[10] L. J. Rogers, *Second memoir on the expansion of certain infinite products*, Proc. London Math. Soc. 25 (1894), pp. 318–343.
[11] — *On a type of modular relation*, ibid. (2) 19 (1921), pp. 387–397.
[12] A. Selberg, *Über einige arithmetische Identitäten*, Avhandlinger. utgitt. Norske. Vid. Akademi, Oslo 1936, No. 8, 23 pages.
[13] A. Selberg and S. D. Chowla, *On Epstein's Zeta function*, Journ. für Math. 227 (1967), pp. 86–110.
[14] C. L. Siegel, *Advanced analytic number theory*, Tata Institute of Fundamental Research, Bombay 1980.
[15] G. N. Watson, *Theorems stated by Ramanujan VII, Theorems on continued fractions*, Journ. London Math. Soc. 4 (1929), pp. 39–48.
[16] — *Theorems stated by Ramanujan IX, Two theorems on continued fractions*, ibid. 4 (1929), pp. 231–237.
[17] — *Über eigenschaften des Ramanujanschen Kettenbrüche*, Monat. für Math. 48 (1939), pp. 516–530.
[18] H. Weber, *Lehrbuch der Algebra*, Bd. III, Braunschweig 1908.

TATA INSTITUTE OF FUNDAMENTAL RESEARCH, Bombay
and
THE INSTITUTE FOR ADVANCED STUDY, Princeton

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Pairs of additive equations IV. Sextic equations

by

R. J. Cook (Sheffield)

1. Introduction. H. Davenport and D. J. Lewis [10] remarked that it should be possible in principle to show that the equations, with integer coefficients,

$$F(x) = a_1x_1^k + \ldots + a_Nx_N^k = 0,\quad G(x) = b_1x_1^k + \ldots + b_Nx_N^k = 0 \tag{1}$$

have a non-trivial solution in integers provided that

(i) $N \geqslant 2k^2+1$,

(ii) they have a non-singular real solution,

(iii) for each prime p they have a non-singular p-adic solution,

(iv) if the degree k is even then each form $\lambda F+\mu G$, $(\lambda, \mu) \neq (0, 0)$, contains a reasonable number of variables explicitly.

The condition $N \geqslant 2k^2+1$ is similar to Artin's conjecture for two additive forms. Results of this strength have been established when $k = 2$ ([2]), $k = 3$ ([4], [9] and [13]), $k = 4$ ([4]), $k = 5$ ([6]) and $k \geqslant 18$ ([5] and [11]) since the analytic methods of [5] will also work for even values $k \geqslant 18$. The analytic methods used for quintic equations were based on a method of Davenport [7] for iterating admissible exponents and unfortunately the method just fails to work when $k = 6$. However H. Davenport and P. Erdös ([8], Theorem 2) obtained admissible exponents for 3 sixth powers that improved on the estimates obtainable by Davenport's methods. The basis of the present paper is to establish an analogue for two additive equations of this result of Davenport and Erdös and then to use iterative methods to obtain a sequence of 14 exponents for sixth powers.

THEOREM 1. *Let the equations*

$$F(x) = a_1x_1^6 + \ldots + a_Nx_N^6 = 0,\quad G(x) = b_1x_1^6 + \ldots + b_Nx_N^6 = 0 \tag{2}$$

have integer coefficients. Then they have a non-trivial solution in integers provided that

(i) $N \geqslant 73$,

(ii) *they have non-singular real, 2-adic and 3-adic solutions, and*

(iii) *the ratios a_i/b_i take at least 17 different values with no value occurring more than 18 times.*

In proving the theorem we may make the additional assumptions that $N = 73$ and that for each suffix i either a_i or b_i is non-zero.

2. Exponents for sixth powers. In [6] the s real numbers $\lambda_1, \ldots, \lambda_s$, satisfying

(3) $$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_s > 0$$

were called admissible exponents (for R additive equations of degree k) if:

For any $R\times sR$ integer matrix A, whose columns form s consecutive non-singular $R\times R$ matrices, the number of solutions of

(4) $$AX = AY,$$

(where

(5) $$X = [x_1^k, \ldots, x_{sR}^k]^T, \quad Y = [y_1^k, \ldots, y_{sR}^k]^T$$

are integer vectors satisfying

(6) $$c_j P^{\lambda_\nu} < x_j, y_j < C_j P^{\lambda_\nu}, \quad j = 1, \ldots, sR,$$

with $\nu = 1+[(j-1)/R]$ and $0 < c_j < C_j$), is

(7) $$O(P^{R(\lambda_1+\ldots+\lambda_s)+\varepsilon}) \quad \text{as} \quad P\to\infty$$

for any $\varepsilon > 0$.

If $\lambda_1, \ldots, \lambda_s$ are admissible exponents and $\theta > 0$ then $\theta\lambda_1, \ldots, \theta\lambda_s$ are also admissible exponents. When $R = 1$ and $s = 3$ the methods of Davenport [7] show that for any $\lambda \leqslant (k+1)/(k+2)$ the exponents $1, \lambda, \lambda$ are admissible exponents. For $k \geqslant 6$ this was improved on by Theorem 2 of Davenport and Erdös [8], which states that

(8) $$1,\ 1-k^{-2},\ 1-k^{-2}-k^{-1}$$

are admissible exponents. Recently R. C. Vaughan [14] has extended this result to a sequence of $s > 3$ exponents. Naturally, one might hope to prove that the exponents (8) are admissible exponents for $R > 1$ additive equations, but there appear to be technical difficulties in this. These difficulties might be overcome by altering the definition to require that the columns of A are in general position, but this places severe restrictions on the application of the results.

THEOREM 2. *Let*

(9) $$\lambda = 1-k^{-2}, \quad \mu = 1-k^{-2}-k^{-1}.$$

Suppose that of the ratios c_i/d_i ($i = 1, \ldots, 6$) c_1/d_1, c_2/d_2, c_5/d_5 and c_6/d_6 are distinct and c_3/d_3 and c_4/d_4 are distinct.

Then the number of solutions of the equations

(10) $$\begin{aligned} c_1x_1^k+\ldots+c_6x_6^k &= c_1y_1^k+\ldots+c_6y_6^k,\\ d_1x_1^k+\ldots+d_6x_6^k &= d_1y_1^k+\ldots+d_6y_6^k \end{aligned}$$

in integers $x_1, \ldots, x_6, y_1, \ldots, y_6$ satisfying

(11) $$\Gamma_i P < x_i, y_i < C_i P, \quad i = 1, 2,$$

(12) $$\Gamma_i P^\lambda < x_i, y_i < C_i P^\lambda, \quad i = 3, 4,$$

and

(13) $$\Gamma_i P^\mu < x_i, y_i < C_i P^\mu, \quad i = 5, 6,$$

where $0 < \Gamma_i < C_i$ is

(14) $$O(P^{2(1+\lambda+\mu)+\varepsilon}) \quad as \quad P\to\infty$$

for any $\varepsilon > 0$.

Since $c_1/d_1 \neq c_2/d_2$ we can take suitable linear combinations of the equations (10) to diagonalize the first two columns. Thus we may suppose that

(15) $$c_2 = d_1 = 0$$

in (10).

By 'a solution of (10)' we shall mean a solution which also satisfies (11), (12) and (13). It will be convenient to express the number of such solutions, $\nu(P)$ say, in terms of exponential sums so let

(16) $$S_1(\alpha) = \sum_{(11)} e(c_1\alpha x^k), \quad S_2(\beta) = \sum_{(11)} e(d_2\beta x^k),$$

(17) $$T_i(\alpha, \beta) = \sum_{(12)} e\big((c_i\alpha+d_i\beta)x^k\big), \quad i = 3, 4,$$

and

(18) $$U_i(\alpha, \beta) = \sum_{(13)} e\big((c_i\alpha+d_i\beta)x^k\big), \quad i = 5, 6.$$

Then

(19) $$\nu(P) = \int_0^1\int_0^1 |S_1(\alpha)S_2(\beta)T_3(\alpha,\beta)T_4(\alpha,\beta)U_5(\alpha,\beta)U_6(\alpha,\beta)|^2\,d\alpha\,d\beta.$$

LEMMA 1. *The number of solutions of* (10) *with* $x_1 = y_1$ *and* $x_2 = y_2$ *is*

(20) $$O(P^{2(1+\lambda+\mu)+\varepsilon}).$$

Proof. This contribution to $\nu(P)$ is

$$\ll P^2 \int_0^1 \int_0^1 |T_3(\alpha, \beta) \ldots U_6(\alpha, \beta)|^2 d\alpha d\beta$$

$$\ll P^2 \Big\{\int_0^1 \int_0^1 |T_3(\alpha, \beta) T_4(\alpha, \beta)|^4 d\alpha d\beta\Big\}^{1/2} \Big\{\int_0^1 \int_0^1 |U_5(\alpha, \beta) U_6(\alpha, \beta)|^4 d\alpha d\beta\Big\}^{1/2}$$

and the lemma now follows from our version of Hua's lemma [3].

LEMMA 2. *The number of solutions of* (10) *with* $x_1 \neq y_1$ *and* $x_2 = y_2$, *or* $x_1 = y_1$ *and* $x_2 \neq y_2$, *is*

(21) $$O(P^{2(1+\lambda+\mu)+\varepsilon}).$$

Proof. The number of solutions with $x_1 \neq y_1$ and $x_2 = y_2$ is $O(P\nu_1)$ where ν_1 is the number of solutions of the equations

(22) $$c_1 x_1^k + c_3 x_3^k + \ldots + c_6 x_6^k = c_1 y_1^k + c_3 y_3^k + \ldots + c_6 y_6^k,$$
$$d_3 x_3^k + \ldots + d_6 x_6^k = d_3 y_3^k + \ldots + d_6 y_6^k$$

with $x_1 \neq y_1$. The restrictions on the ratios c_i/d_i imply that d_5 and d_6 are non-zero and that at most one of d_3 and d_4 can be zero. If neither d_3 nor d_4 is zero then we can estimate the number of solutions of the second equation in (22) as

(23) $$\int_0^1 |T_3(0, \beta) T_4(0, \beta) U_5(0, \beta) U_6(0, \beta)|^2 d\beta$$

$$\ll \sum_{i=3}^{4} \sum_{j=5}^{6} \Big\{\int_0^1 |T_i(0, \beta)|^8 d\beta\Big\}^{1/2} \Big\{\int_0^1 |U_j(0, \beta)|^8 d\beta\Big\}^{1/2} \ll P^{5(\lambda+\mu)/2+\varepsilon}$$

by Hua's Lemma [12]. Each such solution fixes the value of $c_1(x_1^k - y_1^k)$ so that the number of possibilities for x_1, y_1 with $x_1 \neq y_1$ is $O(P^\varepsilon)$. Thus in this case the contribution to $\nu(P)$ is

$$O(P^{1+5(\lambda+\mu)/2+2\varepsilon}) \ll P^{2(1+\lambda+\mu)+\varepsilon}$$

since $(\lambda+\mu) < 2$.

Suppose now that one of d_3 and d_4 is zero. If $d_3 = 0$ then the second equation in (22) becomes

(24) $$d_4 x_4^k + d_5 x_5^k + d_6 x_6^k = d_4 y_4^k + d_5 y_5^k + d_6 y_6^k.$$

Now

(25) $$1-k^{-2},\ 1-k^{-2}-k^{-1},\ 1-k^{-2}-k^{-1}$$

are admissible exponents for a single equation of degree k since

$$(1-k^{-2}-k^{-1})/(1-k^{-2}) < (k+1)/(k+2).$$

Therefore the number of solutions of (25) is $O(P^{\lambda+2\mu+\varepsilon})$. For each such solution we then solve the first equation which now has the form

(26) $$c_1 x_1^k + c_3 x_3^k = c_1 y_1^k + c_3 y_3^k + C$$

for some constant C. The number of solutions of (26) is at most

(27) $$\int_0^1 |S_1(\alpha) T_3(\alpha, 0)|^2 d\alpha \leqslant \Big\{\int_0^1 |S_1(\alpha)|^4 d\alpha\Big\}^{1/2} \Big\{\int_0^1 |T_3(\alpha, 0)|^4 d\alpha\Big\}^{1/2}$$
$$= O(P^{1+\lambda+\varepsilon}).$$

In this case the contribution to $\nu(P)$ is

$$\ll P \cdot P^{\lambda+2\mu+\varepsilon} \cdot P^{1+\lambda+\varepsilon} = P^{2(1+\lambda+\mu)+2\varepsilon}$$

and $\varepsilon > 0$ is arbitrary.

We are now left with the solutions of (10) for which $x_1 \neq y_1$ and $x_2 \neq y_2$.

LEMMA 3. *The number of solutions of* (10) *with* $x_5 = y_5$ *and* $x_6 = y_6$ *is*

(28) $$O(P^{2(1+\lambda+\mu)+\varepsilon}).$$

This is proved in the same way as Lemma 1.

LEMMA 4. *The number of solutions of* (10) *for which* $x_1 \neq y_1$, $x_2 \neq y_2$ *and either* $x_5 \neq y_5$ *and* $x_6 = y_6$ *or* $x_5 = y_5$ *and* $x_6 \neq y_6$ *is*

(29) $$O(P^{2(1+\lambda+\mu)+\varepsilon}).$$

Proof. The number of solutions with $x_5 \neq y_5$ and $x_6 = y_6$ is $O(P^\mu \nu_2)$ where ν_2 is the number of solutions of the equations

(30) $$c_1 x_1^k + c_3 x_3^k + \ldots + c_5 x_5^k = c_1 y_1^k + c_3 y_3^k + \ldots + c_5 y_5^k,$$
$$d_2 x_2^k + d_3 x_3^k + \ldots + d_5 x_5^k = d_2 y_2^k + d_3 y_3^k + \ldots + d_5 y_5^k.$$

Suppose first that either d_3 or d_4 is zero, say $d_3 = 0$. Then the second equation becomes

(31) $$d_2 x_2^k + d_4 x_4^k + d_5 x_5^k = d_2 y_2^k + d_4 y_4^k + d_5 y_5^k$$

and since 1, λ and μ are admissible exponents for a single equation the number of solutions of (31) is

(32) $$O(P^{1+\lambda+\mu+\varepsilon}).$$

For each such solution the first equation becomes

(33) $$c_1 x_1^k + c_3 x_3^k = c_1 y_1^k + c_3 y_3^k + C$$

for some constant C. Since c_1 and c_3 are non-zero the number of solutions of (33) is at most

$$\int_0^1 |S_1(\alpha)T_3(\alpha,0)|^2 d\alpha \ll \Big\{\int_0^1 |S_1(\alpha)|^4 d\alpha\Big\}^{1/2}\Big\{\int_0^1 |T_3(\alpha,0)|^4 d\alpha\Big\}^{1/2} \ll P^{1+\lambda+\varepsilon}$$

so in this case the number of solutions is

$$\ll P\cdot P^{1+\lambda+\mu+\varepsilon}\cdot P^{1+\lambda+\varepsilon} = P^{2(1+\lambda+\mu)+2\varepsilon}$$

and $\varepsilon > 0$ is arbitrary.

If neither d_3 nor d_4 is zero then all the coefficients in the second equation are non-zero. Thus the number of solutions of the second equation is at most

$$(34)\qquad \int_0^1 |S_2(\beta)T_3(0,\beta)T_4(0,\beta)U_5(0,\beta)|^2 d\beta$$

$$\ll \sum_{i=3}^{4}\int_0^1 |S_2(\beta)T_i^2(0,\beta)U_5(0,\beta)|^2 d\beta$$

$$\ll \sum_{i=3}^{4}\Big\{\int_0^1 |S_2(\beta)|^8 d\beta\Big\}^{1/4}\Big\{\int_0^1 |T_i(0,\beta)|^8 d\beta\Big\}^{1/2}\Big\{\int_0^1 |U_5(0,\beta)|^8 d\beta\Big\}^{1/4}$$

$$\ll P^{5(1+2\lambda+\mu)/4+\varepsilon}$$

by Hua's Lemma [12]. Each such solution fixes the value of $c_1(x_1^k - y_1^k)$ so the number of possibilities for x_1, y_1 with $x_1 \neq y_1$ is $O(P^\varepsilon)$. Thus the number of solutions in this case is

$$(35)\qquad \ll P^{\mu}\cdot P^{5(1+2\lambda+\mu)/4+\varepsilon}\cdot P^{\varepsilon} \ll P^{2(1+\lambda+\mu)+\varepsilon}$$

since $2\lambda+\mu < 3$.

The final step in the proof of Theorem 2 is to estimate the number of solutions of (10) for which $x_1 \neq y_1$, $x_2 \neq y_2$, $x_5 \neq y_5$ and $x_6 \neq y_6$.

LEMMA 5. *The number of solutions of* (10) *with* $x_1 \neq y_1$, $x_2 \neq y_2$, $x_5 \neq y_5$ *and* $x_6 \neq y_6$ *is*

$$(36)\qquad O(P^{2(1+\lambda+\mu)+\varepsilon}).$$

Proof. Let $\Delta_t(f(x)) = f(x) - f(x-t)$ and

$$y_i = x_i - t(i) \quad \text{for} \quad i = 1, 2.$$

Now

$$(37)\qquad \begin{aligned} c_1\Delta_{t(1)}(x_1^k) &= c_3(y_3^k - x_3^k) + \dots + c_6(y_6^k - x_6^k),\\ d_2\Delta_{t(2)}(x_2^k) &= d_3(y_3^k - x_3^k) + \dots + d_6(y_6^k - x_6^k)\end{aligned}$$

and so

$$(38)\qquad 0 < |t(i)| \ll P^{\theta} \quad \text{for} \quad i = 1, 2$$

where $\theta = 1 - k^{-1}$. For fixed values of $t(1)$, $t(2)$, x_3, x_4, y_3 and y_4 the equations (37) become

$$(39)\qquad \begin{aligned} c_1\Delta_{t(1)}(x_1^k) &= C_1 + O(P^{k-1-k^{-1}}),\\ d_2\Delta_{t(2)}(x_2^k) &= D_1 + O(P^{k-1-k^{-1}}).\end{aligned}$$

For x_1 and x_2 in the range (11)

$$(40)\qquad \Delta_{t,1}(x_i^k) \gg tP^{k-2}$$

and so the number of values of x_1 satisfying (39) is

$$(41)\qquad \ll \frac{P^{k-1-k^{-1}}}{t(1)P^{k-2}} + 1 \ll \frac{P^{\theta}}{|t(1)|}$$

since (from (38)) $|t(1)| \ll P^{\theta}$, and similarly for x_2. Thus the number of possible values of $x_1, \dots, x_4, y_3, y_4, t(1)$ and $t(2)$ satisfying the equations is

$$(42)\qquad \ll P^{4\lambda}\Big(\sum_{0<t\le P^{\theta}} \frac{P^{\theta}}{t}\Big)^2 \ll P^{4\lambda+2\theta+\varepsilon} = P^{2(1+\lambda+\mu)+\varepsilon}.$$

For fixed values of these variables we have

$$(43)\qquad \begin{aligned} c_5(x_5^k - y_5^k) + c_6(x_6^k - y_6^k) &= C_2,\\ d_5(x_5^k - y_5^k) + d_6(x_6^k - y_6^k) &= D_2\end{aligned}$$

for some constants C_2 and D_2. This determines the values $x_5^k - y_5^k$ and $x_6^k - y_6^k$ so the number of possible values with $x_5 \neq y_5$, $x_6 \neq y_6$ is $O(P^{\varepsilon})$ which completes the proof of the lemma and Theorem 2.

Theorem 2 does not quite prove that 1, λ and μ are admissible exponents, as defined in [6], however the proof of Theorem 3 of [6] implies the following result.

LEMMA 6. *Let* A *be an* $R\times sR$ *integer matrix whose columns form* s *consecutive non-singular* $R\times R$ *matrices. Suppose that the number of solutions of*

$$(44)\qquad AX = AY,$$

where

$$(45)\qquad X = [x_1^k, \dots, x_{sR}^k]^T \quad \text{and} \quad Y = [y_1^k, \dots, y_{sR}^k]^T$$

are integer vectors satisfying

$$(46)\qquad \Gamma_j P^{\lambda_\nu} < x_j, y_j < C_j P^{\lambda}$$

with $\nu = 1 + [(j-1)/R]$, $0 < \Gamma_j < C_j$ *and* $\lambda_1 \geq \dots \geq \lambda_s > 0$, *is*

$$(47)\qquad O(P^{R(\lambda_1+\dots+\lambda_s)+\varepsilon}) \quad \text{as} \quad P\to\infty$$

for any $\varepsilon > 0$. *Let*

(48) $$\sigma = \lambda_1 + \ldots + \lambda_s \quad \text{and} \quad \delta = k\lambda_1 - (k-1).$$

Suppose that $\delta > 0$ *and that there exists an integer* l *satisfying*

(49) $$1 \leqslant l \leqslant k-2,$$

(50) $$2^l \delta \leqslant 1,$$

(51) $$(2^l - 1)\delta + \sigma \leqslant l+1.$$

Then for any non-singular $R \times R$ *integer matrix* A_0 *the number of solutions of*

(52) $$A_0 X_0 + AX = A_0 Y_0 + AY,$$

where

(53) $$X_0 = [u_1^k, \ldots, u_R^k]^T, \quad Y_0 = [v_1^k, \ldots, v_R^k]^T$$

are integer vectors satisfying

(54) $$\Gamma_j' P < u_j, v_j < C_j' P, \quad j = 1, \ldots, R,$$

and X, Y *satisfy* (46) *is*

(55) $$O(P^{R(1+\lambda_1+\ldots+\lambda_s)+\varepsilon}) \quad \text{as } P \to \infty$$

for any $\varepsilon > 0$.

Now let $\lambda_1, \ldots, \lambda_s$ be exponents with

(56) $$\lambda_1 = 1 \quad \text{and} \quad \sigma = \sigma(s) = 1 + \lambda_2 + \ldots + \lambda_s.$$

We apply Lemma 6 with $k = 6$. For $\theta > 5/6$ and either

(57) $$l = 1, \quad \theta = \min(7/(6+\sigma), 11/12), \quad \text{or}$$

(58) $$l = 2, \quad \theta = \min(18/(18+\sigma), 21/24), \quad \text{or}$$

(59) $$l = 3, \quad \theta = \min(39/(42+\sigma), 41/48), \quad \text{or}$$

(60) $$l = 4, \quad \theta = \min(80/(90+\sigma), 81/96),$$

we can take $1, \theta, \theta\lambda_2, \ldots, \theta\lambda_s$ as the exponents at the next stage of the iteration and have

(61) $$\sigma(s+1) = 1 + \theta(s)\sigma(s).$$

Beginning with the exponents

(62) $$1, \ \lambda = 35/36, \ \mu = 29/36,$$

so that $\sigma(3) = 25/9$, we obtain the numerical results in Table 1. The essential feature of these results is that $\sigma = \sigma(14)$ satisfies

(63) $$2\sigma + 17/32 = 12.019845\ldots > 12.$$

Table 1

s	$\sigma(s)$	l	θ	$\sigma(s+1)$
3	25/9	2	.866310	3.406417
4	3.406417	3	41/48	3.909647
5	3.909647	3	.849494	4.321224
6	4.321224	4	81/96	4.646033
7	4.646033	4	81/96	4.920090
8	4.920090	4	.842814	5.146722
9	5.146722	4	.840806	5.327398
10	5.327398	4	.839213	5.470822
11	5.470822	4	.837952	5.584288
12	5.584288	4	.836957	5.673813
13	5.673813	4	.836174	5.744297

3. Allocation of variables. Our hypothesis that the equations (2) have a non-singular real solution implies that the linear equations

(64) $$a_1 y_1 + \ldots + a_N y_N = 0, \quad b_1 y_1 + \ldots + b_N y_N = 0$$

have a solution with each $y_i \geqslant 0$ and that $y_i > 0$ for at least 2 values of i for which the corresponding columns of coefficients have rank 2.

LEMMA 7. *Suppose that the equations* (64) *have a real solution with all* $y_i \geqslant 0$ *and with* $y_i > 0$ *for some 2 values of* i *for which the corresponding columns of coefficients have rank 2. Then there exist 2 columns of rank 2, and a further* $S \leqslant 2$ *columns of rank* S *such that the equations* (64) *have a real solution with the unknowns corresponding to these columns positive and the other unknowns zero.*

This is Lemma 13 of Davenport and Lewis [11].

LEMMA 8. *Assuming hypothesis* (iii) *of Theorem* 1, *the* 73 *suffices can be partitioned into two sets* $\mathcal{S}$ *and* $\mathcal{T}$ *such that*

(i) $|\mathcal{S}| = 21$ *and* $|\mathcal{T}| = 52$;

(ii) *the* $2+S$ *suffices corresponding to the columns occurring in Lemma* 7 *are all in* $\mathcal{S}$;

(iii) *no value* a_i/b_i *occurs for more than* 2 *suffices* $i \in \mathcal{S}$; *and*

(iv) *no value* a_i/b_i *occurs for more than* 16 *suffices* $i \in \mathcal{T}$.

Proof. We collect the ratios a_i/b_i into blocks of equal ratios and put the $2+S$ suffices arising from Lemma 7 into $\mathcal{S}$. We take the largest block of equal ratios and allocate suffices to $\mathcal{S}$ so that, together with any suffices from that block already allocated to $\mathcal{S}$, two suffices from that block are placed in $\mathcal{S}$. The remaining ratios of that block, at most 16, are placed in $\mathcal{T}$. We repeat this for the second largest block, then the third and so on. We continue until either we have allocated 21 suffices to $\mathscr{S}$ or we have reached ratios that occur only once.

In the former case we have reached the ninth largest block of equal ratios, so that subsequent ratios can occur at most 8 times. Thus all the remaining suffices can be allocated to $\mathscr{T}$. In the latter case let r denote the number of blocks containing more than one ratio and let s denote the number of blocks consisting of a single ratio. For each repeated ratio we can put two suffices into $\mathscr{S}$ so altogether we can allocate $2r+s$ suffices to $\mathscr{S}$. If $r \geqslant 4$ then since $r+s \geqslant 17$ we have $2r+s \geqslant 21$. If $r \leqslant 3$ then, since no ratio can occur more than 18 times

$$s \geqslant 73-18r \geqslant 22-r$$

and so $2r+s > 21$. Thus we can continue to allocate the suffices so that $\mathscr{S}$ is filled and any remaining suffices can then be allocated to $\mathscr{T}$.

We now renumber the variables so that

$$\mathscr{S} = \{1, 2, \ldots, 21\} \quad \text{and} \quad \mathscr{T} = \{22, \ldots, 73\}.$$

Lemma 9. *The variables in $\mathscr{T}$ can be renumbered so that*

(i) $\dfrac{a_{22}}{b_{22}} \neq \dfrac{a_{23}}{b_{23}}, \ \dfrac{a_{24}}{b_{24}} \neq \dfrac{a_{25}}{b_{25}}, \ldots, \dfrac{a_{72}}{b_{72}} \neq \dfrac{a_{73}}{b_{73}}$;

(ii) *the ratios* $\dfrac{a_{42}}{b_{42}}, \dfrac{a_{43}}{b_{43}}, \dfrac{a_{46}}{b_{46}}, \dfrac{a_{47}}{b_{47}}$ *are distinct; and*

(iii) *the ratios* $\dfrac{a_{68}}{b_{68}}, \dfrac{a_{69}}{b_{69}}, \dfrac{a_{72}}{b_{72}}, \dfrac{a_{73}}{b_{73}}$ *are distinct.*

Proof. Since no ratio occurs more than 16 times in $\mathscr{T}$ there are at least 4 different ratios in $\mathscr{T}$. We take one of each of the 4 most common ratios as

$$\frac{a_{42}}{b_{42}}, \frac{a_{43}}{b_{43}}, \frac{a_{46}}{b_{46}} \text{ and } \frac{a_{47}}{b_{47}}.$$

Since the fifth most common ratio in $\mathscr{T}$ occurs at most 10 times we now have a set $\mathscr{T}_1$ of 48 suffices in which no ratio occurs more than 15 times. Therefore there are at least 4 different ratios in $\mathscr{T}_1$ and we can take one of each of the 4 most common ratios as

$$\frac{a_{68}}{b_{68}}, \frac{a_{69}}{b_{69}}, \frac{a_{72}}{b_{72}} \text{ and } \frac{a_{73}}{b_{73}}.$$

Amongst the remaining 44 ratios no value occurs more than 14 times so the remaining ratios can be arranged into 22 pairs of unequal ratios.

4. Preliminaries to the analytical method. The linear equations

$$\begin{aligned} a_1 y_1 + \ldots + a_{21} y_{21} &= 0, \\ b_1 y_1 + \ldots + b_{21} y_{21} &= 0 \end{aligned} \tag{65}$$

have a solution of rank 2 with each $y_i \geqslant 0$. By making a small perturbation of this solution we obtain a solution with $y_i > 0$ for $1 \leqslant i \leqslant 21$.

Let P be large and positive. For $1 \leqslant j \leqslant 21$ we choose constants p_j and r_j so that, taking $z_j = y_j^{1/6}$,

$$0 < p_j < z_j < r_j \tag{66}$$

and $r_j - p_j$ is suitably small. Let $N(P)$ denote the number of solutions of (2) such that the variables x_j satisfy

$$p_j P < x_j < r_j P \quad \text{for} \quad j = 1, \ldots, 21, \tag{67}$$

$$P^{\lambda_\nu} < x_j < 2P^{\lambda_\nu} \quad \text{for} \quad j = 22, \ldots, 73 \tag{68}$$

where $\nu = [(j-18)/2]$ for $22 \leqslant j \leqslant 47$, $\nu(j) = \nu(j-26)$ for $48 \leqslant j \leqslant 73$ and $1, \lambda_2, \ldots, \lambda_{14}$ are the exponents determined by the process described in Section 2.

Let

$$\Lambda_j = a_j \alpha + b_j \beta, \quad 1 \leqslant j \leqslant 73, \tag{69}$$

and

$$T_j(\Lambda_j) = \sum e(\Lambda_j x^6) \tag{70}$$

where x ranges over an interval of the form (67) or (68) depending on the suffix j. The number $N(P)$ is then given by

$$N(P) = \int_0^1 \int_0^1 \prod_{j=1}^{73} T_j(\Lambda_j)\, d\alpha\, d\beta, \tag{71}$$

and in order to evaluate this integral by the Hardy–Littlewood method we shall need information about the solutions of (2) in p-adic fields.

Lemma 10. *If $p \nmid k$ and $N \geqslant 2k^2+1$ then the equations*

$$\begin{aligned} f(x) &= a_1 x_1^k + \ldots + a_N x_N^k = 0, \\ g(x) &= b_1 x_1^k + \ldots + b_N x_N^k = 0 \end{aligned} \tag{72}$$

have a non-trivial p-adic solution.

Proof. When k is odd this follows from the stronger result of Davenport and Lewis [10]. If k is even we deduce the result from the methods of [10], so we only sketch the argument. Let

$$\Theta = \prod_{i,j} (a_i b_j - a_j b_i),$$

then by p-adic compactness it is sufficient to prove the lemma when $\Theta \neq 0$. If the equations (72) have $\Theta \neq 0$ then they are equivalent to a p-normalized pair of equations

$$f'(\boldsymbol{x}) = 0, \quad g'(\boldsymbol{x}) = 0$$

and by Hensel's Lemma, these equations have a non-trivial p-adic solution if the congruences

$$f'(\boldsymbol{x}) \equiv 0, \quad g'(\boldsymbol{x}) \equiv 0 \bmod p$$

have a solution of rank 2. The forms $f'(\boldsymbol{x})$ and $g'(\boldsymbol{x})$ contain sections

$$f_0(\boldsymbol{x}^*) = a_1' x_1^k + \ldots + a_n' x_n^k,$$

$$g_0(\boldsymbol{x}^*) = b_1' x_1^k + \ldots + b_n' x_n^k$$

(after renumbering the variables) where $n \geqslant 2k+1$ and for $i = 1, \ldots, n$ p does not divide both a_i' and b_i'. By Chevalley's Theorem [1] the congruences

$$f_0(\boldsymbol{x}^*) \equiv 0, \quad g_0(\boldsymbol{x}^*) \equiv 0 \bmod p$$

have a non-trivial solution mod p and it follows from Lemmas 3 and 9 of [10] that these congruences have a solution of rank $2 \pmod p$, which completes the proof of the lemma.

LEMMA 11. *If every form $\lambda f + \mu g$ $(\lambda, \mu \neq 0, 0)$ in the pencil of f and g has at least k^2+1 variables with non-zero coefficients and the equations* (72) *have a non-trivial p-adic solution, then they have a non-singular p-adic solution.*

This is Theorem 2 of Davenport and Lewis [10].

Returning now to the equations (2), the conditions of Theorem 1 ensure that the equations have a non-singular p-adic solution for every prime p.

5. The minor arcs. The unit square is divided up into major arcs $\boldsymbol{M}$, where α and β both have good rational approximations, and the minor arcs $\boldsymbol{m}$ which consist of the rest of the unit square. The major arc $\boldsymbol{M}(A, B, Q)$ consists of those (α, β) which have simultaneous approximations A/Q, B/Q satisfying

$$|\alpha - A/Q| < Q^{-1}P^{-5-\delta}, \quad |\beta - B/Q| < Q^{-1}P^{-5-\delta} \tag{73}$$

where $(A, B, Q) = 1$ and δ is a small positive number, independent of P. The major arcs $\boldsymbol{M}$ are the union of those $\boldsymbol{M}(A, B, Q)$ for which

$$1 \leqslant Q \leqslant P^{1-\delta}, \quad 0 \leqslant A, B < Q, \quad (A, B, Q) = 1. \tag{74}$$

LEMMA 12. *Suppose that $(\alpha, \beta) \in \boldsymbol{m}$. Let i and j be two of the suffices* $1, 2, \ldots, 21$ *for which $a_i b_j - a_j b_i \neq 0$. Then*

$$\textit{either } |T_i(\Lambda_i)| < P^{31/32+2\delta} \quad \textit{or } |T_j(\Lambda_j)| < P^{31/32+2\delta}. \tag{75}$$

This may be proved in the same way as Lemma 6 of [6].

Now let $V(\alpha, \beta)$ be the product of four sums $T_j(\Lambda_j)$, $j \in \mathscr{S}$, with no ratio a_i/b_i occurring more than twice.

LEMMA 13. *For any $\varepsilon > 0$*

$$\int_0^1 \int_0^1 \Big| V(\alpha, \beta) \prod_{j=22}^{73} T_j(\Lambda_j) \Big| d\alpha\, d\beta \ll P^{2\sigma+\varepsilon}, \tag{76}$$

where $\sigma = \sigma(14) = 5.744297\ldots$

This follows from the Cauchy–Schwarz inequality and our choice of $\lambda_2, \ldots, \lambda_{14}$.

LEMMA 14. *For any $\varepsilon > 0$*

$$\iint_{\boldsymbol{m}} \prod_{j=1}^{73} |T_j(\Lambda_j)|\, d\alpha\, d\beta \ll P^{17-17/32+2\sigma+34\delta+\varepsilon}. \tag{77}$$

Proof. The 21 suffices in $\mathscr{S}$ are partitioned into blocks of equal ratios, no block containing more than two suffices. For each suffix j let $\boldsymbol{m}_j$ denote the subset of $\boldsymbol{m}$ for which

$$\max_{i \in \mathscr{S}} |T_i(\Lambda_i)| = |T_j(\Lambda_j)|. \tag{78}$$

Then for all suffices $i \in \mathscr{S}$ except possibly for at most two suffices j we have

$$|T_i(\Lambda_i)| \ll P^{31/32+2\delta} \quad \text{for} \quad (\alpha, \beta) \in \boldsymbol{m}_j. \tag{79}$$

Let $V_j(\alpha, \beta)$ denote a product of four exponential sums, including all those from the block of a_j/b_j, then

$$\iint_{\boldsymbol{m}_j} \Big| \prod_{i=1}^{73} T_i(\Lambda_i) \Big| d\alpha\, d\beta \ll P^{(31/32+2\delta)\cdot 17} \int_0^1 \int_0^1 \Big| V_j(\alpha, \beta) \prod_{i=22}^{73} T_i(\Lambda_i) \Big| d\alpha\, d\beta$$
$$\ll P^{17-17/32+2\sigma+34\delta+\varepsilon}$$

and the lemma now follows on summing over j.

6. The major arcs. For $(\alpha, \beta) \in \boldsymbol{M}(A, B, Q)$ let

$$\varphi = \alpha - A/Q, \quad \psi = \beta - B/Q \tag{80}$$

and

$$c_j/q_j = (a_j A + b_j B)/Q, \quad (c_j, q_j) = 1. \tag{81}$$

Then

$$A_j = a_j\alpha + b_j\beta = c_j/q_j + \gamma_j \tag{82}$$

where

$$\gamma_j = a_j\varphi + b_j\psi. \tag{83}$$

LEMMA 15. *We have*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\prod_{j=1}^{21}\min(P, P^{-5}|\gamma_j|^{-1})\,d\varphi\,d\psi \ll P^9 \tag{84}$$

and for any $\tau > 0$

$$\iint_{D(\tau)}\prod_{j=1}^{21}\min(P, P^{-5}|\gamma_j|^{-1})\,d\varphi\,d\psi \ll P^{9-18\tau} \tag{85}$$

where

$$D(\tau) = \{(\varphi, \psi):\ |\varphi| > P^{-6+\tau},\ |\psi| > P^{-6+\tau}\}. \tag{86}$$

This may be proved in the same way as Lemma 22 of Davenport and Lewis [11], since 20 of the ratios a_i/b_i can be arranged into 10 unequal pairs.

LEMMA 16. *For any* $\varepsilon > 0$

$$\sum_{A,B}\prod_{j=1}^{21} q_j^{-1/6} \ll Q^{-3/2+\varepsilon} \tag{87}$$

where the summation is over

$$0 \leqslant A, B < Q, \quad (A, B, Q) = 1 \tag{88}$$

and q_j *is defined by* (82).

This may be proved in the same way as Lemma 35 of Davenport and Lewis [9] since, in their notation, we have

$$\nu \geqslant 11, \quad \theta_j = 1/6 \text{ or } 1/3 \quad \text{and} \quad \theta_1 + \ldots + \theta_\nu = 7/2.$$

The importance of Lemma 16 is that since the exponent of Q is less than -1 the singular series will converge. However, before passing to the singular series we prune the major arcs. The first step is to take an arbitrary small positive number ω and to estimate the contribution of those major arcs with $Q > P^\omega$.

LEMMA 17. *The contribution made to the integral by all those major arcs* $\mathbf{M}(A, B, Q)$ *with* $Q > P^\omega$ *is*

$$\ll P^{5+4\sigma-\omega/2}. \tag{89}$$

This may be proved in the same way as Lemma 24 of Davenport and Lewis [11], using Lemmas 15 and 16 in place of their Lemmas 22 and 23.

The next step is to replace the remaining major arcs $\mathbf{M}(A, B, Q)$ by truncated major arcs $\mathbf{M}_0(A, B, Q)$ defined by

$$|\alpha - A/Q| < P^{-6+\tau}, \quad |\beta - B/Q| < P^{-6+\tau}, \tag{90}$$

where τ is a small positive constant.

LEMMA 18. *The total difference between the contributions of the major arcs* $\mathbf{M}(A, B, Q)$ *and the truncated major arcs* $\mathbf{M}_0(A, B, Q)$ *with* $Q \leqslant P^\omega$ *is*

$$\ll P^{5+4\sigma-18\tau}. \tag{91}$$

This may be proved in the same way as Lemma 25 of Davenport and Lewis [11], with our Lemma 15 replacing their Lemma 22.

Thus

$$N(P) = \sum_{Q\leqslant P^\omega}\sum_{A,B}\iint_{\mathbf{M}_0(A,B,Q)}\prod_{j=1}^{73} T_j(A_j)\,d\alpha\,d\beta + o(P^{5+4\sigma}). \tag{92}$$

7. Proof of Theorem 1. On the truncated major arcs $\mathbf{M}_0(A, B, Q)$ there is a good approximation to the exponential sums $T_j(A_j)$.

LEMMA 19. *Let* $(\alpha, \beta) \in \mathbf{M}_0(A, B, Q)$. *For* $1 \leqslant j \leqslant 21$

$$T_j(A_j) = q_j^{-1} S(c_j, q_j) I_j(\gamma_j) + O(P^\eta) \tag{93}$$

where

$$S(c, q) = \sum_{x=1}^{q} e_q(cx^6), \tag{94}$$

$$I_j(\gamma_j) = \int_{p_jP}^{r_jP} e(\gamma_j x^6)\,dx \tag{95}$$

and η *is small if* ω *and* τ *are small. If* $22 \leqslant j \leqslant 73$ *then*

$$T_j(\gamma_j) = q_j^{-1} S(c_j, q_j) P^{\lambda_\nu} + O(P^{\eta\lambda_\nu}) \tag{96}$$

where $\nu = [(j-18)/2]$ *for* $22 \leqslant j \leqslant 47$, $\nu(j) = \nu(j-26)$ *for* $48 \leqslant j \leqslant 73$.

This is essentially Lemma 27 of Davenport and Lewis [11].

Substituting these approximations into (92) we obtain, as in [11], Lemma 28

$$N(P) = \mathfrak{S}(P^\omega) I(P^{-6+\tau}) + o(P^{5+4\sigma}) \tag{97}$$

where

$$\mathfrak{S}(P^\omega) = \sum_{Q\leqslant P^\omega}\sum_{A,B}\prod_{j=1}^{73} q_j^{-1} S(c_j, q_j) \tag{98}$$

and

$$I(P^{-6+\tau}) = CP^{4(\sigma-1)} \iint \prod_{j=1}^{21} I_j(\gamma_j)\,d\varphi\,d\psi \tag{99}$$

where C is a positive constant and the integral is taken over the region

$$|\varphi| < P^{-6+\tau}, \quad |\psi| < P^{-6+\tau}.$$

LEMMA 20. *If the equations* (2) *have a non-singular p-adic solution for every prime p then*

$$S(P^{\omega}) = S + o(1) \quad \text{as } P \to \infty \tag{100}$$

where S is a positive constant.

This is essentially a combination of Lemmas 29 and 31 of Davenport and Lewis [11].

LEMMA 21. *We have*

$$I(P^{-6+\tau}) = C_0 P^{5+4\sigma}\big(1+o(1)\big) \quad \text{as } P \to \infty \tag{101}$$

where C_0 is positive and independent of P.

This is essentially Lemma 30 of Davenport and Lewis [11], we have $C_0 > 0$ since the box defined by (67) contains a non-singular real solution of the equations.

Thus

$$N(P) = C_0 S P^{5+4\sigma}\big(1+o(1)\big) \quad \text{as } P \to \infty \tag{102}$$

where $C_0 S > 0$, and this completes the proof of Theorem 1.

References

[1] C. Chevalley, *Démonstration d'une hypothèse de M. Artin*, Abh. Math. Sem. Hamburg, 11 (1935), pp. 73–75.

[2] R. J. Cook, *Simultaneous quadratic equations*, J. London Math. Soc. 4 (1971), pp. 319–326.

[3] — *A note on a lemma of Hua*, Quart. J. Math (Oxford), 23 (1972), pp. 287–288.

[4] — *Pairs of additive equations*, Michigan Math. J. 19 (1972), pp. 325–331.

[5] — *Pairs of additive equations II: large odd degree*, J. Number Theory 17(1983), pp. 80–92.

[6] — *Pairs of additive equations III: quintic equations*, Proc. Edinburgh Math. Soc. 26(1983), pp. 191–211.

[7] H. Davenport, *On sums of positive integral kth powers*, Proc. Roy. Soc. 170 A (1939), pp. 293–299.

[8] H. Davenport and P. Erdös, *On sums of positive integral kth powers*, Ann. of Math. 40 (1939), pp. 533–536.

[9] H. Davenport and D. J. Lewis, *Cubic equations of additive type*, Phil. Trans. Roy. Soc. A261 (1966), pp. 97–136.

[10] H. Davenport and D. J. Lewis, *Two additive equations*, Proc. Sympos. Pure Math. XII, Amer. Math. Soc. 1969, pp. 74–98.

[11] — — *Simultaneous equations of additive type*, Phil. Trans. Roy. Soc. A264 (1969), pp. 557–595.

[12] L. K. Hua, *On Waring's problem*, Quart. J. Math. (Oxford) 9 (1938), pp. 199–202.

[13] R. C. Vaughan, *On pairs of additive cubic equations*, Proc. London Math. Soc. 34 (1977), pp. 354–364.

[14] — *Homogeneous additive equations and Waring's problem*, Acta Arith. 33 (1977), pp. 231–253.

DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF SHEFFIELD
Sheffield 10, England

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