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Power mean-values for Dirichlet's polynomials and the Riemann zeta-function, II

by

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1. Introduction. Statement of the results. In this paper we describe another version of an argument introduced in [4] and substantially refined in the authors paper [2] to estimate the integrals

$$S(T, M) = \int_0^T |\zeta(\tfrac{1}{2}+it)M(it)|^2 dt$$

where $\zeta(s)$ is the Riemann zeta-function and $M(s)$ is a Dirichlet polynomial of length M, i.e.

$$M(s) = \sum_{m \leqslant M} a_m m^{-s}.$$

The final objective would be to give a bound

$$S(T, M) \ll T^{1+\varepsilon} \sum_{m \leqslant M} |a_m|^2 \tag{1}$$

for $M = T$ which in fact is equivalent to the Lindelöf hypothesis. The only unconditional result known hitherto asserts (1) for $M = T^{1/2}$ while the celebrated R^*-conjecture of C. Hooley [3] concerning the order of magnitude of incomplete Kloosterman sums yields (1) for $M = T^{4/7}$ (see [4]).

Here we obtain such result without Hooley's conjecture for polynomials $M(s)$ that are squares, namely we prove the following

Theorem 1. *For any complex numbers a_m with $|a_m| \leqslant 1$ we have*

$$\int_0^T |\zeta(\tfrac{1}{2}+it)|^2 \Big|\sum_{m \leqslant M} a_m m^{-1/2-it}\Big|^4 dt \ll T^{\varepsilon}(T + T^{1/2}M^{7/4} + M^{13/4}). \tag{2}$$

Here and below ε is any positive number, not necessarily the same in each occurrence, and the constant implied in the notation $\ll$ may depend on ε at most.

Theorem 1 will be a consequence of a somewhat more general result.

THEOREM 2. *Let $N \leqslant M$. For any complex numbers a_m, b_n with $|a_m| \leqslant 1$, $|b_n| \leqslant 1$ we have*

$$(3)\qquad \int_0^T |\zeta(\tfrac{1}{2}+it)|^2 \Big|\sum_{m\leqslant M} a_m m^{-1/2-it}\Big|^2 \Big|\sum_{n\leqslant N} b_n n^{-1/2-it}\Big|^2 dt \ll T^{\varepsilon}(T+T^{1/2}M^{3/4}N+T^{1/2}MN^{1/2}+M^{7/4}N^{3/2}).$$

The proof depends strongly on estimates for sums of incomplete Kloosterman sums which the authors established in the memoir [1] on the basis of Kuznetsov sum formula for the Hecke congruence groups. We wish to mention that the same estimates were used in [2] in a different manner to infer the following

THEOREM 3. *For any complex numbers a_m we have*

$$(4)\qquad \int_0^T |\zeta(\tfrac{1}{2}+it)|^4 \Big|\sum_{m\leqslant M} a_m m^{it}\Big|^2 dt \ll T^{\varepsilon}(T+T^{1/2}M^2+T^{3/4}M^{5/4}) \sum_{m\leqslant M} |a_m|^2.$$

Combining Theorems 2 and 3 we shall deduce the following

THEOREM 4. *Let $a_m \equiv \Lambda(m)$ or $a_m \equiv \mu(m)$ and $M(s) = \sum_{m\leqslant M} a_m m^{-s}$. We then have*

$$(5)\qquad \int_0^T |\zeta(\tfrac{1}{2}+it)M(\tfrac{1}{2}+it)|^2 dt \ll T^{\varepsilon}(T+T^{1/2}M^{7/8}+M^{5/3}).$$

Hence (1) *holds for $M(s)$ subject to $M \leqslant T^{4/7}$.*

In a letter to the second author D.R. Heath-Brown has informed that he was able to prove the second assertion of Theorem 4 for polynomials $M(s) = \sum_{m\leqslant M} \mu(m) m^{-s}$ of length $M \leqslant T^{8/15}$ and a few other results included in our Theorem 2. He made use of A. Weil estimate for Kloosterman sums as it had been done originally in [4].

In this paper we do not give asymptotic formulas for our integrals. They are important for problems concerning the distribution of zeros of the Riemann zeta-function. It is therefore interesting to point out that R. Balasubramanian and B. Conrey have succeeded to show the formula (oral communication)

$$\int_0^T |\zeta(\tfrac{1}{2}+it)M(\tfrac{1}{2}+it)|^2 dt \sim T \sum_{m_1,m_2\leqslant M} \frac{a_{m_1}a_{m_2}}{[m_1, m_2]}\left(\log \frac{T(m_1, m_2)^2}{2\pi m_1 m_2}+2\gamma-1\right)$$

if $a_m = \mu(m)f(m)$, f is a smooth function $\ll 1$ and $M < T^{8/15-\varepsilon}$. Consequently by refining the method of Levinson they showed that at least 38% of complex zeros of $\zeta(s)$ lay on the critical line.

2. Preliminary transformations. Our first aim is to prove Theorem 2. It is clearly sufficient to show that

$$(6)\qquad \mathscr{J}(L, M, N, T) := \int_T^{2T} |L(it)M(it)N(it)|^2 dt \ll T^{4\varepsilon}(T+T^{1/2}M^{3/4}N+M^{7/4}N^{3/2})LMN$$

for $L < L_1 \leqslant 2L \leqslant T$, $N < M$ where

$$L(s) = \sum_{L<l\leqslant L_1} l^{-s},\qquad M(s) = \sum_{M<m\leqslant 2M} a_m m^{-s},\qquad N(s) = \sum_{N<n\leqslant 2N} b_n n^{-s}.$$

We could restrict ourselves to $L \leqslant T^{1/2}$ by the approximate functional equation for $\zeta(s)$ but this is not necessary. However, it will be useful to assume that

$$(7)\qquad LMN > T^{1+4\varepsilon},$$

in the opposite case (6) following from the classical mean value theorem. For notational simplicity let us put $B = MN$ and

$$B(s) = M(s)N(s) = \sum_{B<b\leqslant 4B} \beta_b b^{-s}$$

where

$$(8)\qquad \beta_b = \sum_{\substack{M<m\leqslant 2M,\\ mn=b}} \sum_{N<n\leqslant 2N} a_m b_n.$$

In this section we express the integral $\mathscr{J}(L, M, N, T)$ by means of exponential sums that are alike Kloosterman sums. We begin with introducing smooth weights $f(t)$ and $g(l)$ such that

$$f(t) \geqslant 0,\quad f(t) = 1 \quad \text{if} \quad T < t \leqslant 2T,\quad f(t) = 0 \quad \text{unless } \tfrac{1}{2}T < t < \tfrac{5}{2}T,$$
$$g(l) \geqslant 0,\quad g(l) = 1 \quad \text{if} \quad L < l \leqslant 2L,\quad g(l) = 0 \quad \text{unless } \tfrac{1}{2}L < l < \tfrac{5}{2}L$$

and

$$f^{(\nu)}(t) \ll T^{-\nu},\qquad g^{(\nu)}(l) \ll L^{-\nu}$$

for any $\nu \geqslant 0$, the constant implied in $\ll$ depending on ν alone. By a standard device we can replace $L(s)$ by

$$G(s) = \sum g(l) l^{-s}$$

and we can introduce the kernel $f(t)$ in the integration over t. To be precise, letting

$$\mathscr{J}(f, g; M, N) := \int f(t)|G(it)B(it)|^2 dt$$

it can be shown (by applying Perron's formula for example) that

$$\mathcal{J}(L, M, N, T) \ll (T+MN)LMN+\max \mathcal{J}(f, g; M, N)\log T$$

where the maximum is taken over sequences $(a_m)_{M<m\leqslant 2M}$ and $(b_n)_{N<n\leqslant 2N}$ with $|a_m| \leqslant 1$ and $|b_n| \leqslant 1$. Hence it remains to prove (6) for $\mathcal{J}(f, g; M, N)$.

Squaring out the sum over the variables l and b involved in $|G(it)B(it)|^2$ one obtains terms $g(l)g(l_1)\beta_b\bar{\beta}_{b_1}(lb/l_1b_1)^{it}$ to be integrated over t with weight $f(t)$. For the terms placed far enough from the diagonal, precisely for those with

$$lb = l_1b_1+h, \qquad |h| \geqslant H := LMNT^{\varepsilon-1}$$

we get

$$\int f(t)\left(\frac{lb}{l_1b_1}\right)^{it} dt \ll h^{-2}T^{-100}$$

by iterated partial integration, whence such terms contribute to $\mathcal{J}(f, g; M, N)$ at most $O(MN)$. Also the terms on the diagonal, namely those with $lb = l_1b_1$ contribute to $\mathcal{J}(f, g; M, N)$ an admissible quantity $O(TLMN)$. From the above discussion we conclude that

$$\mathcal{J}(f, g; M, N) = \int f(t) \sum_{0<|h|\leqslant H} \sum_{b,b_1} \beta_b\bar{\beta}_{b_1} \sum_{lb=l_1b_1+h} g(l)g(l_1)\left(1-\frac{h}{lb}\right)^{-it} + O(TLMN).$$

Here we insert the approximations

$$g(l_1) = g(lb/b_1)+O(T^{\varepsilon-1})$$

and

$$\left(1-\frac{h}{lb}\right)^{-it} = e^{iht/lb}+O(T^{\varepsilon-1})$$

with the effect that

$$\mathcal{J}(f, g; M, N) = \int f(t) \sum_{0<|h|\leqslant H} \sum_{b,b_1} \beta_b\bar{\beta}_{b_1} \sum_{lb\equiv h[b_1]} g(l)g(lb/b_1)e^{iht/lb} + O\big(T^{\varepsilon}(T+MN)LMN\big).$$

The terms with $\delta = (b, b_1) > \Delta := LMNT^{-\varepsilon-1}$ contribute to $\mathcal{J}(f, g; M, N)$ at most

$$4T \sum_{0<|h|\leqslant H} \sum_{\delta|h,\delta>\Delta} \sum_{(b,b_1)=\delta} |\beta_b\beta_{b_1}|2\delta Lb_1^{-1} \ll LB^{-1}TH \sum_{\delta>\Delta} \sum_{(b,b_1)=\delta} |\beta_b\beta_{b_1}| \ll L\Delta^{-1}TH \sum_b |\beta_b|^2\tau(b) \ll T^{1+2\varepsilon}LMN(\log 2MN)^8.$$

The remaining terms can be arranged as follows

$$\mathcal{J}(f, g; M, N) = \int f(t) \sum_{0<\delta\leqslant\Delta} \sum_{0<\delta|h|\leqslant H} \sum_{(b,b_1)=\delta} \beta_b\bar{\beta}_{b_1} \sum_{l\equiv h\bar{a}[c]} g(l)g(lb/b_1)e^{i\delta ht/lb} + O\big(T^{3\varepsilon}(T+MN)LMN\big).$$

Here and below we denote $a = \delta^{-1}b$ and $c = \delta^{-1}b_1$, so $(a, c) = 1$. For the innermost sum, we apply the Poisson formula

$$\sum_l = \sum_k e\left(-hk\frac{\bar{a}}{c}\right)\hat{g}(h, k)$$

where

$$\hat{g}(h, k) = \int g(\xi a)g(\xi c)e\left(\frac{ht}{2\pi\xi ac}+k\xi\right)d\xi.$$

If $\delta|k| > K := L^{-1}MNT^{2\varepsilon}$, then by iterated partial integration it follows that $\hat{g}(h, k) \ll k^{-2}T^{-100}$ so the total contribution of the 'tail' $\delta|k| > K$ is absorbed by the earlier error term.

With the constant terms ($k = 0$), we proceed as follows

$$\sum_{0<|h|\leqslant\delta^{-1}H} \hat{g}(h, 0) = \int g(\xi a)g(\xi c)\Big(\sum_{0<|h|\leqslant\delta^{-1}H} e^{iht/\xi ac}\Big)d\xi = \int g(\xi a)g(\xi c)\chi(x, y)d\xi+O(\delta LB^{-1})$$

where $\chi(x, y) = (e^{ixy}-e^{-ixy})(e^{iy}-1)^{-1}$ with $x = \delta^{-1}H$ and $y = t/\xi ac$ by the elementary formula

$$\sum_{0<|h|\leqslant x} e^{ihy} = \chi(x, y)+O(1).$$

Notice that $y \ll \Delta T/LB = T^{-\varepsilon}$ and $xy \gg HT/LB = T^{\varepsilon}$, therefore by iterated partial integration with respect to ξ we get

$$\int g(\xi a)g(\xi c)\chi(x, y)d\xi \ll T^{-100}$$

which shows that the total contribution of the constant terms is absorbed by the earlier error term.

Finally we arrive at the following general formula

$$\mathcal{J}(f, g; M, N) = \sum_{0<\delta\leqslant\Delta} \sum_{0<\delta|h|\leqslant H} \sum_{0<\delta|k|\leqslant K} \sum_{(a,c)=1} \beta_{\delta a}\bar{\beta}_{\delta c}\, e\left(-hk\frac{\bar{a}}{c}\right)F(h, k, a, c) + O\big(T^{3\varepsilon}(T+MN)LMN\big)$$

where

(9) $$F(h, k, a, c) = 2\pi ac \iint \xi g(\xi a)g(\xi c)f(2\pi\xi\eta ac)e(\xi k+\eta h)d\xi d\eta.$$

Now, let us specify β_b to that given by (8). We have $b = \delta a = mn$, hence writing $m = \mu d$ with $\mu | \delta^\infty$, $(d, \delta) = 1$, we see that $n = \nu r$ and $a = \varrho r d$ where $\nu = \delta/(\mu, \delta)$ and $\varrho = \mu/(\mu, \delta)$. With this notation, we get

$$\mathscr{I}(f, g; M, N) = \sum_{0<\delta\leqslant\Delta} \sum_{\mu|\delta^\infty} \sum_{0<\delta|h|\leqslant H} \sum_{0<\delta|k|\leqslant K} \sum_{(c,e)=1} \bar{\beta}_{\delta c} \times \tag{10}$$

$$\times \sum_{(d,\delta c)=1} a_{\mu d} \sum_{(r,c)=1} b_{\nu r} e\left(-hk \frac{\overline{\varrho r d}}{c}\right) F(h, k, \varrho r d, c) + O\left(T^{3\varepsilon}(T+MN)LMN\right).$$

3. An upper bound for sums of Kloosterman type. To proceed further we need the following

Lemma 1. *Let $C, D, U, V \geqslant 1$ and $|c(u, v)| \leqslant 1$. We then have*

$$\sum_{\substack{1\leqslant c\leqslant C\\(c,\varrho d)=1}} \sum_{1\leqslant d\leqslant D} \left| \sum_{1\leqslant u\leqslant U} \sum_{\substack{1\leqslant v\leqslant V\\(v,c)=1}} c(u, v) e\left(u \frac{\overline{\varrho v d}}{c}\right)\right|$$

$$\ll (CDUV)^{1/2+\varepsilon}\{(CD)^{1/2} + (U+V)^{1/4}[CD(U+\varrho V)(C+\varrho V^2) + \varrho U V^2 D^2]^{1/4}\}.$$

Proof. Split up the outer sum into sums of the type

$$\mathscr{K}(C, D, U, V) = \sum_{C<c\leqslant 2C} \sum_{D<d\leqslant 2D} \left|\sum_u \sum_v\right|.$$

By the Cauchy–Schwarz inequality we get

$$\mathscr{K}^2(C, D, U, V) \leqslant CD \sum_{(c,d)=1} \sum g(c, d) \left|\sum_u \sum_v\right|^2$$

$$= CD \sum_{\substack{1\leqslant u_1,u_2\leqslant U\\1\leqslant v_1,v_2\leqslant V}} c(u_1, v_1)\overline{c(u_2, v_2)} \sum_{(c,\varrho v_1 v_2 d)=1} g(c, d) e\left((u_1 v_2 - u_2 v_1)\frac{\overline{\varrho v_1 v_2 d}}{c}\right)$$

$$= CD \sum_{1\leqslant r\leqslant R} \sum_{|n|\leqslant N} b(n, r) \sum_{(c,\varrho r)=1} \sum_{(d,c)=1} g(c, d) e\left(n \frac{\overline{rd}}{c}\right)$$

say, where $g(c, d)$ is any function which majorizes the characteristic function of the cube $[C, 2C]\times[D, 2D]$, $R = \varrho V^2$, $N = 2UV$ and

$$b(n, r) = \sum_{\substack{1\leqslant v_1,v_2\leqslant V\\\varrho v_1 v_2 = r}} \sum_{\substack{1\leqslant u_1,u_2\leqslant U\\u_1 v_2 - u_2 v_1 = n}} c(u_1, v_1)\overline{c(u_2, v_2)}.$$

The terms with $n = 0$ contribute at most

$$4C^2 D^2 \sum_r |b(0, r)| \ll C^2 D^2 \sum_{1\leqslant q\leqslant UV} \tau^2(q) \ll C^2 D^2 UV(\log UV)^4.$$

For estimating the remaining sum we appeal to Theorem 12 of [1] giving the upper bound

$$(CDUV)^\varepsilon CD[C(R+N)(C+DR) + C^2 D\sqrt{(R+N)R} + D^2 NR]^{1/2} \times$$
$$\times \Big(\sum_{n,r} |b(n, r)|^2\Big)^{1/2}.$$

We have

$$\sum_{n,r} |b(n, r)|^2 \ll V^\varepsilon \sum_{1\leqslant v_1,v_2\leqslant V} \sum_n \Big|\sum_{\substack{1\leqslant u_1,u_2\leqslant U\\u_1 v_2 - u_2 v_1 = n}} 1\Big|^2$$

$$= V^\varepsilon \sum_{1\leqslant v_1,v_2\leqslant V} \sum_{\substack{1\leqslant u_1,u_2,u_3,u_4\leqslant U\\(u_1-u_3)v_2 = (u_2-u_4)v_1}} 1$$

$$\ll (UV)^{3\varepsilon} U^2 V(U+V).$$

Finally, by elementary calculations one completes the proof of Lemma 1.

4. Proof of Theorem 2. Lemma 1 is almost applicable to the sum $\mathscr{I}_{\delta\mu}(f, g; M, N) = \sum_h \sum_k \sum_c \sum_d \sum_r$ from (10) with only a minor objection, that the variables c, d, r in the weight function $F(h, k, \varrho r d, c)$ need be separated. This is, however, an easy problem which can be solved in various standard ways. The technique based on applying Mellin's transform for the kernel functions $g(\xi\varrho r d)$, $g(\xi c)$ and $f(2\pi\xi\eta\varrho r d c)$ in the integral representation (9) is standard, so we skip details. Having done it, we apply Lemma 1 with $C = \delta^{-1}MN$, $D = \mu^{-1}M$, $U = \delta^{-2}HK = \delta^{-2}M^2N^2T^{3\varepsilon-1}$ and $V = \nu^{-1}N$ giving

$$\mathscr{I}_{\delta\mu}(f, g; M, N) \ll T^\varepsilon(\delta M^{-1}N^{-1}LT)(\varrho^{-1/2}\delta^{-2}M^2N^2T^{-1/2}) \times$$

$$\times\left\{\left(\frac{M^2N}{\delta\mu}\right)^{1/2} + \left(\frac{M^2N^2}{\delta^2 T} + \frac{N}{\nu}\right)^{1/4}\left[\frac{M^2N}{\delta\mu}\left(\frac{M^2N^2}{\delta^2T} + \frac{\varrho N}{\nu}\right)\left(\frac{MN}{\delta} + \frac{\varrho N^2}{\nu^2}\right) + \right.\right.$$
$$\left.\left. + \frac{\varrho M^2N^4}{\delta^2\nu^2\mu^2 T}\right]^{1/4}\right\}$$

$$\ll \delta^{-5/4}\mu^{-1/4}T^{3\varepsilon}LMN\{T^{1/2}M^{3/4}N + T^{1/2}MN^{1/2} + M^{7/4}N^{3/2}\}.$$

Hence, by (10), the summation over $\mu|\delta^\infty$ and $\delta \leqslant \Delta$ completes the proof of Theorem 2.

5. Proof of Theorem 4. We first rearrange the polynomial $M(s)$ by applying the celebrated identity of R. C. Vaughan [5]. Since the arguments are well-known we shall be very brief. Thus $M(s)$ can be regarded as the product of two Dirichlet's polynomials having either the shape

$$M_1(s)N_1(s) \tag{11}$$

or

$$M_2(s)L_2(s), \tag{12}$$

where $M_1(s)$, $N_1(s)$, $M_2(s)$ and $L_2(s)$ are Dirichlet's polynomials of length M_1, N_1, M_2, L_2 respectively satisfying

$$M_1N_1 \leqslant M, \quad W < N_1 < M_1,$$
$$M_2L_2 \leqslant M, \quad M_2 < W^2$$

and with coefficients bounded by the divisor function. Moreover $L_2(s)$ is a partial sum of the Riemann zeta-function. Here W is any parameter at our disposal. We choose $W = T^{1/5}$, so for polynomial (11), by Theorem 2 we get

$$\int_0^T |\zeta M_1 N_1(\tfrac{1}{2}+it)|^2 dt \ll T^\varepsilon(T + T^{1/2}M^{3/4}N_1^{1/4} + T^{1/2}MN_1^{-1/2} + M^{7/4}N_1^{-1/4})$$
$$\ll T^\varepsilon(T + T^{1/2}M^{7/8} + T^{2/5}M + T^{-1/20}M^{7/4})$$

which implies (5). For the polynomial (12), if $M_2 > W$ the arguments are very similar to those of the first one, and if $M_2 < W$ we apply Theorem 3 getting

$$\int_0^T |\zeta M_2 L_2(\tfrac{1}{2}+it)|^2 dt \ll T^{1+\varepsilon}.$$

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Reducibility of lacunary polynomials, IV

by

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The aim of this paper is to make a further contribution to the problem of reducibility of polynomials

$$f(x) = a_0 + \sum_{j=1}^{k} a_j x^{n_j} \quad (0 = n_0 < n_1 < \ldots < n_k,\ a_0a_k \neq 0) \tag{1}$$

for fixed integral coefficients a_j and variable exponents n_j. The non-reciprocal irreducible factors of $f(x)$ can be found by means of Theorem 2 in [3] and as to reciprocal factors the conjecture proposed in [2] implies the existence of a constant $C(a_0, a_1, \ldots, a_k)$ such that either all reciprocal irreducible factors of f are cyclotomic or $\sum_{j=1}^{k}\gamma_j n_j = 0$ for suitable integers γ_j satisfying

$$0 < \max_{1\leqslant j\leqslant k} |\gamma_j| \leqslant C(a_0, a_1, \ldots, a_k).$$

We shall prove

Theorem. *If f is given by* (1) *with a_j integral, then either all reciprocal irreducible factors of f are cyclotomic or there exist integers $\gamma_1, \ldots, \gamma_k$ satisfying*

$$\sum_{j=1}^{k} \gamma_j n_j = 0, \tag{2}$$

$$0 < \max_{j=1,\ldots,k} |\gamma_j| \leqslant \max_{0\leqslant j\leqslant k} \frac{\log a_j^2}{\log 2} \tag{3}$$

and the number of reciprocal non-cyclotomic factors of f does not exceed the total number of prime factors of (a_0, a_k) or finally the following system of