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Linear forms in members of a binary recursive sequence

by

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1. For any sequence of rational integers $u_0, u_1, \ldots, u_m, \ldots$ satisfying

$$u_m = ru_{m-1} + su_{m-2}, \quad m = 2, 3, \ldots$$

where r and s are integers with $r^2+4s \neq 0$, we have

$$u_m = a\alpha^m + b\beta^m, \quad m = 0, 1, 2, \ldots$$

where α and β are roots of the polynomial x^2-rx-s and

$$a = \frac{u_0\beta - u_1}{\beta-\alpha}, \quad b = \frac{u_1 - u_0\alpha}{\beta-\alpha}.$$

Put

$$R = \max(|u_0|, |u_1|, 3).$$

Observe that the heights(1) of a and b do not exceed C_1R^2 where $C_1 > 0$ is an effectively computable number depending only on α and β. The sequence $\{u_m\}$ is said to be a *non-degenerate binary recursive sequence* if a, b, α, β are non-zero and α/β is not a root of unity. Further x^2-rx-s is called the *polynomial associated to the sequence* $\{u_m\}$. By $\{u_m\}$ we shall always mean a non-degenerate binary recursive sequence. The letters r, s, α, β, a, b and R will always have the meaning, as described above, with reference to the sequence $\{u_m\}$. Further the roots α and β of the polynomial associated to $\{u_m\}$ are ordered to satisfy $|\alpha| \geqslant |\beta|$. Since α/β is not a root of unity, we find that $|\alpha| > 1$.

Parnami and the author [7] proved that $u_m \neq u_n$ whenever $m \neq n$ and $\max(m, n) \geqslant C_2$ where $C_2 > 0$ is an effectively computable number depending only on the sequence $\{u_m\}$. We state notations for a generalization which includes this result. Let a_1, a_2, a_3 and a_4 be non-zero algebraic numbers of degrees at most D and heights not exceeding H, where we assume that $H \geqslant 3$. Assume that $A \neq 0$ and B are algebraic numbers of

(1) The *height* of an algebraic number is defined as the maximum of the absolute values of its minimal polynomial with relatively prime integer coefficients.

degrees at most D and heights at most H', where $H' \geqslant 3$. Suppose that λ and μ are non-zero algebraic numbers. For $m = 0, 1, 2, \ldots$, put

$$x_m = a_1\lambda^m + a_2\mu^m, \quad y_m = a_3\lambda^m + a_4\mu^m. \tag{1}$$

Further set

$$\tau = \max(\|\lambda\|, \|\mu\|)$$

where $\|\lambda\|$ and $\|\mu\|$ denote the maximum of the absolute values of the conjugates of λ and μ respectively.

THEOREM 1. *Suppose λ/μ is not a root of unity and $\tau > 1$. For non-negative integers m and n, the equation*

$$x_m = y_n \tag{2}$$

with

$$a_1\lambda^m \neq a_3\lambda^n \tag{3}$$

implies that

$$\max(m, n) \leqslant C_3 \log H$$

for some effectively computable number $C_3 > 0$ depending only on D, λ and μ.

If λ, μ are algebraic integers and λ/μ is not a root of unity, the assumption $\tau > 1$ is satisfied. Further we shall give a quantitative version of Theorem 1.

THEOREM 2. *Suppose $|\lambda| \geqslant |\mu|$, $|\lambda| > 1$ and λ/μ is not a root of unity. There exist effectively computable numbers $C_4 > 0$ and $C_5 > 0$ depending only on D, λ and μ such that for all non-negative integers m, n with $m \geqslant n$, $m \geqslant C_4\log(HH')$ and $Aa_1\lambda^m \neq Ba_3\lambda^n$, we have*

$$|Ax_m - By_n| \geqslant |\lambda|^m e^{-C_5\nu}$$

where $\nu = (\log m \log H + \log H')\log(n+2)$.

Putting $a_1 = a_3 = a$, $a_2 = a_4 = b$, $\lambda = \alpha$, $\mu = \beta$, $x_m = u_m$ and $y_n = u_n$ in Theorem 2, we obtain

COROLLARY 1. *There exist effectively computable numbers $C_6 > 0$ and $C_7 > 0$ depending only on D, α and β such that for all pairs of non-negative integers m and n satisfying $m \geqslant n$, $m \geqslant C_6\log(RH')$ and $A\alpha^m \neq B\alpha^n$, we have*

$$|Au_m - Bu_n| \geqslant |\alpha|^m e^{-C_7\nu_1}$$

where $\nu_1 = (\log m \log R + \log H')\log(n+2)$.

Since α/β is not a root of unity and $\alpha\beta \neq 0$, the equations $A\alpha^m = B\alpha^n$ and $A\beta^m = B\beta^n$ with $m \neq n$ cannot hold simultaneously. Thus, if $|\alpha| = |\beta|$, we can interchange, if necessary, α and β to derive the following result from Corollary 1.

COROLLARY 2. *Suppose that $|\alpha| = |\beta|$. Then*

$$|Au_m - Bu_n| \geqslant |\alpha|^m e^{-C_7\nu_1}$$

whenever $m > n$ and $m \geqslant C_6\log(RH')$.

For given non-zero algebraic numbers A, B and a given sequence $\{u_m\}$ whose associated polynomial has non real roots, it follows from Corollary 2 that $|Au_m - Bu_n| \to \infty$ whenever $\max(m, n)$ tends to infinity through non-negative integers m and n with $m \neq n$. This need not be the case with the sequences $\{u_m\}$ whose associated polynomials have real roots. For example, the Fibonacci sequence $u_0 = 0$, $u_1 = 1$ and $u_m = u_{m-1} + u_{m-2}$ for $m \geqslant 2$ satisfies

$$|u_m - \alpha u_{m-1}| = |\beta|^{m-1} = |\alpha|^{-m+1}, \quad m = 1, 2, \ldots$$

For further results in this direction, see Kiss [4]. Putting $A = B = 1$ in Corollary 1 and observing that $|\alpha| > 1$, we have

COROLLARY 3. *There exist effectively computable numbers $C_8 > 0$ and $C_9 > 0$ depending only on α and β such that for all pairs of non-negative integers m and n with $m > n$ and $m \geqslant C_8\log R$, we have*

$$|u_m - u_n| \geqslant |\alpha|^m e^{-C_9\nu_2}$$

where $\nu_2 = \log m \log R \log(n+2)$.

In particular, we obtain the following result from Corollary 3.

COROLLARY 4. *For distinct non-negative integers m and n, the equation*

$$u_m = u_n$$

implies that

$$\max(m, n) \leqslant C_8 \log R. \tag{4}$$

Corollary 4 includes the result of Parnami and the author [7] already stated. Compare the corollary with the results of Kubota [5] and Beukers [3]. Further the estimate (4) is best possible with respect to R. For example, consider non-degenerate binary recursive sequences

$$\{u_m^{(n)}\}, \quad n = 1, 2, \ldots$$

given by

$$u_m^{(n)} = a^{(n)}3^m - b^{(n)}2^m, \quad m = 0, 1, 2, \ldots$$

with

$$a^{(n)} = 2^n - 1 \quad \text{and} \quad b^{(n)} = 3^n - 1.$$

For $n = 0, 1, 2, \ldots$, we have

$$u_n^{(n)} = u_0^{(n)} \quad \text{and} \quad 0 < \max(|u_0^{(n)}|, |u_1^{(n)}|) \leqslant 2\cdot 3^n.$$

This example is due to Tijdeman [12]. If a, b, α and β with $(a, \beta) = (b, \alpha) = 1$ are rational integers, we shall show that $\max(m, n)$, in Corollary 4, is bounded by a number depending only on the greatest prime factor of ab. For an integer x with $|x| > 1$, denote by $P(x)$ the greatest prime factor of x. Put $P(1) = P(-1) = 0$. For $0 \neq x \in \boldsymbol{Q}$, write $x = a/b$ where a and b are relatively prime integers and define $P(x) = P(ab)$. We shall prove:

THEOREM 3. *Suppose that a', b', x, y with $|x| \neq |y|$ and $(a', y) = (b', x) = 1$ are non-zero rational integers and m, n are distinct non-negative rational integers. Then there exists an effectively computable number $C_{10} > 0$ depending only on $P(a'b')$ such that the equation*

$$a'x^m + b'y^m = a'x^n + b'y^n \tag{5}$$

implies that

$$\max(m, n) \leqslant C_{10}.$$

Theorem 3 with a', b' fixed is a particular case of Theorem 1 of [10]. The relation

$$3 \cdot 2^{m-1} 2^m - 4^m = 3 \cdot 2^{m-1} 2^{m-1} - 4^{m-1}, \quad m = 1, 2, \ldots$$

shows that the restriction $(a', y) = (b', x) = 1$, in Theorem 3, is necessary.

For non-negative integers m and n, put

$$\Lambda_{m,n} = \max\left(\frac{\max(m, n)}{\log R}, 3\right).$$

Further set $d_1 = [\boldsymbol{Q}(\alpha) : \boldsymbol{Q}]$. Finally we shall apply Theorem 1 and Theorem C (see § 2) to strengthen Corollary 4 as follows:

THEOREM 4. *Let m and n be distinct non-negative integers such that u_m and u_n are non-zero. Then there exist effectively computable numbers $C_{11} > 0$ and $C_{12} > 0$ depending only on α and β such that the inequality*

$$P\left(\frac{u_m}{u_n}\right) \leqslant C_{11}\left(\frac{\Lambda_{m,n}}{\log \Lambda_{m,n}}\right)^{1/(d_1+1)} (*)$$

implies that

$$\Lambda_{m,n} \leqslant C_{12}.$$

Since α/β is not a root of unity, we find that the equations $u_m = 0$ and $u_n = 0$ with $m \neq n$ cannot hold simultaneously. Further, by Corollary 1 with $A = 1$ and $B = 0$, it follows that the equation $u_m = 0$ implies that $m \leqslant C_{13} \log R$ for some effectively computable number $C_{13} > 0$ depending only on α and β.

Applying Theorem 4 with the least integer n such that $u_n \neq 0$ (n is either zero or one), we derive

$$P(u_m) \geqslant C_{14}(m/\log m)^{1/(d_1+1)}, \quad m \geqslant C_{15}, \tag{6}$$

(*) Added in proofs. The arguments of Theorem 4 allow to replace $P\left(\frac{u_m}{u_n}\right)$ by $P\left(\frac{u_m}{(u_m, u_n)}\right)$ with $m > n$.

where $C_{14} > 0$ and $C_{15} > 0$ are effectively computable numbers depending only on the sequence $\{u_m\}$. Mahler [6] proved, ineffectively, that $P(u_m)$ tends to infinity with m. Schinzel [9] gave an effective and quantitative version of Mahler's result. Stewart [11] proved (6) with C_{14} and C_{15} depending only on a and b.

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2. In this section, we state the results that we shall require from other sources. Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers. Put $K = \boldsymbol{Q}(\alpha_1, \ldots, \alpha_n)$ and $[K:\boldsymbol{Q}] = d$. Let the heights of $\alpha_1, \ldots, \alpha_{n-1}$ and α_n be at most A' and A ($\geqslant 2$) respectively. All the results of this paper depend on the following result of Baker [2] on linear forms in logarithms.

THEOREM A. *There exists an effectively computable number $C > 0$ depending only on n, d and A' such that, for any δ with $0 < \delta < 1/2$, the inequalities*

$$0 < |b_1 \log \alpha_1 + \ldots + b_n \log \alpha_n| < (\delta/B')^{C \log A} e^{-\delta B}$$

have no solution in rational integers $b_1, \ldots, b_{n-1}$ and b_n ($\neq 0$) with absolute values at most B and B' respectively. It is assumed that the logarithms have their principal values.

Putting $\delta = 1/B$ and $B' = B$, Theorem A includes the following result which is also due to Baker [1].

THEOREM B. *There exists an effectively computable number $C' > 0$ depending only on n, d and A' such that the inequalities*

$$0 < |\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1| < \exp(-C' \log A \log B)$$

have no solution in rational integers $b_1, \ldots, b_n$ with absolute values at most B ($\geqslant 2$).

For the proof of Theorem 4, we shall also require the following p-adic analogue, due to van der Poorten [8].

THEOREM C. *Let $\mathfrak{p}$ be a prime ideal of K lying above a rational prime p. Suppose that $b_1, \ldots, b_{n-1}$ and $b_n = -1$ are rational integers of absolute values at most B. There exists an effectively computable number $C'' > 0$ depending only on n, d and A' such that for every δ with $0 < \delta < 1$, the inequality*

$$\infty > \operatorname{ord}_{\mathfrak{p}}(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) > \delta B$$

implies that

$$B \leqslant C'' \delta^{-1} p^d \log(\delta^{-1} p^d) \log A.$$

3. In this section, we shall prove Theorem 1. Let $\lambda, \mu, \alpha_1, \alpha_2, \alpha_3$ and α_4 be non-zero algebraic numbers. Suppose that $\alpha_1, \alpha_2, \alpha_3$ and α_4 have degrees at most D and heights not exceeding H ($\geqslant 3$). Denote by F the field generated by $\lambda, \mu, a_1, a_2, a_3$ and a_4 over Q. Let x_m and y_m be given by (1). For $1 \leqslant i \leqslant 4$, observe that

$$\max_\sigma |\sigma(a_i)| \leqslant DH \tag{7}$$

and

$$\min_\sigma |\sigma(a_i)| \geqslant (DH)^{-1} \tag{8}$$

where maximum and minimum are taken over all the embeddings σ of F. Further for every prime ideal $\mathfrak{p}$ in the ring of integers of F, we have

$$|\mathrm{ord}_\mathfrak{p}(a_i)| \leqslant c \log H \quad (1 \leqslant i \leqslant 4) \tag{9}$$

for some effectively computable number $c > 0$ depending only on D. We denote by $c_1, c_2, \ldots$ effectively computable positive numbers depending only on D, λ and μ. Observe that $[F:Q] \leqslant c_1$.

We apply Theorem A to obtain the following estimate for $|x_m|$.

LEMMA 1. *Suppose λ/μ is not a root of unity. There exist c_2 and c_3 such that for every δ with $0 < \delta < 1/2$, we have*

$$|x_m| \geqslant \big(\max(|\lambda|, |\mu|)\big)^m \exp\big(-c_2 \log(1/\delta) \log H - \delta m\big) \tag{10}$$

whenever $m \geqslant c_3 \log H$.

Proof of Lemma 1. Suppose λ/μ is not a root of unity. We, first, prove that the equation $x_m = 0$ implies that $m < c_3 \log H$. Suppose that $x_m = 0$. Then

$$(\lambda/\mu)^m = -a_2/a_1. \tag{11}$$

If λ/μ is not a unit, there exists a prime ideal $\mathfrak{p}$ in the ring of integers of F such that $\mathrm{ord}_\mathfrak{p}(\lambda/\mu)$ is non-zero. Then, by (11),

$$m \leqslant m|\mathrm{ord}_\mathfrak{p}(\lambda/\mu)| \leqslant |\mathrm{ord}_\mathfrak{p}(a_1)| + |\mathrm{ord}_\mathfrak{p}(a_2)|. \tag{12}$$

The assertion follows from (9) and (12). Thus we may assume that λ/μ is a unit. Since λ/μ is not a root of unity, we can find an embedding σ of F such that $|\sigma(\lambda/\mu)| > 1$. Further, by taking images under σ on both the sides in (11), we have

$$|\sigma(\lambda/\mu)|^m = |\sigma(a_2)/\sigma(a_1)|$$

and the assertion follows from (7) and (8).

We assume that $m \geqslant c_3 \log H$, so that $x_m \neq 0$. Now we complete the proof of Lemma 1 by applying (8) and Theorem A with $n = 3$, $d \leqslant c_1$, $\log A' = c_4$, $\log A = c_5 \log H$, $B = m+2$, $B' = 1$ to

$$\left| l\log(-1) + m\log\left(\frac{\lambda}{\mu}\right) + \log\left(\frac{a_1}{a_2}\right)\right|$$

where l with $|l| \leqslant m+2$ is an integer and logarithms have their principal values.

Further we shall prove:

LEMMA 2. *Suppose λ/μ is not a root of unity. For non-negative integers m and n with $m \geqslant n$, the equation (2) with (3) implies that*

$$n \leqslant c_6\big((m-n) + \log H\big). \tag{13}$$

Proof of Lemma 2. Suppose λ/μ is not a root of unity. Let m and n with $m \geqslant n$ be non-negative integers satisfying (2) and (3). Re-writing (2),

$$\lambda^n(a_1\lambda^{m-n} - a_3) = -\mu^n(a_2\mu^{m-n} - a_4). \tag{14}$$

It follows from (3) and (14) that $a_1\lambda^{m-n} - a_3$ and $a_2\mu^{m-n} - a_4$ are non-zero. If λ/μ is not a unit, we can find a prime ideal $\mathfrak{p}$ in the ring of integers of F such that $\mathrm{ord}_\mathfrak{p}(\lambda/\mu)$ is non-zero. Then, by (14),

$$n \leqslant n|\mathrm{ord}_\mathfrak{p}(\lambda/\mu)| \leqslant |\mathrm{ord}_\mathfrak{p}(a_1\lambda^{m-n} - a_3)| + |\mathrm{ord}_\mathfrak{p}(a_2\mu^{m-n} - a_4)|$$

and (13) follows from (9). Thus we may assume that λ/μ is a unit. Since λ/μ is not a root of unity, we can find an embedding σ of F such that $|\sigma(\lambda/\mu)| > 1$. Further, by taking images under σ on both the sides in (14), we have

$$|\sigma(\lambda/\mu)|^n = \left|\frac{\sigma(a_2\mu^{m-n} - a_4)}{\sigma(a_1\lambda^{m-n} - a_3)}\right|.$$

Now inequality (13) follows from (7) and Liouville type argument. This completes the proof of Lemma 2.

COROLLARY 6. *Put $c_7 = 2(c_6+1)$. Suppose λ/μ is not a root of unity. For non-negative integers m and n with $m \geqslant n$ and*

$$m - n \leqslant c_7^{-1} m, \tag{15}$$

the equation (2) with (3) implies that

$$m \leqslant 2c_6 \log H.$$

Proof of Corollary 6. By (13) and (15),

$$n \leqslant c_7^{-1} c_6 m + c_6 \log H$$

which, together with (15), implies that

$$m \leqslant c_7^{-1}(c_6+1)m + c_6 \log H = 2^{-1}m + c_6 \log H.$$

Hence $m \leqslant 2c_6 \log H$. This completes the proof of Corollary 6.

Proof of Theorem 1. Suppose $\tau > 1$ and λ/μ is not a root of unity. Let m and n be non-negative integers satisfying (2) and (3). There is no loss of generality in assuming that $m \geqslant n$. Further there exists an embedding σ of F such that

$$\tau = \max\big(|\sigma(\lambda)|, |\sigma(\mu)|\big).$$

Thus, by considering the equation $\sigma(x_m) = \sigma(y_n)$ in place of (2), it involves no loss of generality in supposing that $\max(|\lambda|, |\mu|) > 1$. Write

(16) $$\log\max(|\lambda|, |\mu|) = c_8.$$

We may assume that $m \geqslant c_9 \log H$ with c_9 sufficiently large. Let $c_9 > \max(c_3, 2c_6)$. Then the assertion of Lemma 1 is valid and, by Corollary 6,

(17) $$m-n > c_7^{-1} m.$$

Further, by (7) and (16), we have

(18) $$|y_n| \leqslant 2DHe^{c_8 n}.$$

Now it follows from (2), (10) with $\delta = \min(c_8/2c_7, 1/4)$, (16) and (18) that

(19) $$m-n \leqslant (2c_7)^{-1} m + c_{10} \log H.$$

Combining (17) and (19), we obtain $m \leqslant 2c_7 c_{10} \log H$. This completes the proof of Theorem 1.

4. Proof of Theorem 2. Let $\lambda, \mu, a_1, a_2, a_3, a_4, D, H, x_m, y_n, A, B$ and H' be as in Theorem 2. Suppose that m and n with $m \geqslant n$ are non-negative integers satisfying $Aa_1\lambda^m \neq Ba_3\lambda^n$. Denote by $c_{11}, c_{12}, \ldots$ effectively computable positive numbers depending only on D, λ and μ. Put

$$f = Ax_m - By_n.$$

We assume that $m \geqslant c_{11} \log(HH')$ with c_{11} sufficiently large. Let $c_{11} > c_3$ so that the assertion of Lemma 1 is valid. For $B \neq 0$, observe that

(20) $$\max(|A|, |B|) \leqslant DH' \quad \text{and} \quad \min(|A|, |B|) \geqslant (DH')^{-1}.$$

If $|Ax_m| \geqslant 2|By_n|$, then

$$|f| \geqslant |Ax_m| - |By_n| \geqslant |Ax_m|/2$$

and the theorem follows from (20) and (10) with $\delta = 1/m$. Thus we may assume that

(21) $$|Ax_m| < 2|By_n|.$$

Further, by (7),

(22) $$|y_n| \leqslant 2DH|\lambda|^n.$$

Now it follows from (21), (10) with $\delta = 1/m$, (22), (20) and $\max(|\lambda|, |\mu|) = |\lambda| > 1$ that

(23) $$m-n \leqslant c_{12}(\log m \log H + \log H').$$

If c_{11} is sufficiently large, it follows from Theorem 1 that f is non-zero. Further, re-writing f, we obtain

$$0 \neq |f| = |\lambda^n(Aa_1\lambda^{m-n} - Ba_3) + \mu^n(Aa_2\mu^{m-n} - Ba_4)|.$$

Since $Aa_1\lambda^{m-n} - Ba_3 \neq 0$, we may write

$$0 \neq |f| = |\lambda^n(Aa_1\lambda^{m-n} - Ba_3)|\,\Delta$$

where

$$\Delta = \left| -\left(\frac{\mu}{\lambda}\right)^n \frac{Aa_2\mu^{m-n} - Ba_4}{Aa_1\lambda^{m-n} - Ba_3} - 1 \right|$$

and Δ is non-zero. We apply Theorem B with $n = 3$, $d \leqslant c_{13}$, $B = n+2$, $\log A' = c_{14}$ and $\log A \leqslant c_{15}\big((m-n) + \log(HH')\big)$ which, together with (23), implies that $\log A \leqslant c_{16}(\log m \log H + \log H')$. We obtain

$$\Delta \geqslant e^{-c_{17}\nu}.$$

Further, by (23), (20), (7) and a Liouville type argument, we obtain

$$|\lambda^n(Aa_1\lambda^{m-n} - Ba_3)| \geqslant |\lambda|^m \exp\big(-c_{18}(\log m \log H + \log H')\big).$$

Hence

$$|f| \geqslant |\lambda|^m e^{-c_{19}\nu}.$$

This completes the proof of Theorem 2.

5. Proof of Theorem 4. For an integer x in $Q(\alpha)$, denote by $[x]$ the ideal generated by x in the ring of integers of $Q(\alpha)$. We have

$$([\alpha^2], [\beta^2]) = [k]$$

where k is a positive rational integer. In fact $k = (r^2+2s, s)$. Put

$$\alpha_1 = \alpha^2/k, \quad \beta_1 = \beta^2/k.$$

Then α_1 and β_1 are non-zero algebraic integers such that the ideals $[\alpha_1]$ and $[\beta_1]$ are relatively prime. Further observe that $|\alpha_1| \geqslant |\beta_1|$, α_1/β_1 is not a root of unity and α_1, β_1 are roots of a quadratic monic polynomial with rational integers as coefficients. Consequently, we find that $|\alpha_1| > 1$. For $m' = 0, 1, 2, \ldots$ and $\delta' = 0, 1$, we write

(24) $$u_{2m'+\delta'} = k^{m'} v_{2m'+\delta'}$$

where

$$v_{2m'+\delta'} = a\alpha^{\delta'}\alpha_1^{m'} + b\beta^{\delta'}\beta_1^{m'}.$$

We denote by $k_1, k_2, \ldots$ effectively computable positive numbers depending only on α and β.

Let m and n be distinct non-negative integers such that u_m and u_n are non-zero. There is no loss of generality in assuming that $m > n$. Write

(25) $$u_m/u_n = B_1/A_1$$

where $A_1 > 0$ and B_1 are relatively prime non-zero integers. Further write

$$m = 2m_1 + \delta_1, \quad n = 2n_1 + \delta_2$$

where δ_1 and δ_2 are either zero or one. Since $m > n$, observe that $m_1 \geqslant n_1$. Further, by (25) and (24),

$$A_1 k^{m_1-n_1} v_m = B_1 v_n.$$

Cancelling the common factors of $A_1 k^{m_1-n_1}$ and B_1, we can find non-zero rational integers A_2, B_2 with $(A_2, B_2) = 1$ and

(26) $$P(A_2B_2) \leqslant P(A_1B_1k)$$

such that

(27) $$A_2 v_m = B_2 v_n.$$

We apply Theorem 1 with $a_1 = A_2 a\alpha^{\delta_1}$, $a_2 = A_2 b\beta^{\delta_1}$, $a_3 = B_2 a\alpha^{\delta_2}$, $a_4 = B_2 b\beta^{\delta_2}$, $\lambda = \alpha_1$, $\mu = \beta_1$, $x_{m_1} = A_2 v_{m'}$, $y_{n_1} = B_2 v_n$ and

$$\log H \leqslant k_1(\log|A_2B_2| + \log R).$$

Since $|\alpha_1| > 1$, we see $\tau > 1$. If $a_1\lambda^{m_1} = a_3\lambda^{n_1}$, then we notice, by (27), $a_2\mu^{m_1} = a_4\mu^{n_1}$ and consequently we find that

$$(\alpha/\beta)^{m-n} = 1$$

which is not possible, since α/β is not a root of unity and $m \neq n$. Further λ/μ is not a root of unity. Thus all the assumptions of Theorem 1 are satisfied. Hence, by Theorem 1, we conclude that

(28) $$m \leqslant 2m_1 + 1 \leqslant k_2(\log|A_2B_2| + \log R).$$

We assume that $m > k_3 \log R$ with k_3 sufficiently large. Let $k_3 > 2k_2$. Then, by (28),

(29) $$m < 2k_2 \log|A_2B_2|.$$

Write $P = P(A_2B_2)$. By (29), we find that $P \geqslant 2$. For a prime p dividing A_2 and for a prime q dividng B_2, it follows from (27) that

(30) $$\operatorname{ord}_p(A_2) \leqslant \operatorname{ord}_p(v_n), \quad \operatorname{ord}_p(B_2) \leqslant \operatorname{ord}_p(v_m),$$

since A_2 and B_2 are relatively prime. Further it follows from $(A_2, B_2) = 1$ and (30) that

(31) $$\log|A_2B_2| = \sum_{p|A_2B_2} \operatorname{ord}_p(A_2B_2)\log p \leqslant \log P \sum_{p\leqslant P} \max\big(\operatorname{ord}_p(A_2), \operatorname{ord}_p(B_2)\big) \leqslant \log P \sum_{p\leqslant P} \max\big(\operatorname{ord}_p(v_m), \operatorname{ord}_p(v_n)\big).$$

Thus, by (29) and (31), we can find a prime $p_0 \leqslant P$ such that

$$\max\big(\operatorname{ord}_{p_0}(v_m), \operatorname{ord}_{p_0}(v_n)\big) > \big(2k_2\pi(P)\log P\big)^{-1} m.$$

Since $\pi(P) \leqslant 2P/\log P$, we have

$$\max\big(\operatorname{ord}_{p_0}(v_m), \operatorname{ord}_{p_0}(v_n)\big) > (4k_2P)^{-1} m.$$

Let $\mathfrak{p}_0$ be a prime ideal in the ring of integers of $Q(\alpha)$ lying above p_0. Then

(32) $$\max\big(\operatorname{ord}_{\mathfrak{p}_0}(v_m), \operatorname{ord}_{\mathfrak{p}_0}(v_n)\big) > (4k_2P)^{-1} m.$$

Since the ideals $[\alpha_1]$ and $[\beta_1]$ are relatively prime, we see that $\mathfrak{p}_0$ is prime to at least one of the ideals $[\alpha_1]$ and $[\beta_1]$. For simplicity, we assume that $\mathfrak{p}_0$ and $[\alpha_1]$ are relatively prime. Put

$$\Delta_1 = \operatorname{ord}_{\mathfrak{p}_0}\left(\left(\frac{\beta}{\alpha}\right)^m \frac{b}{a} + 1\right), \quad \Delta_2 = \operatorname{ord}_{\mathfrak{p}_0}\left(\left(\frac{\beta}{\alpha}\right)^n \frac{b}{a} + 1\right).$$

Then

(33) $$\max\big(\operatorname{ord}_{\mathfrak{p}_0}(v_m), \operatorname{ord}_{\mathfrak{p}_0}(v_n)\big) \leqslant \max(\Delta_1, \Delta_2) + k_4 \log R.$$

By (33) and (32), we find that

$$\max(\Delta_1, \Delta_2) > (4k_2P)^{-1} m - k_4 \log R.$$

We may assume that $m > 8k_2k_4 P \log R$, otherwise the theorem follows from (26). Then

$$\infty > \max(\Delta_1, \Delta_2) > (8k_2P)^{-1} m,$$

since $v_m v_n \neq 0$. Now we apply Theorem C with $p = p_0 \leqslant P$, $n = 3$, $d = d_1$, $\log A' = k_5$, $\log A = k_6 \log R$, $B = m$ and $\delta = \min\big((8k_2P)^{-1}, 2^{-1}\big)$ to $\max(\Delta_1, \Delta_2)$. We obtain

$$m \leqslant k_7 P^{d_1+1} \log P \log R$$

which, together with (26), completes the proof of Theorem 4.

6. In this section, we shall prove Theorem 3. We, therefore, assume that the conditions of Theorem 3 are satisfied. The plan for the proof is similar to that of Theorem 1 of [10]. There is no loss of generality in assuming that $(a', b') = 1$, $m > n$ and $|x| > |y| > 0$. Further notice that

(34) $$a'x^m + b'y^m \neq 0.$$

Put $R_1 = \max(|a'|, |b'|, 2)$. Denote by $w_1, w_2, \ldots$ effectively computable positive numbers depending only on $P(a'b')$. We can assume that $m \geqslant w_1$ with w_1 sufficiently large. Now we prove:

LEMMA 3.

(35) $$\log R_1 \leqslant w_2\big(\log|x| + \log(m-n)\big).$$

Proof of Lemma 3. Re-writing (5), we have

$$a'x^n(x^{m-n} - 1) = b'y^n(1 - y^{m-n}).$$

For a prime p dividing a',

$$\operatorname{ord}_p(a') \leqslant \operatorname{ord}_p(1 - y^{m-n}) \leqslant w_3\big(\log|y| + \log(m-n)\big).$$

Similarly for a prime q dividing b',

$$\operatorname{ord}_q(b') \leqslant w_4(\log|x| + \log(m-n)).$$

Now the lemma follows immediately.

LEMMA 4.

$$(36)\qquad m-n \leqslant w_5 \log m.$$

Proof of Lemma 4. Apply (34), Theorem A with $n = w_6$, $d = 1$, $B = 2\log R_1$, $B' = m$, $A' = w_7$, $A = |x|$, $\delta = 1/4$ and the inequality (35) to obtain

$$(37)\qquad |a'x^m + b'y^m| \geqslant |a'|\,|x|^{m-w_8\log m}.$$

Further observe that

$$(38)\qquad |a'x^n + b'y^n| \leqslant 2\max(|a'x^n|, |b'y^n|) \leqslant 2R_1|x|^n.$$

Now the lemma follows by combining (5), (37), (38) and (35).

LEMMA 5. *If*

$$(39)\qquad |y| \leqslant \tfrac{2}{3}|x|,$$

then

$$m \leqslant w_9 \log|x|.$$

Proof of Lemma 5. Re-writing the equation (5), we have

$$(40)\qquad \left(\frac{x}{y}\right)^n = \frac{b'(1-y^{m-n})}{a'(x^{m-n}-1)}.$$

By (40) and (39), we obtain

$$(41)\qquad n \leqslant w_{10}\log R_1.$$

Now combine (36), (41) and (35) to complete the proof of Lemma 5.

Write $g = (|x|, |y|)$ and $\theta = (\log m)^{-2}$. Then we have

COROLLARY 7. *If* $g > |x|^{1-\theta}$, *then*

$$(42)\qquad (\log m)^2 < \log|x|.$$

Proof of Corollary 7. In view of Lemma 5, we may assume that $|y| > \frac{2}{3}|x|$. Then

$$|x|^{1-\theta} < g \leqslant |x| - |y| < \tfrac{1}{3}|x|,$$

which implies (42).

LEMMA 6.

$$(43)\qquad g \leqslant |x|^{1-\theta}.$$

Proof. We assume that $g > |x|^{1-\theta}$. Observe that

$$\max\left(\frac{|x|}{g}, \frac{|y|}{g}\right) = \frac{|x|}{g} < |x|^{\theta}$$

and further, by (42), $|x|^{\theta} > e$. Now apply (34) and Theorem A with $n = w_{11}$, $d = 1$, $B = 2\log R_1$, $B' = m$, $A' = w_{12}$, $A = |x|^{\theta}$ and $\delta = 1/m$ to conclude that

$$(44)\qquad |a'x^m + b'y^m| \geqslant \max(|a'x^m|, |b'y^m|)\,|x|^{-w_{13}\theta\log m} R_1^{-2/m}.$$

Further combine (44) and (38) to obtain

$$|y|^{m-n} \leqslant \frac{\max(|a'x^m|, |b'y^m|)}{\max(|a'x^n|, |b'y^n|)} \leqslant 2\,|x|^{w_{13}\theta\log m} R_1^{2/m}$$

which, together with (35) and (42), gives

$$(45)\qquad 1 \leqslant m-n \leqslant \frac{w_{14}}{\log m}\,\frac{\log|x|}{\log|y|}.$$

If $|x| > |y^2|$, then it follows from (40) and (35) that

$$n \leqslant w_{15}\left(1 + \frac{\log m}{\log|x|}\right) \leqslant w_{16}\log m$$

and this, in view of (36), implies that $m \leqslant w_{17}$. Thus we can assume that $|x| \leqslant |y|^2$ and hence, by (45), we conclude that

$$1 \leqslant m-n \leqslant 2w_{14}(\log m)^{-1}.$$

This is not possible if w_1 is large enough and hence the proof of Lemma 6 is finished.

Proof of Theorem 3. Re-writing equation (5), we have

$$a'x^n(x^{m-n}-1) = b'y^n(1-y^{m-n}),$$

i.e.

$$a'\left(\frac{x}{g}\right)^n (x^{m-n}-1) = b'\left(\frac{y}{g}\right)^n (1-y^{m-n})$$

which, together with (43) and $(b', x) = 1$, gives

$$(46)\qquad |x|^{n\theta} \leqslant \left(\frac{|x|}{g}\right)^n \leqslant |1-y^{m-n}| \leqslant 2\,|x|^{m-n}.$$

Combining (46) and (36), we obtain

$$n \leqslant w_{18}(\log m)^3.$$

Now apply (36) again to conclude the proof of Theorem 3.

7. *Remarks.* (i) Suppose that the polynomial associated to the sequence $\{u_m\}$ has complex roots. Let $f(x, y)$ be a binary form with integer coefficients of degree $h \geqslant 1$. Assume that $f(1, 0) \neq 0$. Suppose that the

maximum of the absolute values of the coefficients of f does not exceed H_1, where $H_1 \geqslant 3$. Then there exist effectively computable numbers $C_{16} > 0$ and $C_{17} > 0$ depending only on h, α and β such that for all pairs of non-negative integers m and n with $m > n$ and $m \geqslant C_{16}\log(RH_1)$, we have

$$|f(u_m, u_n)| \geqslant |\alpha|^{mh} e^{-C_{17}\nu_3}$$

where

$$\nu_3 = (\log m \log R + \log H_1)\log(n+2).$$

This follows from Corollary 2.

(ii) Let $P \geqslant 2$ and denote by S the set of all non-zero integers composed of primes not exceeding P. Then we can apply Theorem 3 and Theorem A to derive the following result: There are only finitely many solutions of the equation (5) in integers $a' \in s$, $b' \in s$, $x \in s$, $y \in s$, $n \geqslant 0$ and m with $(a', b') = (a', y) = (b', x) = 1$, $|x| \neq |y|$ and $n < m$. Further effective bounds for $|a'|$, $|b'|$, $|x|$, $|y|$, m and n can be given in terms of P. If a' and b' are fixed, this follows from Theorem 4 of [10].

(iii) Put $A_m = A_{m,m}$. We can apply Theorem A and Theorem C to prove: There exist effectively computable numbers $C_{18} > 0$ and $C_{19} > 0$ depending only on α and β such that

$$P(u_m) \geqslant C_{18}\left(\frac{A_m}{\log A_m}\right)^{1/(d_1+1)}, \quad A_m \geqslant C_{19}.$$

Here $d_1 = [\boldsymbol{Q}(\alpha):\boldsymbol{Q}]$. We give a sketch of the proof. By considering the subsequence $\{u_{2m}\}$ and $\{u_{2m+1}\}$ separately and observing that $([\alpha^2], [\beta^2])$ is an ideal generated by a non-zero rational integer, there is no loss of generality in assuming that the ideals $[\alpha]$ and $[\beta]$ are relatively prime. We assume that $A_m \geqslant C_{19}$ with C_{19} sufficiently large. Then, by Lemma 1, we obtain

$$|u_m| > |\alpha|^{m/2}.$$

Writing $P_1 = P(u_m)$, we have

$$\frac{m}{2}\log|\alpha| < \log|u_m| = \sum_{p|u_m} \operatorname{ord}_p(u_m)\log p \leqslant \log P_1 \sum_{p \leqslant P_1} \operatorname{ord}_p(u_m).$$

Consequently there exists a prime $p_1 \leqslant P_1$ such that

$$\operatorname{ord}_{p_1}(u_m) > (4P_1 \log|\alpha|)^{-1} m.$$

Now we apply Theorem C, as in the proof of Theorem 4, to complete the proof of the assertion.

(iv) Suppose that the associated polynomial to a recursive sequence $\{U_m\}$ of order 3 has none of the roots equal to 0, ± 1 and none of the ratios of its roots a root of unity. Mignotte and the author have used Theorem 2 in proving that $U_m \neq U_n$ whenever $m \neq n$ and $\max(m, n)$ exceeds a certain effectively computable number depending only on the sequence $\{U_m\}$.

Added in proofs. We can combine Theorem 1 of [10] with the effective result on the greatest prime factor of a binary form in place of ineffective theorems of Roth and Schinzel on the equation $f(x, y) = g(x, y)$. Then we obtain: Let $A, B, C, D \in \boldsymbol{Z}$ with $AB \neq 0$. The equation

$$Ax^m + By^m = Cx^n + Dy^n$$

has only finitely many solutions in integers x, y, m, n with $|x| \neq |y|$, $0 \leqslant n < m$, $m > 2$, $(m, n) \neq (4, 2)$ and $Ax^m \neq Cx^n$ if the binary forms $AX^m + AY^m$ and $CX^n + DY^n$ are not divisible in $\boldsymbol{Z}[X, Y]$ by a common linear factor. Furthermore the result is effective.

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