 < 2^{-0.25}$. With the exception of these possibilities, $\mathrm{Bd}(\eta) < 2R_L(\eta)$ always holds (see Table II). As an upper bound $\mathrm{Bd}(\eta)$ in the general case, we may take as follows:

$$\mathrm{Bd}(\eta) = \begin{cases} \min\left\{2, \left[\dfrac{5}{\log(f\sqrt{|D_K|}/32)}\right]^4\right\} R_L(\eta), & \text{for } f\sqrt{|D_K|} > 32, \\ \text{the associated value}, & \text{for 17 cases in Tables I, II}, \\ 2R_L(\eta), & \text{otherwise}. \end{cases}$$

Table I. Possibilities that $C_0 < 50^{-0.25}$

(Q_1, Q_2, Q_3, Q_4)	$D(\theta)$	θ	$\lvert z_1\rvert^2$	$\lvert z_2\rvert^2$	$C_0 >$	D_K	h_K	f	$\mathrm{Bd}(\eta)$
$(-2, 2, 1, 0)$	47^2	0.5764714	1.5764714	1.1003633	0.22759	-47	5	1	$373R_L(\eta)$
$(1, 0, 1, 3)$	119^2	0.3881968	1.6345126	1.5760126	0.24567	-119	10	1	$275R_L(\eta)$
$(1, 1, 2, 3)$	79^2	0.4418042	1.7914862	1.2634457	0.29152	-79	5	1	$139R_L(\eta)$
$(1, -1, 0, 3)$	143^2	0.3510372	1.5409205	1.8487004	0.30724	-143	10	1	$114R_L(\eta)$
$(-2, 1, 1, -1)$	47^2	0.9087907	0.5764714	1.9087907	0.32323	-47			
$(-2, 3, -3, 1)$	103^2	0.3745937	1.3745937	1.9420711	0.33187	-103	5	1	$83R_L(\eta)$
$(1, -2, -1, 3)$	127^2	0.3234887	2.0912979	1.4781719	0.36889	-127	5	1	$54R_L(\eta)$
$(1, 2, 3, 3)$	47^2	0.5238918	2.1003633	0.9087907	0.37105	-47			

Table II. Possibilities that $C_0 < 2^{-0.25}$

(D_K, f)	h_K	$C_0 >$	$\mathrm{Bd}(\eta)$
$(-319, 1)$	10	0.46822	$21R_L(\eta)$
$(-159, 1)$	10	0.52422	$14R_L(\eta)$
$(-239, 1)$	15	0.53332	$13R_L(\eta)$
$(-3439, 1)$	30	0.62272	$7R_L(\eta)$
$(-439, 1)$	15	0.62754	$7R_L(\eta)$
$(-131, 1)$	5	0.65186	$6R_L(\eta)$

(D_K, f)	h_K	$C_0 >$	$\mathrm{Bd}(\eta)$
$(-479, 1)$	25	0.68056	$5R_L(\eta)$
$(-344, 1)$	10	0.69928	$5R_L(\eta)$
$(-303, 1)$	10	0.75210	$4R_L(\eta)$
$(-179, 1)$	5	0.78726	$3R_L(\eta)$
$(-7, 11)$	1	0.79503	$3R_L(\eta)$

Now by Remark 3, the 5-primary part of $[E_L\colon \mu_L\eta(\mathfrak{f}_1)^{Z[G]}]$ must be equal to a square of rational integer. Hence by Proposition 1 each prime factor p of $[E_L\colon \mu_L\eta(\mathfrak{f}_1)^{Z[G]}]$ may be restricted as follows:

$$p < \mathrm{Bd}(\eta(\mathfrak{f}_1))^{1/2} \quad \text{for} \quad p \equiv \pm 1 \pmod 5,$$

$$p < \mathrm{Bd}(\eta(\mathfrak{f}_1))^{1/4} \quad \text{for} \quad p \not\equiv \pm 1 \pmod 5.$$

Remark 7. According to Remark 1, we put

$$|\varepsilon_L(\mathfrak{f}_1)|^2 = \alpha^{24h_K} \quad \text{and} \quad |\varepsilon_L(\mathfrak{f}_1^2)|^2 = \beta^{24h_K},$$

with real positive units α, β in Ω. (Note that $\alpha = \eta(\mathfrak{k}_1)$.) Then $R_L(\eta(\mathfrak{k}_1))$ is given by

$$R_L(\eta(\mathfrak{k}_1)) = 5(a^2-3ab+b^2)^2,$$

where $a = \log\alpha$ and $b = \log\beta$.

EXAMPLE 1. Let $K = Q(\sqrt{-7})$ and L be the cyclic extension of degree 5 over K which is the subfield of the ring class field N_{11} modulo 11 over K. Here each class in $\mathfrak{R}(11)$ is represented by one of the following ten O_{11}-ideals

$$[11\omega, 1] \quad \text{and} \quad [\omega+t, 11] \quad (0 \leqslant t \leqslant 10,\ t \neq 4, 6)$$

where $\omega = (1+\sqrt{-7})/2$. Two classes in $\mathfrak{U}$ are represented by $[11\omega, 1]$ and $[\omega+5, 11]$. We let $\mathfrak{k}_1$ be equal to the class of $[\omega, 11]$. Then

$$\eta(\mathfrak{k}_1) \sim 38.888287830,$$
$$\eta(\mathfrak{k}_1)^{S^{\pm1}} \sim -0.225596309 \mp i\,0.237796817,$$
$$\eta(\mathfrak{k})^{S^{\pm2}} \sim -0.218547605 \pm i\,0.437692368.$$

And the minimal polynomial of $\eta(\mathfrak{k}_1)$ over Q (also over K) is $X^5-38X^4-34X^3-21X^2-6X-1$. We examined that $\eta(\mathfrak{k}_1)^{(1-S)^3}$ and $\eta(\mathfrak{k}_1)^{(1-S)^2}$ are contained in E_L^5, but $\eta(\mathfrak{k}_1)^{1-S}$ is not in E_L^5. Namely $\eta(\mathfrak{k}_1)^{(1-S)^3} = \varepsilon_1^5$ and $\eta(\mathfrak{k}_1)^{(1-S)^2} = \varepsilon_2^5$ with some units ε_1 and ε_2 in E_L, whose minimal polynomial over K are respectively

$$X^5-(5+11\omega)X^4+(65+22\omega)X^3-(87-22\omega)X^2+(16-11\omega)X-1,$$
$$X^5-4X^4+13X^3-17X^2+9X-1.$$

Therefore the 5-part of $[E_L: \pm\eta(\mathfrak{k}_1)^{Z[G]}]$ is 25. Since $\mathrm{Bd}(\eta(\mathfrak{k}_1)) = 3R_L(\eta(\mathfrak{k}_1)) = 4128.65\ldots$, any other prime factor of $[E_L: \pm\eta(\mathfrak{k}_1)^{Z[G]}]$ must be less than 12. For each p of 2, 3 and 7, we examined that $\eta(\mathfrak{k}_1)$ is not in $\pm E_L^p$. For $p = 11$, we examined that none of $\eta(\mathfrak{k}_1)^{\Pi_i}$ $(i = 1, 2, 3, 4)$ is in E_L^{11}, where

$$\Pi_1 = (2+S^2)(2+S^3)(2+S^4), \qquad \Pi_2 = (2+S)(2+S^3)(2+S^4),$$
$$\Pi_3 = (2+S)(2+S^2)(2+S^4), \qquad \Pi_4 = (2+S)(2+S^2)(2+S^3).$$

Therefore we can conclude that $[E_L: \pm\eta(\mathfrak{k}_1)^{Z[G]}] = 25$, i.e. $h_L = 5$, and consequently $E_L = \pm\varepsilon_2^{Z[G]}$.

In Table III, D_K are all discriminants of imaginary quadratic number fields K such that $D_K > -1000$ and K has an unramified cyclic extension L of degree 5. For each of the all cases in Table III, we examined that $[E_L: \pm\eta(\mathfrak{k}_1)^{Z[G]}] = 1$, i.e. $h_L = h_K/5$ and $E_L = \pm\eta(\mathfrak{k}_1)^{Z[G]}$. In Table III, we listed up the class numbers h_L, h_Ω and the coefficients of the minimal polynomial $X^5-Q_1X^4+Q_2X^3-Q_3X^2+Q_4X-1$ of $\eta(\mathfrak{k}_1)$ over Q for the

Table III

D_K	h_K	Q_1	Q_2	Q_3	Q_4	$\mathfrak{k}_1$	h_L	h_Ω
−47	5	0	−1	2	−1	[ω, 2]	1	1
−79	5	3	2	1	1	[ω, 2]	1	1
−103	5	1	−3	3	−2	[ω, 2]	1	1
−119	10	3	1	0	1	[ω, 2]	2	1
−127	5	3	−1	−2	1	[ω, 2]	1	1
−131	5	5	3	−1	−1	[ω, 3]	1	1
−143	10	3	0	−1	1	[ω, 2]	2	1
−159	10	4	−2	1	−2	[ω, 2]	2	1
−179	5	6	1	5	2	[ω, 3]	1	1
−227	5	9	9	9	5	[ω, 3]	1	1
−239	15	2	−5	4	−2	[ω, 2]	3	1
−296	10	29	−34	6	5	[ω+1, 3]	2	1
−303	10	10	−5	−5	1	[ω, 2]	2	1
−319	10	6	−3	−1	−1	[ω, 2]	2	1
−344	10	43	8	19	−9	[ω+1, 3]	2	1
−347	5	13	−27	21	−7	[ω, 3]	1	1
−415	10	13	9	0	1	[ω, 2]	2	1
−439	15	8	−4	−2	0	[ω, 2]	3	1
−443	5	22	−17	−3	4	[ω, 3]	1	1
−455	20	12	−16	7	0	[ω, 2]	4	1
−479	25	8	15	12	4	[ω, 2]	5	1
−488	10	109	78	22	−11	[ω+1, 3]	2	1
−523	5	64	1	59	−6	[ω+1, 7]	1	1
−571	5	66	−45	41	8	[ω+1, 5]	1	1
−599	25	19	10	−5	−1	[ω, 2]	5	1
−611	10	31	−7	−3	3	[ω, 3]	2	1
−615	20	45	−13	14	−3	[ω, 2]	4	1
−619	5	81	−77	51	11	[ω, 5]	1	1
−635	10	64	−63	31	−2	[ω, 3]	2	1
−664	10	657	858	458	−31	[ω+2, 5]	2	1
−671	30	23	45	27	4	[ω, 2]	6	1
−683	5	56	41	−5	−6	[ω, 3]	1	1
−691	5	110	−3	85	−18	[ω+1, 5]	1	1
−699	10	321	−481	275	−9	[ω, 5]	2	1
−724	10	937	−1030	794	−55	[ω+2, 5]	2	1
−739	5	134	161	117	18	[ω, 5]	1	1
−751	15	18	−5	4	0	[ω, 2]	3	1
−776	20	311	−196	0	11	[ω+1, 3]	4	1
−779	10	49	−3	13	1	[ω, 3]	2	1
−787	5	205	33	177	25	[ω+2, 7]	1	1
−788	10	519	392	128	−1	[ω+1, 3]	2	1
−803	10	69	−11	21	1	[ω, 3]	2	1
−815	30	36	50	29	8	[ω, 2]	6	1
−824	20	531	−16	172	15	[ω+1, 3]	4	1
−836	20	399	−112	80	−17	[ω+1, 3]	4	1
−851	10	60	73	39	10	[ω, 3]	2	1
−872	10	759	−132	152	19	[ω+1, 3]	2	1
−916	10	2695	3824	1760	−73	[ω+1, 5]	2	1
−920	20	1599	−1588	536	−37	[ω+1, 3]	4	1
−923	10	93	−38	2	5	[ω, 3]	2	1
−947	5	125	103	7	−5	[ω, 3]	1	1
−971	15	107	23	43	9	[ω, 3]	3	1

suitably chosen $\mathfrak{f}_1$. Therein ω means $(1+\sqrt{-D_K})/2$, for $D_K \equiv 1 \pmod 4$ and $\sqrt{-D_K}/2$, for $D_K \equiv 0 \pmod 4$.

Appendix. For the computation of the approximate values of the elliptic units, we used the q-expansion formula of the Dedekind eta-function $\eta(z)$ as follows:

$$\eta(z)^{-1} = e^{\frac{\pi i}{12}z} \sum_{m=0}^{\infty} p(m) e^{2\pi i m z},$$

where $p(m)$ denotes the number of partitions of m. This infinite series converges very fast, when $\mathrm{Im}(z)$ is large. For example, if $\mathrm{Im}(z) \geqslant \sqrt{3}/2$, the error value

$$\left|\eta(z)^{-1} - e^{\frac{\pi i}{12}z} \sum_{m=0}^{\infty} p(m) e^{2\pi i m z}\right|$$

is always less than 10^{-17}. Hence the approximate value of each $\varepsilon_L(\mathfrak{f})$ can be computed as exactly as desired. And by using the formula (1.2), the approximate values of $\eta(\mathfrak{f}_1)$ and its all algebraic conjugates $\eta(\mathfrak{f}_1)^{S^i}$ ($i = 1, 2, \ldots$) are computable as exactly as desired, up to a power of the $24h_K$-th root of unity. Considering the fact that the coefficients of the minimal polynomial of $\eta(\mathfrak{f}_1)$ over Q are rational integers and are expressible by means of the fundamental symmetric polynomials on $\eta(\mathfrak{f}_1)$ and $\eta(\mathfrak{f}_1)^{S^i}$ ($i = 1, 2, \ldots, l-1$), we are able to completely decide the minimal polynomial of $\eta(\mathfrak{f}_1)$ and at the same time the true approximate values of $\eta(\mathfrak{f}_1)$ and its algebraic conjugates $\eta(\mathfrak{f}_1)^{S^i}$ by the finite amount of computations.

Example. Let $K = Q(\sqrt{-7})$ and L is the ring class field over K modulo 7. Then each class in $\mathfrak{R}(7)$ is represented by one of the following seven O_7-ideals

$$[7\omega, 1] \quad \text{and} \quad [\omega+t, 7] \quad (0 \leqslant t \leqslant 6,\ t \neq 3),$$

where $\omega = (1+\sqrt{-7})/2$. Let $\mathfrak{f}_1$ be represented by $[\omega, 7]$ and $S = \sigma(\mathfrak{f}_1^{-1})$. Then

$$\eta(\mathfrak{f}_1) = \frac{1}{7}\frac{\left|\eta\left(\frac{\omega}{7}\right)\right|^2}{|\eta(7\omega)|^2}, \qquad \eta(\mathfrak{f}_1)^S \equiv \sqrt{7}\,\frac{\eta\left(\frac{\omega+5}{7}\right)\eta(7\omega)}{\eta\left(\frac{\omega}{7}\right)^2},$$

$$\eta(\mathfrak{f}_1)^{S^2} \equiv \frac{\eta\left(\frac{\omega}{7}\right)\eta\left(\frac{\omega+2}{7}\right)}{\eta\left(\frac{\omega+5}{7}\right)^2}, \qquad \eta(\mathfrak{f}_1)^{S^3} \equiv \frac{\eta\left(\frac{\omega+4}{7}\right)\eta\left(\frac{\omega+5}{7}\right)}{\eta\left(\frac{\omega+2}{7}\right)^2}.$$

Herein $\equiv$ indicates the equation, with disregard at most of the multiple of the 24-th roots of unity. Hence by the finite amount of computations it can be confirmed that the minimal polynomial of $\eta(\mathfrak{f}_1)$ over Q is $X^7 - 7X^6 - 7X^5 - 7X^4 - 1$ and at the same time

$$\eta(\mathfrak{f}_1) \sim 7.98626081,$$
$$\eta(\mathfrak{f}_1)^{S^{\pm 1}} \sim 0.42166812 \pm i\,0.34605432,$$
$$\eta(\mathfrak{f}_1)^{S^{\pm 2}} \sim -0.54678212 \mp i\,0.49621329,$$
$$\eta(\mathfrak{f}_1)^{S^{\pm 3}} \sim -0.36801640 \mp i\,0.79775336.$$

For computing three tables (Table I, II and III), the author used the electronic computer NEAC MS-50 installed in Kyushu-Toukai University.

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