Hence

$$\sum_{n=1}^{\infty}\sum_{m=1}^{n-1} N^{+}_{m,n}(a) < a^2+\frac{4}{3}a\left(\frac{\log\frac{3}{2}a}{\log\frac{3}{2}}\right)^2, \tag{32}$$

and $N_1(a) = O(a^2)$ follows from (26), (30), (31) and (32).

Remark 5. By a modification of the above argument one could get an asymptotic formula for the number of triples in question.

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An application of the Fouvry–Iwaniec theorem

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The celebrated prime number theorem of Bombieri [1] and A. I. Vinogradov [7] states that for any $A > 0$ there is a constant B for which

$$Q \leqslant x^{1/2}(\log x)^{-B}$$

implies

$$\sum_{q\leqslant Q}\max_{(a,q)=1}\sup_{y\leqslant x}\left|\pi(y;q,a)-\frac{1}{\varphi(q)}\operatorname{li} y\right| \ll \frac{x}{(\log x)^A}.$$

It would be of great interest to extend the range for Q, even at the cost of the maxima over a and y. The first step in this direction has been taken by Fouvry and Iwaniec [4],

$$\sum_{\substack{q\leqslant Q\\(a,q)=1}}\lambda(q)\left\{\pi(x;q,a)-\frac{1}{\varphi(q)}\operatorname{li} x\right\} \ll \frac{x}{(\log x)^A}, \tag{1}$$

for

$$Q \leqslant x^{9/17-\varepsilon},$$

where $\lambda(q)$ satisfies extremely technical conditions, and the implied constant depends on a as well as on A and ε. A study of their paper indicates that we may replace the bound on the right hand side of (1) by

$$\ll \frac{|a|^{1/2}x}{(\log x)^A},$$

the implied constant now being independent of a. The Corollary in [4] stated only for $a = 2$, can now be extended as follows.

Let $\pi_2(x, a)$ denote the number of pairs of primes p, $p+a$ such that $p \leqslant x$. We then have with $B = 34/9$

$$\pi_2(x,a) \leqslant (B+\varepsilon)H(a)\frac{x}{(\log x)^2} \tag{2}$$

uniformly for $a \leqslant (\log x)^c$. The arithmetic constant is given by

$$H(a) = \big(1+(-1)^a\big)\prod_{\substack{p|a\\ p>2}} \frac{p-1}{p-2} \prod_{p>2}\left(1-\frac{1}{(p-1)^2}\right).$$

For large a this does not replace Chen's estimate [3] which requires only $a = O(x)$. Chen obtained the larger constant 3.9171... in place of 34/9, and for the Goldbach problem. His techniques also apply to the twin primes problem.

The uniform bound (2) can be applied to the estimation of

$$E_r = \liminf \frac{p_{n+r}-p_n}{\log p_n}$$

using ideas of Bombieri and Davenport [2]. The method of [6] gives

$$E_r \leqslant \frac{2r-1}{4Br}\left\{Br+(Br-1)\frac{\theta}{\sin\theta}\right\}$$

where θ is the smallest positive solution of

$$\theta+\sin\theta = \frac{\pi}{Br}, \quad \sin\theta < (\pi+\theta)\cos\theta, \tag{3}$$

and B is a constant for which (2) is valid, uniformly for $a \leqslant (\log x)^2$.

The equations (3) are soluble for $Br > 1.34952$, which is valid for all positive integers r when $B = 34/9$. When $r = 1$ the numerical improvement is from $E_1 < 0.4426$ in [6] to $E_1 < 0.4394$.

Similarly another result of [5] can be expressed as

$$\liminf_{n\to\infty} \frac{\max(p_{n+1}-p_n,\, p_n-p_{n-1})}{\log p_n} \leqslant \frac{3}{4} + \frac{\{(B-1)(9B-8)\}^{1/2}}{4B},$$

and the constant is improved from 1.3624 to 1.3124. Some further improvement is possible, as the weight function used in [5] was not optimal.

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